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On the Weak Solutions of the 3D MHD Equations and 3D Magneto-Micropolar Equations

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ABSTRACT: In our current line of investigation, we examine the finite-time regularity of generalised solutions to the 3D MHD equations in anisotropic Lebesgue space as well as the 3D magneto-micropolar equations in anisotropic Lorentz space. Using the pressure term and its gradient as a foundation, the new regularity results are presented. We concluded by demonstrating the requirements in terms of magnetic field and velocity components.

Key Words: 3D viscous MHD equations, logarithmic pressure regularity, generalised solutions, anisotropic Lebesgue spaces, 3D magneto-micropolar equations, partial component regularity, anisotropic Lorentz spaces.

Contents

1. Introduction

The first system we will analyse in the space $\mathbb{R}^3 \times \mathbb{R}_+$ for the conditional regularity is the following system of 3D viscous MHD equations:

$$\begin{cases} \frac{\partial v}{\partial t} + v \cdot \nabla v - \triangle v + \nabla \rho + \frac{1}{2}\nabla |w|^2 - w \cdot \nabla w = 0, \\ \frac{\partial w}{\partial t} - \triangle w + v \cdot \nabla w - w \cdot \nabla v = 0, \\ \nabla \cdot v = 0, \ \nabla \cdot w = 0, \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \end{cases} \tag{1.1}$$

where $v(x,t)$ and $w(x,t)$ are the velocity and magnetic fields of the flow, $\rho(x,t)$ is the scalar pressure, while v_0 and w_0 are the given initial velocity and magnetic fields with $\nabla \cdot v_0 = 0$ and $\nabla \cdot w_0 = 0$ in the distributional sense.

For the Navier-Stokes equations Berselli and Galdi [1] showed the solution is regular on [0,T), if Serrin-type regularity conditions

$$\int_0^T \|\rho\|_{L^l}^m \, dt < \infty, \quad \text{where } \frac{3}{l} + \frac{2}{m} \leq 2, \ \ \frac{3}{2} < l \leq \infty,$$

and

$$\int_0^T \|\nabla\rho\|_{L^l}^m \, dt < \infty, \quad \text{where } \frac{3}{l} + \frac{2}{m} \leq 3, \ \ 1 < l \leq \infty,$$

are satisfied. Similarly, Zhou [2] presented the new results, given as

$$\rho \in L^{l,m}, \quad w \in L^{2l,2m}, \text{ or } \|\rho\|_L^{\infty,\frac{3}{l}}, \quad \|w\|_L^{\infty,3}, \tag{1.2}$$

$$\text{where } \frac{2}{l} + \frac{3}{m} \leq 2, \ \ \frac{3}{2} < m \leq \infty,$$

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and

$$\nabla\rho \in L^{l,m}, \quad w \in L^{3l,3m}, \text{ or } \|\nabla\rho\|_L^{\infty,3}, \quad \|\nabla w\|_L^{\infty,3}, \tag{1.3}$$
$$\text{where } \frac{2}{l}+\frac{3}{m} \leq 3, \quad 1 < m \leq \infty.$$

Later on, conditions (1.2) and (1.3) were improved by Duan [3] in terms of only pressure and its gradient

$$\rho \in L^{\frac{2n}{2n-3}}(0,T,L^n(\mathbb{R}^3)) \text{ with } \frac{3}{2} < n \leq \infty,$$

and

$$\nabla\rho \in L^{\frac{2n}{2n-3}}(0,T,L^n(\mathbb{R}^3)) \text{ with } 1 < n \leq \infty.$$

These results have been logarithmically improved by Zhou and Fan [4] given as

$$\int_0^T \frac{\|\rho(\cdot,t)\|_{L^n}^{\frac{2n}{2n-3}}}{1+\ln(e+\|\rho(\cdot,t)\|_{L^n})}\,dt < \infty \quad \text{with} \quad n > \frac{3}{2},$$

also for the limiting case $n=\infty$,

$$\int_0^T \frac{\|\rho(\cdot,t)\|_{L^\infty}}{1+\ln(e+\|\rho(\cdot,t)\|_{L^m})}\,dt < \infty \quad \text{for} \quad 1 < m < \infty,$$

and in terms of gradient pressure,

$$\int_0^T \frac{\|\nabla\rho(\cdot,t)\|_{L^n}^{\frac{2n}{3n-3}}}{1+\ln(e+\|\nabla\rho(\cdot,t)\|_{L^n})}\,dt < \infty \quad \text{with} \quad n > 1,$$

also for the limiting case $n=\infty$,

$$\int_0^T \frac{\|\nabla\rho(\cdot,t)\|_{L^\infty}^{\frac{2}{3}}}{1+\ln(e+\|\rho(\cdot,t)\|_{L^m})}\,dt < \infty \quad \text{for} \quad 1 < m < \infty.$$

For the regularity criterion in BMO, critical Besov, anisotropic, Triebel–Lizorkin and Multiplier spaces for various fluid models. We refer to the following readings [5,6,7,8,9,10,11,12,13] and references therein.

The second mathematical model we analyse is the following 3D magneto-micropolar system investigated in $\mathbb{R}^3 \times \mathbb{R}_+$:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{U} - \triangle\mathcal{U} + \nabla(\xi+\mathcal{V}^2) - \nabla\times\mathcal{W} - \mathcal{V}\cdot\nabla\mathcal{V} = 0, \\ \frac{\partial \mathcal{W}}{\partial t} - \triangle\mathcal{W} + \mathcal{U}\cdot\nabla\mathcal{W} - \nabla\times\mathcal{U} + 2\mathcal{W} - \nabla div\ \mathcal{W} = 0, \\ \frac{\partial \mathcal{V}}{\partial t} - \triangle\mathcal{V} + \mathcal{U}\cdot\nabla\mathcal{V} - \mathcal{V}\cdot\nabla\mathcal{U} = 0, \\ \nabla\cdot\mathcal{U} = 0, \quad \nabla\cdot\mathcal{V} = 0, \\ \mathcal{U}(x,0) = \mathcal{U}_0(x), \quad \mathcal{W}(x,0) = \mathcal{W}_0(x), \quad \mathcal{V}(x,0) = \mathcal{V}_0(x). \end{cases} \tag{1.4}$$

The fluid velocity and magnetic fields are described by $\mathcal{U}$ and $\mathcal{V}$, Where $\mathcal{W}$, ξ are the micro-rotation velocity and scalar pressure. Whereas, $\mathcal{U}_0$, $\mathcal{V}_0$ and $\mathcal{W}_0$ are the given initial data with $\nabla\cdot\mathcal{U}_0 = 0$ and $\nabla\cdot\mathcal{V}_0 = 0$ in the distributional sense, C a generic constant could vary from line to line.
It is generally recognised that the problem of regularity for weak solutions to the 3D magneto-micropolar equations is one of the most prominent unresolved problems in applied analysis due to the existence of the magnetic field $\mathcal{V}$ and micro-rotation velocity $\mathcal{W}$, which complicates nonlinear components. Rojas-Medar and Boldrini [14] provided the global weak solution to system (1.4), whereas Rojas-Medar [15] establishes the local strong solutions. But the question of the regularity of the weak solutions that preserved for all time has remained key important problem.

For the system (1.4), Yuan [16] presented the fundamental regularity criteria, given as

$$\int_0^T \|\mathcal{U}\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l}+\frac{2}{m} = 1, \quad 3 < l \leq \infty,$$

and

$$\int_0^T \|\nabla \mathcal{U}\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l} + \frac{2}{m} = 2, \ \ \frac{3}{2} < l \leq \infty.$$

Later on, Ni. et al. [17] presented a result for the horizontal components for 3D MHD system given as

$$\int_0^T \|\nabla_h \mathcal{U}\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l} + \frac{2}{m} \leq 2, \ \ \frac{3}{2} < l \leq \infty,$$

and

$$\int_0^T \|\nabla_h \mathcal{V}\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l} + \frac{2}{m} \leq 2, \ \ \frac{3}{2} < l \leq \infty,$$

where $\nabla_h = (\partial_1, \partial_2)$.

In [18] Jia. improved the above regularity criteria given as

$$\int_0^T \|\nabla_h \mathcal{U}_h\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l} + \frac{2}{m} \leq 2, \ \ \frac{3}{2} < l \leq \infty,$$

and

$$\int_0^T \|\nabla_h \mathcal{V}_h\|_{L^l}^m d\tau < \infty, \text{ with } \frac{3}{l} + \frac{2}{m} \leq 2, \ \ \frac{3}{2} < l \leq \infty,$$

where $\nabla_h = (\partial_1, \partial_2)$ and $\mathcal{U}_h = (\mathcal{U}_1, \mathcal{U}_2)$.

Recently, Xu et al. [19] have revised the above sufficient condition required only on $(\partial_1\mathcal{U}_1, \partial_1\mathcal{V}_1)$, $(\partial_2\mathcal{U}_2, \partial_2\mathcal{V}_2)$, $(\partial_3\mathcal{U}_3, \partial_3\mathcal{V}_3)$, for the 3D incompressible MHD system. Given as

$$L_j^{m,l}(T) := \int_0^T (\|\partial_j \mathcal{U}_j\|_{L^l}^m + \|\partial_j \mathcal{V}_j\|_{L^l}^m) d\tau < \infty,$$

where $\frac{2}{m} + \frac{3}{l} = 2$, and $\frac{3}{2} < l \leq \infty$.

Motivated by the above results we will present their logarithmic improvements that are refined for the weak solutions of system (1.4). Our regularity criteria logarithmically improved the results presented in the partial derivative of the partial components velocity and magnetic fields. The regularity criteria is fine in the sense that it only needs to estimate conditions on the velocity and magnetic fields for the regularity of weak solutions.

2. Preliminaries

We will review definitions, lemmas, and inequalities in this section. We will also give a few outcomes that will help to support our primary findings.

Definition 1. Let $1 \leq l, m, n \leq \infty$. A measurable function f belongs to an anisotropic Lebesgue spaces $L^l(\mathbb{R}_{x_1}; L^m(\mathbb{R}_{x_2}; L^n((\mathbb{R}_{x_3})))$ if the norm

$$\left\|\left\|\left\|f\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}} = \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |f(x_1,x_2,x_3)|^l dx_1\Big)^{\frac{m}{l}} dx_2\Big)^{\frac{n}{m}} dx_3\Big)^{\frac{1}{n}} < \infty.$$

Remark 1. Througout the paper we take $B^+ = v + w$ and $B^- = v - w$ for the system (1.1), and $\mathcal{R}^+ = \mathcal{U} + \mathcal{V}$ and $\mathcal{R}^- = \mathcal{U} - \mathcal{V}$ for the sytem (1.4) to help prove our main resutls.
A class of global Leray-Hopf weak solutions (B^+, B^-) to system (1.1) was constructed by Durant and Lions [20] which satisfies energy inequality.

Remark 2. Thanks to

$$\int_0^T \frac{\Big\|\Big\|\big\|\rho(\cdot,t)\big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|^{\eta}_{L^n_{x_3}}}{1+\ln(e+\|\rho(\cdot,t)\|^2_{L^2})}\,dt \leq \int_0^T \Big\|\Big\|\big\|\rho(\cdot,t)\big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|^{\eta}_{L^n_{x_3}}\,dt,$$

with conditions $\frac{2}{\eta}+\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=2,\quad 2\leq l,m,n\leq\infty,$ and $1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})\geq 0$. It would be easy to derive the pressure regularity criterion for 3D viscous MHD equations.

Lemma 1. [21] Let $2\leq l,m,n\leq\infty$ and $1-\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)\geq 0,$ then exists $C>0,$ such that $\forall\ B^+\in C_0^\infty(\mathbb{R}^3)$

$$\Big\|\Big\|\big\|B^+\big\|_{L^{\frac{2m}{m-2}}_{x_2}}\Big\|_{L^{\frac{2n}{n-2}}_{x_3}}\Big\|_{L^{\frac{2l}{l-2}}_{x_1}} \leq C\|\partial_1 B^+\|^{\frac{1}{m}}_{L^2}\|\partial_2 B^+\|^{\frac{1}{n}}_{L^2}\|\partial_3 B^+\|^{\frac{1}{l}}_{L^2}\|B\|^{1-(\frac{1}{m}+\frac{1}{n}+\frac{1}{l})}_{L^2},$$

here we let $\frac{2m}{m-2}=\infty$ when $m=2$.
Similarly, let $1\leq l,m,n\leq\infty$ and $1-\left(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}\right)\geq 0$ then exists $C>0,$ such that $\forall\ B^+\in C_0^\infty(\mathbb{R}^3)$

$$\Big\|\Big\|\big\|B^+\big\|_{L^{\frac{2m}{m-1}}_{x_2}}\Big\|_{L^{\frac{2n}{n-1}}_{x_3}}\Big\|_{L^{\frac{2l}{l-1}}_{x_1}} \leq C\|\partial_1 B^+\|^{\frac{1}{2m}}_{L^2}\|\partial_2 B^+\|^{\frac{1}{2n}}_{L^2}\|\partial_3 B^+\|^{\frac{1}{2l}}_{L^2}\|B\|^{1-(\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2l})}_{L^2},$$

here we let $\frac{2m}{m-1}=\infty$ when $m=1$.
Similar statement holds for B^-.

Definition 2. [22] Let $l=(l_1,l_2,l_3)$ and $m=(m_1,m_2,m_3)$ with $0<l_i\leq\infty$, $0<m_i\leq\infty$. If $l_i=\infty$ then $m_i=\infty$ for every $i=1,2,3$. An anisotropic Lorentz space $L^{l_1,m_1}(\mathbb{R}_{x_1};L^{l_2,m_2}(\mathbb{R}_{x_2};L^{l_3,m_3}((\mathbb{R}_{x_3}))$ is defined as

$$\Big\|\Big\|\big\|f\big\|_{L^{l_1,m_1}_{x_1}}\Big\|_{L^{l_2,m_2}_{x_2}}\Big\|_{L^{l_3,m_3}_{x_3}} := \Big(\int_0^\infty\Big(\int_0^\infty\Big(\int_0^\infty [t_1^{\frac{1}{l_1}}t_2^{\frac{1}{l_2}}t_3^{\frac{1}{l_3}}f^{*_1,*_2,*_3}(t_1,t_2,t_3)]^{m_1}\frac{dt_1}{t_1}\Big)^{\frac{m_2}{m_1}}\frac{dt_2}{t_2}\Big)^{\frac{m_3}{m_2}}\frac{dt_3}{t_3}\Big)^{\frac{1}{m_3}}<\infty.$$

For the detailed study on the anisotropic Lorentz spaces and mixed norm spaces (see [23,24]).

Lemma 2. [23,22] If $1\leq l_1,l_2,m_1,m_2\leq\infty$, then $\forall\ f\in L^{l_1,m_1}(\mathbb{R}^n),\ g\in L^{l_2,m_2}(\mathbb{R}^n)$,

$$\|fg\|_{L^{l,m}(\mathbb{R}^n)}\leq C\ \|f\|_{L^{l_1,m_1}(\mathbb{R}^n)}\|g\|_{L^{l_2,m_2}(\mathbb{R}^n)},$$

where $\frac{1}{l}=\frac{1}{l_1}+\frac{1}{l_2}$ and $\frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}$.

Lemma 3. [23,22] Let $1<l<\infty, 1\leq m\leq\infty$ and $\frac{1}{l}+\frac{1}{l'}=1,\ \frac{1}{m}+\frac{1}{m'}=1$ with $1<l<l'$ and $m'\leq m\leq\infty$. If $\frac{1}{l_2}+1=\frac{1}{l}+\frac{1}{l_1}$ and $\frac{1}{m_2}=\frac{1}{m}+\frac{1}{m_1'}$ then the convolution operator

$$*:L^{l,m}(\mathbb{R}^n)\times L^{l_1,m_1}(\mathbb{R}^n)\mapsto L^{l_2,m_2}(\mathbb{R}^n),$$

is a bounded bilinear operator.

Lemma 4. [25] There exists a positive constant C such that

$$\Big\|\Big\|\big\|f\big\|_{L^{\frac{2m}{m-2},2}_{x_2}}\Big\|_{L^{\frac{2n}{n-2},2}_{x_3}}\Big\|_{L^{\frac{2l}{l-2},2}_{x_1}} \leq C\|\partial_1 f\|^{\frac{1}{m}}_{L^2}\|\partial_2 f\|^{\frac{1}{n}}_{L^2}\|\partial_3 f\|^{\frac{1}{l}}_{L^2}\|f\|^{1-(\frac{1}{m}+\frac{1}{n}+\frac{1}{l})}_{L^2},$$

where $2\leq l,m,n\leq\infty$ and $\forall\ \ f\in C_0^\infty,\ 1-\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)\geq 0.$

3. Proofs of the main results

This section is devoted to the proofs of our main results, the first two theroems are results on system (1.1) in anisotropic Lebesgue space. The last theorem is result on system (1.4) in anisotropic Lorentz space.

Theorem 1. Assume the initial data $B_0^+, B_0^- \in H^1(\mathbb{R}^3)$ with $(\nabla \cdot B_0^+) = (\nabla \cdot B_0^-) = 0$ in the sense of distributions. Let (B^+, B^-) be the weak solution to (1.1) on $\mathbb{R}^3 \times (0,T)$ if

$$\int_0^T \frac{\left\|\left\|\left\|\rho(\cdot,t)\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^\eta_{L^n_{x_3}}}{1+\ln(e+\|\rho(\cdot,t)\|^2_{L^2})}\,dt < \infty,$$

where $\frac{2}{\eta}+\frac{1}{l}+\frac{1}{m}+\frac{1}{n}$=2 and $2 \le l,m,n \le \infty$ and 1-$(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}) \ge 0$,
then the weak solution is regular on [0,T).
The alternative interpretation of our regularity criteria is, if the solution blows-up on t=T, then

$$\int_0^T \frac{\left\|\left\|\left\|\rho(\cdot,t)\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^\eta_{L^n_{x_3}}}{1+\ln(e+\|\rho(\cdot,t)\|^2_{L^2})}\,dt = \infty.$$

Proof. Converting (1.1) into mathematical symmetric form

$$\partial_t B^+ + B^- \cdot \nabla B^+ - \nabla B^+ + \nabla\rho = 0, \tag{3.1}$$

$$\partial_t B^- + B^+ \cdot \nabla B^- - \nabla B^- + \nabla\rho = 0, \tag{3.2}$$

$$\nabla\cdot B^+ = 0, \qquad \nabla\cdot B^- = 0, \tag{3.3}$$

$$B^+(x,0) = B_0^+(x), \quad B^-(x,0) = B_0^-(x), \tag{3.4}$$

where $B^+(x) = v_0(x) + w_0(x)$ and $B^-(x) = v_0(x) - w_0(x)$.
Multiplying (3.1) with $B^+|B^+|^2$ and (3.2) with $B^-|B^-|^2$ integrating by parts and adding, we get

$$\frac{1}{4}\frac{d}{dt}\Big(\|B^+\|^4_{L^4}+\|B^-\|^4_{L^4}\Big)+\frac{1}{2}\Big(\|\nabla|B^+|^2\|^2_{L^2}+\|\nabla|B^-|^2\|^2_{L^2}\Big)+$$

$$\Big(\||B^+||\nabla B^+|\|^2_{L^2}+\||B^-||\nabla B^-|\|^2_{L^2}\Big) \tag{3.5}$$

$$= \int_{\mathbb{R}^3}\rho B^+\cdot\nabla|B^+|^2dx + \int_{\mathbb{R}^3}\rho B^-\cdot\nabla|B^-|^2dx = I_1 + I_2. \tag{3.6}$$

Now, we estimate I_1

$$I_1 = \int_{\mathbb{R}^3}\rho B^+\cdot\nabla|B^+|^2dx \le \frac{1}{4}\|\nabla|B^+|^2\|^2_{L^2} + C\int_{\mathbb{R}^3}|\rho|^2|B^+|^2dx.$$

Similarly, estimating I_2

$$I_2 = \int_{\mathbb{R}^3}\rho B^-\cdot\nabla|B^-|^2dx \le \frac{1}{4}\|\nabla|B^-|^2\|^2_{L^2} + C\int_{\mathbb{R}^3}|\rho|^2|B^-|^2dx.$$

Estimating the second term on the right hand side of I_1

$$\int_{\mathbb{R}^3}|\rho|^2|B^+|^2dx = \int_{\mathbb{R}^3}|\rho||\rho||B^+|^2dx$$

$$\le C\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}\left\|\left\|\left\|\rho\right\|_{L^{\frac{2l}{l-2}}_{x_1}}\right\|_{L^{\frac{2m}{m-2}}_{x_2}}\right\|_{L^{\frac{2n}{n-2}}_{x_3}}\||B^+|^2\|_{L^2}$$

$$\leq \left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}} \|\rho\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\rho\|_{L^2}^{\frac{1}{l}}\|\partial_2\rho\|_{L^2}^{\frac{1}{m}}\|\partial_3\rho\|_{L^2}^{\frac{1}{n}}\|B^+\|_{L^4}^2$$

$$\leq C\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}} \|\rho\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\rho\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|B^+\|_{L^4}^2$$

$$\leq C\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}} \||B^+|\nabla B^+\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|B^+\|_{L^4}^{4-(\frac{2}{l}+\frac{2}{m}+\frac{2}{n})}$$

$$\leq \frac{1}{4}\||B^+|\nabla B^+\|^2 + C\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} \|B^+\|_{L^4}^4 \ . \tag{3.7}$$

By Similar estimation for I_2, we achieve

$$\leq \frac{1}{4}\||B^-|\nabla B^-\|^2 + C\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} \|B^-\|_{L^4}^4 \ . \tag{3.8}$$

Adding estimates (3.7) and (3.8)

$$\frac{1}{4}\Big(\|B^+\|_{L^4}^4 + \|B^-\|_{L^4}^4\Big) + \frac{1}{2}\Big(\|\nabla|B^+|^2\|_{L^2}^2 + \|\nabla|B^-|^2\|_{L^2}^2\Big)$$
$$\Big(\||B^+||\nabla B^+|^2\|_{L^2}^2 + \||B^-||\nabla B^-|^2\|_{L^2}^2\Big)$$
$$\leq \Bigg(\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Bigg)(\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4).$$

Now, for our desired result, we use the relations between temperature and velocity and Lemma 1.
Let $L(t) = \|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4 \leq e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4$.
Also we use inequality
$1+\ln(e+\|\rho\|_{L^2}^2) \leq 1+\ln(e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)$,

$$\frac{d}{dt}L(t) \leq C\Bigg(\frac{\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)(e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)(1+\ln(e+\|\rho\|_{L^2}^2)$$

$$\frac{d}{dt}L(t) \leq C\Bigg(\frac{\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)L(t)(1+\ln L(t))$$

$$\implies \frac{d}{dt}(1+\ln L(t)) \leq C\Bigg(\frac{\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)(1+\ln L(t)).$$

By applying Gronwall's inequality in the interval $[0,T)$

$$\ln L(t) \leq (1+\ln L(0))exp\Big\{C\Bigg(\frac{\left\|\left\|\left\|\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|_{L^n_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)dt\Big\}.$$

Which together with given theorem

$$\Longrightarrow \sup_{0\le t\le T}(\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)\le\infty.$$

Which completes the proof of Theorem 1.

Theorem 2. Assume the initial data $B_0^+, B_0^- \in H^1(\mathbb{R}^3)$ with $(\nabla\cdot B_0^+)=(\nabla\cdot B_0^-)=0$ in the sense of distributions. Let (B^+,B^-) be the weak solution to (1.1) on $\mathbb{R}^3\times(0,T)$ if

$$\int_0^T \frac{\left\|\left\|\left\|\nabla\rho(\cdot,t)\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^{\eta}_{L^n_{x_3}}}{1+\ln(e+\|\rho(\cdot,t)\|_{L^2}^2)}dt<\infty,$$

where $\frac{2}{\eta}+\frac{1}{l}+\frac{1}{m}+\frac{1}{n}$=3 and $1\le l,m,n\le\infty$ and 1-$(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n})\ge 0$,
then the weak solution is regular on [0,T).
Proof. Now, we continue from (3.6), and estimate I_1

$$I_1\le C\int_{\mathbb{R}^3}\nabla\rho|B^+|^3dx.$$

$$=C\int_{\mathbb{R}^3}|\nabla\rho|^{\frac12}|\nabla\rho|^{\frac12}|B^+||B^+|^2dx$$

$$\le C\left\|\left\|\left\||\nabla\rho|^{\frac12}\right\|_{L^{2l}_{x_1}}\right\|_{L^{2m}_{x_2}}\right\|_{L^{2n}_{x_3}}\left\|\left\|\left\||B^+|^2\right\|_{L^{\frac{2l}{l-2}}_{x_1}}\right\|_{L^{\frac{2m}{m-2}}_{x_2}}\right\|_{L^{\frac{2n}{n-2}}_{x_3}}\||\nabla\rho|^{\frac12}\|_{L^4}\|B^+\|_{L^4}$$

$$\le C\left\|\left\|\left\|\nabla\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^{\frac12}_{L^n_{x_3}}\||B^+|^2\|_{L^2}^{1-(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n})}\|\partial_1|B^+|^2\|_{L^2}^{\frac{1}{2l}}\|\partial_2|B^+|^2\|_{L^2}^{\frac{1}{2m}}\|\partial_3|B^+|^2\|_{L^2}^{\frac{1}{2n}}$$

$$\|\nabla\rho\|_{L^2}^{\frac12}\|B^+\|_{L^4}$$

$$\le C\left\|\left\|\left\|\nabla\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^{\frac12}_{L^n_{x_3}}\|\nabla|B^+|^2\|_{L^2}^{\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}}\||B^+||\nabla B^+|\|_{L^2}^{\frac12}\|B^+\|_{L^4}^{3-(\frac1l+\frac1m+\frac1n)}.$$

$$\le C\left\|\left\|\left\|\nabla\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^{\frac{2}{3-(\frac1l+\frac1m+\frac1n)}}_{L^n_{x_3}}\|B^+\|_{L^4}^4+\frac14(\|\nabla|B^+|^2\|_{L^2}^2\||B^+||\nabla B^+|\|_{L^2}^2). \tag{3.9}$$

similarly we would get estimates for I_2

$$I_2\le C\int_{\mathbb{R}^3}\nabla\rho|B^-|^3dx$$

$$=C\int_{\mathbb{R}^3}|\nabla\rho|^{\frac12}|\nabla\rho|^{\frac12}|B^-||B^-|^2dx$$

$$\le C\left\|\left\|\left\|\nabla\rho\right\|_{L^l_{x_1}}\right\|_{L^m_{x_2}}\right\|^{\frac{2}{3-(\frac1l+\frac1m+\frac1n)}}_{L^n_{x_3}}\|B^-\|_{L^4}^4+\frac14(\|\nabla|B^-|^2\|_{L^2}^2\||B^-||\nabla B^-|\|_{L^2}^2). \tag{3.10}$$

Now adding estimates (3.9) and (3.10)

$$\frac14\Big(\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4\Big)+\frac12\Big(\|\nabla|B^+|^2\|_{L^2}^2+\|\nabla|B^-|^2\|_{L^2}^2\Big) \tag{3.11}$$

$$\Big(\||B^+||\nabla B^+|^2\|_{L^2}^2+\||B^-||\nabla B^-|^2\|_{L^2}^2\Big)$$

$$\leq C\Big(\Big\|\Big\|\Big\|\nabla\rho\Big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|_{L^n_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)(\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4).$$

Let $L(t)=\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4\leq e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4$.
Here, we will use the following inequality and Lemma 1.
$1+\ln(e+\|\rho\|_{L^2}^2)\leq 1+\ln(e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)$,

$$\frac{d}{dt}L(t)\leq C\Bigg(\frac{\Big\|\Big\|\Big\|\nabla\rho\Big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|_{L^n_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)(e+\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)(1+\ln(e+\|\rho\|_{L^2}^2)$$

$$\frac{d}{dt}L(t)\leq C\Bigg(\frac{\Big\|\Big\|\Big\|\nabla\rho\Big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|_{L^n_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)L(t)(1+\ln L(t))$$

$$\implies\frac{d}{dt}(1+\ln L(t))\leq C\Bigg(\frac{\Big\|\Big\|\Big\|\nabla\rho\Big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|_{L^n_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)(1+\ln L(t)).$$

By applying Gronwall's inequality in the interval $[0,T)$

$$\sup_{0\leq t\leq T}\ln L(t)\leq(1+\ln L(0))exp\Big\{C\Bigg(\frac{\Big\|\Big\|\Big\|\nabla\rho\Big\|_{L^l_{x_1}}\Big\|_{L^m_{x_2}}\Big\|_{L^n_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\rho\|_{L^2}^2)}\Bigg)dt\Big\},$$

$$\implies\sup_{0\leq t\leq T}(\|B^+\|_{L^4}^4+\|B^-\|_{L^4}^4)\leq\infty.$$

These sharp estimates ensure the smoothness of solutions on interval $[0,T)$. Which completes the proof of Theorem 2. **Theorem 3.** Assume that $\mathcal{U}_0\in H^1(\mathbb{R}^3)$ and $(\mathcal{V}_0,\mathcal{W}_0)\in H^2(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0=\nabla\cdot\mathcal{V}_0=0$ in the sense of distributions. Let $(\mathcal{U},\mathcal{V},\mathcal{W})$ is the weak solution to system (1.4). If

$$\int_0^T\frac{\Big\|\Big\|\Big\|\partial_i\mathcal{U}_i\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\Big\|\Big\|\Big\|\partial_i\mathcal{V}_i\Big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\mathcal{U}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4)}dt<\infty,$$

where $2<l,m,n\leq\infty$ and $1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})\geq 0$, then the solution remains its smoothness up time $t^*<T$. *Proof.* The symmetric form of system (1.4) is given as by setting $\mathcal{R}^+=\mathcal{U}+\mathcal{V}$ and $\mathcal{R}^-=\mathcal{U}-\mathcal{V}$.

$$\partial_t\mathcal{R}^++\mathcal{R}^-\cdot\nabla\mathcal{R}^++\nabla\xi-\nabla\times\mathcal{W}-\Delta\mathcal{R}^+=0,\tag{3.12}$$

$$\partial_t\mathcal{W}+\frac{1}{2}(\mathcal{R}^++\mathcal{R}^-)\cdot\nabla\mathcal{W}-\frac{1}{2}\nabla\times(\mathcal{R}^++\mathcal{R}^-)-\Delta\mathcal{W}-\nabla div\mathcal{W}+2\mathcal{W}=0,\tag{3.13}$$

$$\partial_t\mathcal{R}^-+\mathcal{R}^+\cdot\nabla\mathcal{R}^-+\nabla\xi-\nabla\times\mathcal{W}-\Delta\mathcal{R}^-=0,\tag{3.14}$$

$$\nabla\cdot\mathcal{R}^+=0,\qquad\nabla\cdot\mathcal{R}^-=0.\tag{3.15}$$

Where $\mathcal{R}_0^+(x,0)=\mathcal{U}_0(x)+\mathcal{V}_0(x)$, $\mathcal{R}_0^-(x,0)=\mathcal{U}_0(x)-\mathcal{V}_0(x)$ and $\mathcal{W}(x,0)=\mathcal{W}_0(x)$.

For desired bounds, taking inner product of 3.12, 3.13 and 3.14 with $\mathcal{R}_j^+|\mathcal{R}_j^+|^2$, $\mathcal{W}^+|\mathcal{W}^+|^2$ and $\mathcal{R}_j^-|\mathcal{R}_j^-|^2$ integrating by parts, adding and using the following very helpful identities due to divergence free condition:

$$|\nabla\times\mathcal{R}^+|\le|\nabla\mathcal{R}^+|,\qquad |\nabla|\mathcal{R}^+||\le|\nabla\mathcal{R}^+|,$$

$$\int_{\mathbb{R}^3}\nabla\times\mathcal{W}\cdot|\mathcal{R}^+|^2|\mathcal{R}^+|dx=-\int_{\mathbb{R}^3}|\mathcal{R}^+|^2\mathcal{W}\cdot\nabla\times\mathcal{R}^+dx-\int_{\mathbb{R}^3}\mathcal{W}\cdot\nabla|\mathcal{R}^+|^2\times\mathcal{R}^+dx,$$

$$\int_{\mathbb{R}^3}(\mathcal{R}^+\cdot\nabla\mathcal{R}^+)\cdot|\mathcal{R}^+|^2|\mathcal{R}^+|dx=\frac{1}{4}\int_{\mathbb{R}^3}\mathcal{R}^+\cdot\nabla|\mathcal{R}^+|^4dx=0.$$

Trivial to say that these inequalities also hold for $\mathcal{R}^-$. We evaluate by setting $\mathcal{R}=(R_1,R_2,R_3)=R_j$

$$\frac{1}{4}\frac{d}{dt}\big(\|\mathcal{R}_j^+\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{R}_j^-\|_{L^4}^4\big)+2\|\mathcal{W}\|_{L^4}^4+\||\mathcal{W}|div\mathcal{W}\|_{L^2}^2+\||\mathcal{R}_j^+|\nabla\mathcal{R}_j^+\|_{L^2}^2+\||\mathcal{W}|\nabla\mathcal{W}\|_{L^2}^2$$
$$+\||\mathcal{R}_j^-|\nabla\mathcal{R}_j^-\|_{L^2}^2+\frac{1}{2}\big(\|\nabla|\mathcal{R}_j^+|^2\|_{L^2}^2+\|\nabla|\mathcal{W}|^2\|_{L^2}^2+\|\nabla|\mathcal{R}_j^-|^2\|_{L^2}^2\big)$$
$$=\int_{\mathbb{R}^3}\mathcal{R}_j^-|\mathcal{R}_j^-|^2\cdot\nabla\times\mathcal{W}dx+\int_{\mathbb{R}^3}\mathcal{R}_j^+|\mathcal{R}_j^+|^2\cdot\nabla\times\mathcal{W}dx+\frac{1}{2}\int_{\mathbb{R}^3}\mathcal{W}|\mathcal{W}|^2\cdot(\nabla\times(\mathcal{R}_j^++\mathcal{R}_j^-))dx$$
$$-\int_{\mathbb{R}^3}(div\mathcal{W})(\mathcal{W}\cdot\nabla|\mathcal{W}|^2dx+\int_{\mathbb{R}^3}\xi\mathcal{R}_j^+\cdot\nabla|\mathcal{R}_j^+|^2dx+\int_{\mathbb{R}^3}\xi\mathcal{R}_j^-\cdot\nabla|\mathcal{R}_j^-|^2dx.$$
$$:=B_1+B_2+B_3+B_4+B_5+B_6. \tag{3.16}$$

Now, we will estimate above equation, for that, we will find bounds for all terms. Here by using Young's and Holder's inequality

$$|B_1|\le C\big(\||\mathcal{R}_j^-|\nabla\mathcal{R}_j^-\|_{L^2}^2+\|\mathcal{R}_j^-\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4\big).$$
$$|B_2|\le C\big(\||\mathcal{R}_j^+|\nabla\mathcal{R}_j^+\|_{L^2}^2+\|\mathcal{R}_j^+\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4\big).$$
$$|B_3|\le C\big(\||\mathcal{W}|\nabla\mathcal{W}\|_{L^2}^2+\|\mathcal{R}_j^+\|_{L^4}^4+\|\mathcal{R}_j^-\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4\big).$$
$$|B_4|\le C\big(\||\mathcal{W}|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla|\mathcal{W}|^2\|_{L^2}^2\big).$$
$$|B_5|\le C\Bigg\|\Bigg\|\Big\|\partial_j\mathcal{R}_j^+\Big\|_{L_{x_1}^{l,\infty}}\Bigg\|_{L_{x_2}^{m,\infty}}\Bigg\|_{L_{x_3}^{n,\infty}}\Bigg\|\Bigg\|\Big\|\xi\Big\|_{L_{x_1}^{\frac{2l}{l-2},2}}\Bigg\|_{L_{x_2}^{\frac{2m}{m-2},2}}\Bigg\|_{L_{x_3}^{\frac{2n}{n-2},2}}\||\mathcal{R}_j^+|^2\|_{L^2}.$$

Using above inequalities, Lemma 4, Holder's and Young's inequality, we get estimates

$$\le\delta\Big(\|\mathcal{R}^+\cdot\nabla\mathcal{R}^-\|_{L^2}^2+\|\mathcal{R}^-\cdot\nabla\mathcal{R}^+\|_{L^2}^2\Big)+C\Bigg\|\Bigg\|\Big\|\partial_j\mathcal{R}_j^+\Big\|_{L_{x_1}^{l,\infty}}\Bigg\|_{L_{x_2}^{m,\infty}}\Bigg\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$
$$\Big(\|\mathcal{R}^+\|_{L^4}^4+\|\mathcal{R}^-\|_{L^4}^4\Big).$$
$$\le\delta\Big(\|\mathcal{R}^+\cdot\nabla\mathcal{R}^-\|_{L^2}^2+\|\mathcal{R}^-\cdot\nabla\mathcal{R}^+\|_{L^2}^2\Big)+C\Bigg\|\Bigg\|\Big\|\partial_j\mathcal{R}_j^+\Big\|_{L_{x_1}^{l,\infty}}\Bigg\|_{L_{x_2}^{m,\infty}}\Bigg\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$
$$\Big(\|\mathcal{R}^+\|_{L^4}^4+\|\mathcal{R}^-\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4\Big).$$

Similarly,

$$|B_6|\le\delta\Big(\|\mathcal{R}^+\cdot\nabla\mathcal{R}^-\|_{L^2}^2+\|\mathcal{R}^-\cdot\nabla\mathcal{R}^+\|_{L^2}^2\Big)+C\Bigg\|\Bigg\|\Big\|\partial_j\mathcal{R}_j^-\Big\|_{L_{x_1}^{l,\infty}}\Bigg\|_{L_{x_2}^{m,\infty}}\Bigg\|_{L_{x_3}^{n,\infty}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$

$$\Big(\|\mathcal{R}^+\|_{L^4}^4 + \|\mathcal{R}^-\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big).$$

Adding all the estimates, putting them in (3.16), simplifications yield

$$\frac{1}{4}\frac{d}{dt}\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big) + \Big(\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \|\nabla|\mathcal{V}|^2\|_{L^2}^2 + \|\nabla|\mathcal{W}|^2\|_{L^2}^2\Big) + \Big(\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2$$

$$+\||\mathcal{V}||\nabla\mathcal{U}|\|_{L^2}^2 + \||\mathcal{U}||\nabla\mathcal{V}|\|_{L^2}^2 + \||\mathcal{V}||\nabla\mathcal{V}|\|_{L^2}^2 + \||\mathcal{W}||\nabla\mathcal{W}|\|_{L^2}^2\Big)$$

$$\leq C\sum_{j=1}^{3}\Bigg(\Big\|\Big\|\big\|\partial_j\mathcal{U}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \Big\|\Big\|\big\|\partial_j\mathcal{V}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Bigg)\Big(\|\mathcal{U}\|_{L^4}^4$$

$$+\|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big). \tag{3.17}$$

The use of following simple inequality will extend our space for the required results

$$\frac{\Big\|\Big\|\big\|\partial_j\mathcal{U}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \Big\|\Big\|\big\|\partial_j\mathcal{V}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4)}$$

$$\leq \Big\|\Big\|\big\|\partial_j\mathcal{U}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \Big\|\Big\|\big\|\partial_j\mathcal{V}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}.$$

Now, let $S(t) = e + \|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4$.
Continuing from equation (3.17), we get

$$\leq C\sum_{j=1}^{3}\Bigg(\frac{\Big\|\Big\|\big\|\partial_j\mathcal{U}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \Big\|\Big\|\big\|\partial_j\mathcal{V}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4)}\Bigg)S(t)$$

$$\Big(1+\ln S(t)\Big).$$

Applying Gronwall's inequality, we finally get

$$\sup_{0\leq t\leq T}\ln S(t) \leq (1+\ln S(0))$$

$$\exp\int_0^T C\sum_{j=1}^{3}\Bigg(\frac{\Big\|\Big\|\big\|\partial_j\mathcal{U}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \Big\|\Big\|\big\|\partial_j\mathcal{V}_j\big\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4)}\Bigg)dt\} < \infty. \tag{3.18}$$

$$\Longrightarrow \sup_{0\leq t\leq T}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) \leq C. \tag{3.19}$$

These bounds ensure the smoothness of weak solutions in the proposed time interval in anisotropic Lorentz function space. Hence, the proof of our result.

Remark 3. The result of Theorem 3 holds true for anisotropic Lebesgue space, classical Lebesgue space, and Lorentz space as well.

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