Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.66553

# Approximate Controllability of Fractional Stochastic Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay

A. Lahmoudi*[1] and E. Lakhel[2]

ABSTRACT: In this paper, by using Schaefer's fixed point theorem and fractional power of operators, we study the approximate controllability of fractional stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. As an application, an example is provided to illustrate the obtained results.

Key Words: Stochastic fractional differential equations, approximate controllability, fractional powers of closed operators, infinite delay, fractional Brownian motion.

## Contents



## 1. Introduction

Fractional Brownian motion (fBm) $\{B^H(t) : t \in \mathbb{R}\}$ is a Gaussian stochastic proces, which depends on a parameter $H \in (0, 1)$ called the Hurst index. This stochastic process has self-similarity, stationary increments, and long-range dependence properties. It is known that fBm is a generalization of Brownian motion that reduces to a standard Brownian motion when $H = \frac{1}{2}$. Fractional Brownian motion is not a semimartingale if $H \neq \frac{1}{2}$ (see Biagini *al.* *[14]), so the classical Itô theory cannot be used to construct a stochastic calculus with respect to fBm. For additional details on the fBm, we refer the reader to [9].*

*Stochastic differential equations driven by fBm have attracted significant interest. See the recent papers [10,11,12,13]. Moreover, Dung studied the existence and uniqueness of impulsive stochastic Volterra integro-differential equation driven by fBm in [20]. Boufoussi et al. [8] proved the existence and uniqueness of a mild solution to a related problem and studied the dependence of the solution on the initial condition in infinite dimensional space, using the Riemann-Stieltjes integral. More recently, Li [17] investigated the existence of mild solution to a class of stochastic delay fractional evolution equations driven by fBm.*

*Fractional differential equations are a generalization of ordinary differential equations and integration to arbitrary non-integer orders. Fractional calculus is widely used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through the formulation of fractional derivative (see [15]). Concurrently, the theory of fractional differential equations have attracted many scientists and mathematicians [14,16,17,23] because it has numerous applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry, and biological sciences. See [27,1] for more details.*

*Deterministic models often fluctuate due to noise or stochastic perturbation. Generally, the noise or perturbation of a system is modelled by a Brownian motion (Wiener process). There has been significant work on fractional stochastic differential equations (FSDEs) with Brownian motion in recent years. However, as many researchers have found [2,3], it is insufficient to use a standard Brownian motion to*

---

* Corresponding author.
2020 *Mathematics Subject Classification:* 34K40, 93B05, 60H20.
Submitted January 04, 2023. Published December 04, 2025

*model many phenomena which have long memory, such as telecommunication and asset prices. As an extension of Brownian motion, fractional Brownian motion is a family of Gaussian processes introduced by Kolmogorov [4]. Controllability is one of the most fundamentally significant concepts in mathematical control theory, which can be categorized into two kinds: exact (complete) controllability and approximate controllability. Exact controllability means that under some admissible control input, a system can be steered from an arbitrary given initial state to an arbitrary desired final state, while approximate controllability means the system can be steered to arbitrary small neighbourhoods of final state. The study of the latter for control systems is more appropriate since the conditions of the former are usually too strong in infinite-dimensional spaces [18,21]. Recently, many efforts focused on the approximate controllability of FSDEs with Brownian motion; see [23,22,25,26,19,24,6]. However, little has been done on approximate controllability of FSDEs with fBm.*

*Motivated by the above considerations, we study the approximate controllability of FSDEs with fBm of the form*

$$\begin{cases} d[J_t^{1-\alpha}(x(t)-\varphi(0)] = [Ax(t)+f(t,x_t)+Bu(t)]dt+\sigma(t)dB^H(t),\ t\in[0,T], \\ x(t)=\varphi(t)\in L^2(\Omega,\mathcal{B}_h),\ \textit{for a.e. } t\in(-\infty,0], \end{cases} \tag{1.1}$$

*where* $\frac{1}{2}<\alpha<1$, $J^{1-\alpha}$ *is the* $(1-\alpha)-$*order Riemann-Liouville fractional integral operator,* $A$ *is the infinitesimal generator of an analytic semigroup of bounded linear operators,* $(S(t))_{t\geq 0}$, *in a Hilbert space* $X$; $B^H$ *is a fBm with* $H\in(\frac{1}{2},1)$ *on a real and separable Hilbert space* $Y$; *and the control function* $u(\cdot)$ *takes values in* $L^2([0,T],U)$, *the Hilbert space of admissible control functions for a separable Hilbert space* $U$; *and* $B$ *is a bounded linear operator from* $U$ *into* $X$. *The history* $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, *belongs to an abstract phase space* $\mathcal{B}_h$ *defined axiomatically, and* $f:[0,T]\times\mathcal{B}_h\to X$ *and* $\sigma:[0,T]\to\mathcal{L}_2^0(Y,X)$ *are appropriate functions, where* $\mathcal{L}_2^0(Y,X)$ *denotes the space of all Q-Hilbert-Schmidt operators from* $Y$ *into* $X$ *(see section 2 below). The outline of this paper is as follows: In the next section, some necessary notations and concepts are provided. In Section 3, we derive the approximate controllability of fractional stochastic differential systems driven by a fBm. Finally, in Section 4, we conclude with an example to illustrate the applicability of the general theory.*

## 2. Preliminaries

*We collect some notions, concepts and lemmas concerning the Wiener integral with respect to an infinite dimensional fractional Brownian, and we recall some basic results which will be used throughout the paper.*

*Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a complete probability space. A standard fBm* $\{\beta^H(t),t\in\mathbb{R}\}$ *with Hurst parameter* $H\in(0,1)$ *is a zero mean Gaussian process with continuous sample paths such that*

$$R_H(t,s)=\mathbb{E}[\beta^H(t)\beta^H(s)]=\frac{1}{2}\big(t^{2H}+s^{2H}-|t-s|^{2H}\big),\qquad s,t\in\mathbb{R}. \tag{2.1}$$

*Let* $X$ *and* $Y$ *be two real, separable Hilbert spaces and let* $\mathcal{L}(Y,X)$ *be the space of bounded linear operator from* $Y$ *to* $X$. *For convenience, we shall use the same notation to denote the norms in* $X,Y$ *and* $\mathcal{L}(Y,X)$. *Let* $Q\in\mathcal{L}(Y,Y)$ *be an operator defined by* $Qe_n=\lambda_n e_n$ *with finite trace* $trQ=\sum_{n=1}^{\infty}\lambda_n<\infty$, *where* $\lambda_n\geq 0$ $(n=1,2...)$ *are non-negative real numbers and* $\{e_n\}$ $(n=1,2...)$ *is a complete orthonormal basis in* $Y$.

*We define the infinite dimensional fBm on* $Y$ *with covariance* $Q$ *as*

$$B^H(t)=B_Q^H(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^H(t),$$

*where* $\beta_n^H$ *are real, independent one dimensional fBm's. This process is Gaussian, it starts from* $0$, *has zero mean and covariance:*

$$E\langle B^H(t),x\rangle\langle B^H(s),y\rangle=R(s,t)\langle Q(x),y\rangle\ \textit{ for all } x,y\in Y \textit{ and } t,s\in[0,T].$$

*In order to define Wiener integrals with respect to the Q-fBm, we introduce the space* $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ *of all Q-Hilbert-Schmidt operators* $\psi : Y \to X$. *Recall that* $\psi \in \mathcal{L}(Y, X)$ *is called a Q-Hilbert-Schmidt operator, if*

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

*and that the space* $\mathcal{L}_2^0$ *equipped with the inner product* $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ *is a separable Hilbert space. Let* $\phi(s);\ s \in [0, T]$ *be a function with values in* $\mathcal{L}_2^0(Y, X)$, *such that* $\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty$. *The Wiener integral of* $\phi$ *with respect to* $B^H$ *is defined by*

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n d\beta_n^H(s). \tag{2.2}$$

*We conclude this subsection with the following result which is fundamental to proving our result.*

***Lemma 2.1*** *[7] If* $\psi : [0, T] \to \mathcal{L}_2^0(Y, X)$ *satisfies* $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, *then the above sum in (2.2) is well-defined as an* $X$*-valued random variable and*

$$\mathbb{E}\|\int_0^t \psi(s) dB^H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

*The study of theory of differential equation with infinite delays depends on the choice of the abstract phase space. We assume that the phase space* $\mathcal{B}_h$ *is a linear space of functions mapping* $(-\infty, 0]$ *into* $X$, *endowed with a norm* $\|.\|_{\mathcal{B}_h}$. *We shall introduce some basic definitions, notations and lemma used in this paper.*

*First, we present the abstract phase space* $\mathcal{B}_h$. *Assume that* $h : (-\infty, 0] \longrightarrow [0, +\infty)$ *is a continuous function with* $l = \int_{-\infty}^0 h(s) ds < +\infty$. *We define the abstract phase space* $\mathcal{B}_h$ *by*

$$\begin{aligned}\mathcal{B}_h = \quad &\{\psi : (-\infty, 0] \longrightarrow X \textit{ for any } \tau > 0, (\mathbb{E}\|\psi\|^2)^{\frac{1}{2}} \textit{ is bounded and measurable} \\ &\textit{function on } [-\tau, 0] \textit{ and } \textstyle\int_{-\infty}^0 h(t) \sup_{t \leq s \leq 0} (\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}} dt < +\infty\}.\end{aligned}$$

*If we equip this space with the norm*

$$\|\psi\|_{\mathcal{B}_h} := \int_{-\infty}^0 h(t) \sup_{t \leq s \leq 0} (\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}} dt,$$

*then it is clear that* $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ *is a Banach space.*

*Next, we consider the space* $\mathcal{B}_T$ *given by*

$$\mathcal{B}_T = \{x : x \in \mathcal{C}((-\infty, T], X), \textit{ with } x_0 \in \mathcal{B}_h\},$$

*where* $\mathcal{C}((-\infty, T], X)$ *denotes the space of all continuous* $X-$*valued stochastic processes* $\{x(t),\ t \in (-\infty, T]\}$. *The function* $\|.\|_{\mathcal{B}_T}$ *defined by*

$$\|x\|_{\mathcal{B}_T} = \|x_0\|_{\mathcal{B}_h} + \sup_{0 \leq t \leq T} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}$$

*is a semi-norm in* $\mathcal{B}_T$. *The following lemma is an important common property of phase spaces.*

***Lemma 2.2*** *[28] Suppose* $x \in \mathcal{B}_T$, *then for all* $t \in [0, T]$ , $x_t \in \mathcal{B}_h$ *and*

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{B}_h} \leq l \sup_{0 \leq s \leq t} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

*where* $l = \int_{-\infty}^0 h(s) ds < \infty$.

*The following well-known definitions are related to fractional order differentiation and integration.*

***Definition 2.1*** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

*where $\Gamma(.)$ is the Gamma function.*

***Definition 2.2*** *The Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of a function $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$$D_t^\alpha f(t) = \frac{d}{dt} J_t^{1-\alpha} f(t).$$

***Definition 2.3*** *The Caputo fractional derivative of order $\alpha \in (0,1)$ of $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$${}^C D_t^\alpha f(t) = D_t^\alpha (f(t) - f(0)).$$

*We refer the reader to [1], for more details on fractional calculus.*

## 3. Approximate Controllability Result

*Before stating the main result, we introduce the concepts of a mild solution of the problem (1.1) and the meaning of approximate controllability of a fractional stochastic functional differential equation.*

***Definition 3.1*** *An $X$-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1.1) if*

1. *$x(t)$ is continuous on $[0,T]$ almost surely,*
2. *for every $t \in [0,T]$,*

$$\begin{aligned} x(t) \quad &= T_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) Bu(s) ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s) dB^H(s),\ \mathbb{P}-a.s.;\ \text{and} \end{aligned} \tag{3.1}$$

3. *$x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|^2_{\mathcal{B}_h} < \infty$,*

*and*

$$T_\alpha(t)x = \int_0^\infty \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta,\ t \geq 0,\ x \in X.$$

$$S_\alpha(t)x = \alpha \int_0^\infty \theta \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta,\ t \geq 0,\ x \in X,$$

*and*

$$\eta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\alpha\pi), \quad \theta \in (0, \infty),$$

*$\eta_\alpha$ is a probability density function defined on $(0, \infty)$.*

***Remark 3.1*** *(See [29])*

$$\int_0^\infty \theta \eta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{3.2}$$

*The following properties of $T_\alpha$ and $S_\alpha$ are useful.*

***Lemma 3.1*** *(See [29]) Under the previous assumptions on $S(t)$, $t \geq 0$ and $A$, the operators $T_\alpha(t)$ and $S_\alpha(t)$ have the following properties:*

*(i) For any $x \in X$, $\|T_\alpha(t)x\| \leq M\|x\|$, $\|S_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$.*

*(ii)* $\{T_\alpha(t),\ t \geq 0\}$ *and* $\{S_\alpha(t),\ t \geq 0\}$ *are strongly continuous.*

*(iii)* *For any* $t > 0$, $T_\alpha(t)$ *and* $S_\alpha(t)$ *are also compact operators if* $S(t)$ *is compact.*

*Our main result in this paper is based on the following fixed-point theorem.*

***Theorem 3.1 (Schaefer's fixed point theorem)*** *Let* $\mathcal{V}$ *be a Banach space, and* $\Psi : \mathcal{V} \longrightarrow \mathcal{V}$ *be a completely continuous operator. Then, either*
- $\Psi$ *has a fixed point or*
- *the set* $\Theta = \{x \in \mathcal{V} : x = \beta\Psi(x), 0 < \beta < 1\}$ *is unbounded.*

*In order to establish the approximate controllability of (1.1), we impose the following conditions on the data of the problem:*

$(\mathcal{H}.1)$ *The analytic semigroup,* $(S(t))_{t\geq 0}$*, generated by* $A$ *is compact for* $t > 0$*, and there exists* $M \geq 1$ *such that*

$$\sup_{t\geq 0} \|S(t)\| \leq M.$$

$(\mathcal{H}.2)$ *The function* $t \longmapsto f(t, x)$ *is measurable and there exists a constant* $M_f > 0$ *such that for every* $x, y \in \mathcal{B}$*, for every* $t \in [0, T]$*,*

$$\mathbb{E}\|f(t, x)\|^2 \leq M_f(1 + \mathbb{E}\|x\|^2),$$

$$\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq M_f\mathbb{E}\|x - y\|^2.$$

$(\mathcal{H}.3)$ *There exists a constant* $c > 0$ *such that the function* $\sigma : [0, \infty) \to \mathcal{L}_2^0(Y, X)$ *satisfies*

$$\sup_{0\leq s\leq T} \|\sigma(s)\|^2_{\mathcal{L}_2^0} < c, \quad \text{for each } T > 0.$$

*In order to study the approximate controllability for the system (1.1), we introduce the following linear differential system:*

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + Bu(t), & t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{3.3}$$

*It is convenient at this point to introduce the resolvent operators associated with (3.3), namely*

$$\Gamma_0^T = \int_0^T (T - s)^{\alpha-1} S_\alpha(T - s)BB^* S_\alpha^*(T - s)ds,$$

*and*

$$R(\lambda, \Gamma) = (\lambda I + \Gamma)^{-1},$$

*where* $B^*$ *and* $S_\alpha^*$ *denote the adjoint of* $B$ *and* $S_\alpha$*, respectively.*
*Let* $x(T; \varphi, u)$ *be the state value of (1.1) at terminal state* $T$*, corresponding to the control* $u$ *and the initial value* $\varphi$*. Denote by* $R(T, \varphi) = \{x(T; \varphi, u) : u \in L^2([0, T], U)\}$ *the reachable set of system (1.1) at terminal time* $T$*, its closure in* $X$ *is denoted by* $\overline{R(T, \varphi)}$*.*

***Definition 3.2*** *The system (1.1) is said to be approximately controllable on the interval* $[0, T]$ *if* $\overline{R(T, \varphi)} = L^2(\Omega, X)$*.*

***Lemma 3.2*** *[23] The linear control system (3.3) is approximately controllable on* $[0, T]$ *if and only if* $\lambda(\lambda I + \Gamma_0^T)^{-1} \to 0$ *strongly as* $\lambda \to 0^+$*.*

***Lemma 3.3*** *[23] For any* $\bar{x}_T \in L^2(\Omega, X)$ *there exists* $\bar{\varphi} \in L^2(\Omega; L^2([0, T]; L_0^2))$ *such that* $\bar{x}_T = E\bar{x}_T + \int_0^T \bar{\varphi}(s)dB^H(s)$*.*

*For any $\lambda > 0$ and $\bar{x}_T \in L^2(\Omega, X)$, we define the control function by*

$$\begin{array}{rcl} u^\lambda(t,x) & = & B^* S_\alpha^*(T-t)(\lambda I + \Gamma_0^T)^{-1}\{E\bar{x}_T - T_\alpha(T)\varphi(0)\} \\ & + & B^* S_\alpha^*(T-t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\ & - & B^* S_\alpha^*(T-t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_s)ds \\ & - & B^* S_\alpha^*(T-t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)\sigma(s)dB^H(s),\ t \in [0,T]. \end{array} \tag{3.4}$$

*Substituting the control $u^\lambda(.)$ into the stochastic control system (3.1) yields a non-linear operator $\Pi^\lambda$ on $\mathcal{B}_T$ given by*

$$\Pi^\lambda(x)(t) = \left\{ \begin{array}{l} \varphi(t), \quad \text{if } t \in (-\infty, 0], \\ T_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,x_s)ds \\ + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)Bu^\lambda(s)ds \\ + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s)dB^H(s), \quad \text{if } t \in [0,T]. \end{array} \right.$$

*The main result of this paper is as follow:*

***Theorem 3.2*** *Suppose that $(\mathcal{H}.1) - (\mathcal{H}.3)$ hold. Then, the system $\Pi^\lambda$ has a fixed point.*

**Proof:** *Let $y : (-\infty, T] \longrightarrow X$ be the function defined by*

$$y(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in (-\infty, 0], \\ T_\alpha(t)\varphi(0), & \text{if } t \in [0,T]. \end{array} \right.$$

*Then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set*

$$x(t) = z(t) + y(t).$$

*Clearly, $x$ is fixed point of $\Pi^\lambda$ if and only if $z$ satisfies $z_0 = 0$ and*

$$\begin{array}{rl} z(t) = & \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s, z_s + y_s)ds + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)Bu^\lambda_{z+y}(s)ds \\ & + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s)dB^H(s), \end{array} \tag{3.5}$$

*where $u^\lambda_{z+y}(t)$ is obtained from (3.4) by replacing $x_t = z_t + y_t$.*
*Set*

$$\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\}.$$

*For any $z \in B_T^0$, we have*

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t\in[0,T]}(\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]}(\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

*Then, $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\widehat{\Pi}^\lambda : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ by*

$$(\widehat{\Pi}^\lambda z)(t) = \left\{ \begin{array}{l} 0 \ \text{ if } t \in (-\infty, 0], \\ \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s, z_s + y_s)ds \\ + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)Bu^\lambda_{z+y}(s)ds \\ + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s)dB^H(s), \qquad \text{if } t \in [0,T]. \end{array} \right. \tag{3.6}$$

*It is clear that the operator $\widehat{\Pi}^\lambda$ has a fixed point if and only if $\widehat{\Pi}^\lambda$ has a fixed point. so, it suffices to show that $\widehat{\Pi}^\lambda$ has a fixed point. To this end, we shall prove that $\widehat{\Pi}^\lambda$ is completely continuous and that $\{x = \lambda\widehat{\Pi}(x) : 0 < \lambda < 1)\}$ bounded.*
**Step 1.** *$\widehat{\Pi}^\lambda$ is completely continuous.*
**Claim 1.** *$\widehat{\Pi}^\lambda$ is continuous.*
*Let the $z^n$ be a sequence such that $z^n \longrightarrow z$ in $\mathcal{B}_T^0$. Then, for $t \in [0,T]$, by Lemma 3.1 (i), $(\mathcal{H}.2)$, and the Hölder inequality, we have*

$$
\begin{aligned}
\mathbb{E}\|(\widehat{\Pi}^{\lambda}z^{n})(t)-(\widehat{\Pi}^{\lambda}z)(t)\|^{2}\leq\ & 2\mathbb{E}\|\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)B[u^{\lambda}_{z^{n}+y}-u^{\lambda}_{z+y}]ds\|^{2}\\
&+2\mathbb{E}\|\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)[f(s,z^{n}_{s}+y_{s})-f(s,z_{s}+y_{s})]ds\|^{2}\\
&\leq\frac{2M^{2}M_{b}^{2}}{\Gamma^{2}(\alpha+1)}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{T}\mathbb{E}\|u^{\lambda}_{z^{n}+y}(s)-u^{\lambda}_{z+y}(s)\|^{2}ds\\
&+\frac{2M^{2}}{\Gamma^{2}(\alpha+1)}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{T}\mathbb{E}\|f(s,z^{n}_{s}+y_{s})-f(s,z_{s}+y_{s})\|^{2}ds\\
&\leq\frac{2M^{2}M_{b}^{2}}{\Gamma^{2}(\alpha+1)}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{T}\mathbb{E}\|u^{\lambda}_{z^{n}+y}(s)-u^{\lambda}_{z+y}(s)\|^{2}ds\\
&+\frac{2M^{2}M_{f}^{2}}{\Gamma^{2}(\alpha)}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{T}\|z^{n}_{s}-z_{s}\|^{2}_{\mathcal{B}_{h}}ds
\end{aligned}
$$

*where*

$$
\begin{aligned}
\mathbb{E}\|u^{\lambda}_{z^{n}+y}(s)-u^{\lambda}_{z+y}(s)\|^{2}&\leq\frac{M^{2}M_{b}^{2}}{\lambda^{2}\Gamma^{2}(\alpha)}\mathbb{E}\|\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)[f(s,z^{n}_{s}+y_{s})-f(s,z_{s}+y_{s})]ds\|^{2}\\
&\leq\frac{M^{2}M_{b}^{2}}{\lambda^{2}\Gamma^{2}(\alpha)}\frac{M^{2}M_{f}^{2}}{\Gamma^{2}(\alpha+1)}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{T}\|z^{n}_{s}-z_{s}\|^{2}_{\mathcal{B}_{h}}ds.
\end{aligned}
\tag{3.7}
$$

*Observe that* $\mathbb{E}\|(\widehat{\Pi}^{\lambda}z^{n})(t)-(\widehat{\Pi}^{\lambda}z)(t)\|\longrightarrow\infty$ *as* $n\longrightarrow\infty$. *Therefore,* $\widehat{\Pi}^{\lambda}$ *is continuous.*
**Claim 2.** *Now, we prove that* $\widehat{\Pi}^{\lambda}$ *maps* $\mathcal{B}_k$ *bounded sets into equicontinuous sets, where*

$$\mathcal{B}_{k}=\{z\in\mathcal{B}^{0}_{T}:\|z\|^{2}_{\mathcal{B}^{0}_{T}}\leq k\},\qquad \textit{for some } k\geq 0.$$

*Note that* $\mathcal{B}_k\subseteq\mathcal{B}^0_T$ *is a bounded closed convex set. For* $z\in\mathcal{B}_k$,

$$
\begin{aligned}
\|z_{t}+y_{t}\|^{2}_{\mathcal{B}_{h}}&\leq 2(\|z_{t}\|^{2}_{\mathcal{B}_{h}}+\|y_{t}\|^{2}_{\mathcal{B}_{h}})\\
&\leq 4(l^{2}\sup_{0\leq s\leq t}\mathbb{E}\|z(s)\|^{2}+\|z_{0}\|^{2}_{\mathcal{B}_{h}}\\
&+l^{2}\sup_{0\leq s\leq t}\mathbb{E}\|y(s)\|^{2}+\|y_{0}\|^{2}_{\mathcal{B}_{h}})\\
&\leq 4l^{2}(k+M^{2}\mathbb{E}\|\varphi(0)\|^{2})+4\|y\|^{2}_{\mathcal{B}_{h}}\\
&:=q'.
\end{aligned}
\tag{3.8}
$$

*Let* $z\in\mathcal{B}_k$ *and* $|h|$ *be sufficiently small. Then*

$$
\begin{aligned}
&\mathbb{E}\|(\widehat{\Pi}^{\lambda}_{z})(t+h)-(\widehat{\Pi}^{\lambda}_{z})(t)\|^{2}\leq\\
&\leq 3\mathbb{E}\|\int_{0}^{t+h}(t+h-s)^{\alpha-1}S_{\alpha}(t+h-s)f(s,z_{s}+y_{s})ds-\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)f(s,z_{s}+y_{s})ds\|^{2}\\
&+3\mathbb{E}\|\int_{0}^{t+h}(t+h-s)^{\alpha-1}S_{\alpha}(t+h-s)Bu^{\lambda}_{z+y}(s)ds-\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)Bu^{\lambda}_{z+y}(s)ds\|^{2}\\
&+3\mathbb{E}\|\int_{0}^{t+h}(t+h-s)^{\alpha-1}S_{\alpha}(t+h-s)\sigma(s)dB^{H}(s)-\int_{0}^{t}(t-s)^{\alpha-1}S_{\alpha}(t-s)\sigma(s)dB^{H}(s)\|^{2}\\
&:=I_{1}+I_{2}+I_{3}.
\end{aligned}
$$

*We can see that*

$$
\begin{aligned}
I_{1}\leq\ & 9\mathbb{E}\|\int_{0}^{t}[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_{\alpha}(t+h-s)f(s,z_{s}+y_{s})ds\|^{2}\\
&+9\mathbb{E}\|\int_{0}^{t}(t-s)^{\alpha-1}[S_{\alpha}(t+h-s)-S_{\alpha}(t-s)]f(s,z_{s}+y_{s})ds\|^{2}\\
&+9\mathbb{E}\|\int_{t}^{t+h}(t+h-s)^{\alpha-1}S_{\alpha}(t+h-s)f(s,z_{s}+y_{s})ds\|^{2}\\
&:=I_{11}+I_{12}+I_{13}
\end{aligned}
\tag{3.9}
$$

*By Lemma 3.1 (i), ($\mathcal{H}$.2), and the Hölder inequality, we have*

$$
\begin{aligned}
I_{11}&\leq 9\mathbb{E}\|\int_{0}^{t}[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_{\alpha}(t-s)f(s,z_{s}+y_{s})ds\|^{2}\\
&\leq\frac{9M^{2}}{\Gamma^{2}(\alpha)}\int_{0}^{t}[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^{2}ds\int_{0}^{t}\mathbb{E}\|f(s,z_{s}+y_{s})\|^{2}ds\\
&\leq\frac{9M^{2}}{\Gamma^{2}(\alpha)}\int_{0}^{t}[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^{2}ds\int_{0}^{t}M_{f}(1+\mathbb{E}\|z_{s}+y_{s}\|^{2})ds\\
&\leq\frac{9M^{2}M_{f}t(1+q')}{\Gamma^{2}(\alpha)}\int_{0}^{t}[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^{2}ds.
\end{aligned}
\tag{3.10}
$$

*Then* $I_{11}\longrightarrow 0$, *as* $h\longrightarrow 0$.

*Similar computations yield*

$$\begin{aligned} I_{12} &\leq 9\mathbb{E}\| \int_0^t (t-s)^{\alpha-1}[S_\alpha(t+h-s)-S_\alpha(t-s)]f(s,z_s+y_s)ds\|^2 \\ &\leq 9\int_0^t (t-s)^{2\alpha-2}ds\mathbb{E}(\int_0^t \|[S_\alpha(t+h-s)-S_\alpha(t-s)]\|\|f(s,z_s+y_s)\|ds)^2 \\ &\leq \frac{9T^{2\alpha}M_f(1+q')}{2\alpha-1}(\sup_{s\in[0,t]}\|S_\alpha(t+h-s)-S_\alpha(t-s)\|)^2, \end{aligned} \tag{3.11}$$

*and*

$$\begin{aligned} I_{13} &\leq 9\mathbb{E}\| \int_t^{t+h}(t+h-s)^{\alpha-1}S_\alpha(t+h-s)f(s,z_s+y_s)ds\|^2 \\ &\leq 9\int_t^{t+h}(t+h-s)^{2\alpha-2}ds\int_t^{t+h}\mathbb{E}\|S_\alpha(t+h-s)f(s,z_s+y_s)\|^2ds \\ &\leq \frac{9M^2}{\Gamma^2(\alpha)(2\alpha-1)}M_f(1+q')h^{2\alpha}. \end{aligned} \tag{3.12}$$

*Arguing in a similar manner, we see that*

$$\begin{aligned} I_2 \leq\ & 9\mathbb{E}\| \int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_\alpha(t+h-s)Bu^\lambda_{z+y}(s)ds\|^2 \\ &+9\mathbb{E}\| \int_0^t(t-s)^{\alpha-1}[S_\alpha(t+h-s)-S_\alpha(t-s)]Bu^\lambda_{z+y}(s)ds\|^2 \\ &+9\mathbb{E}\| \int_t^{t+h}(t+h-s)^{\alpha-1}S_\alpha(t+h-s)Bu^\lambda_{z+y}(s)ds\|^2 \\ &:= I_{21}+I_{22}+I_{23} \end{aligned} \tag{3.13}$$

*Combining Lemma 3.1 and the Hölder inequality, gives us*

$$\begin{aligned} I_{21} &\leq 9\mathbb{E}\| \int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_\alpha(t-s)Bu^\lambda_{z+y}(s)ds\|^2 \\ &\leq \frac{9M^2}{\Gamma^2(\alpha)}\int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^2ds\int_0^t\mathbb{E}\|Bu^\lambda_{z+y}(s)\|^2ds \\ &\leq \frac{9M^2M_b^2\mathcal{G}}{\Gamma^2(\alpha)}\int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^2ds, \end{aligned} \tag{3.14}$$

*where,*

$$\begin{aligned} \mathbb{E}\|u^\lambda_{z+y}(s)\|^2 \leq\ & 5\frac{M^2M_b^2}{\lambda^2\Gamma(\alpha)}\{\|E\bar{x}_T\|^2+M^2\mathbb{E}\|\varphi(0)\|^2+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_2^0}ds \\ &+\frac{M^2}{\Gamma^2(\alpha)}M_f(1+q')\int_0^T(T-s)^{2\alpha-2}ds \\ &+2\frac{M^2}{\Gamma^2(\alpha)}HT^{2H-1}\int_0^T(T-s)^{(2\alpha-2)}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\} := \mathcal{G}. \end{aligned} \tag{3.15}$$

*By $(\mathcal{H}.2)$, the Hölder inequality and (3.15), we have*

$$\begin{aligned} I_{22} &\leq 9\mathbb{E}\| \int_0^t(t-s)^{\alpha-1}[S_\alpha(t+h-s)-S_\alpha(t-s)]Bu^\lambda_{z+y}(s)ds\|^2 \\ &\leq 9\int_0^t(t-s)^{2\alpha-2}ds\mathbb{E}(\int_0^t\|[S_\alpha(t+h-s)-S_\alpha(t-s)]\|\|Bu^\lambda_{z+y}(s)\|ds)^2 \\ &\leq \frac{9T^{2\alpha}M_b^2\mathcal{G}}{2\alpha-1}(sup_{s\in[0,t]}\|S_\alpha(t+h-s)-S_\alpha(t-s)\|)^2. \end{aligned} \tag{3.16}$$

*Similarly,*

$$\begin{aligned} I_{23} &\leq 9\mathbb{E}\| \int_t^{t+h}(t+h-s)^{\alpha-1}S_\alpha(t+h-s)Bu^\lambda_{z+y}(s)ds\|^2 \\ &\leq 9\int_t^{t+h}(t+h-s)^{2\alpha-2}ds\int_t^{t+h}\mathbb{E}\|S_\alpha(t+h-s)Bu^\lambda_{z+y}(s)\|^2ds \\ &\leq \frac{9M^2}{\Gamma^2(\alpha)(2\alpha-1)}M_b^2\mathcal{G}h^{2\alpha}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} I_3 \leq\ & 9\mathbb{E}\| \int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_\alpha(t+h-s)\sigma(s)dB^H(s)\|^2 \\ &+9\mathbb{E}\| \int_0^t(t-s)^{\alpha-1}[S_\alpha(t+h-s)-S_\alpha(t-s)]\sigma(s)dB^H(s)\|^2 \\ &+9\mathbb{E}\| \int_t^{t+h}(t+h-s)^{\alpha-1}S_\alpha(t+h-s)\sigma(s)dB^H(s)\|^2 \\ &:= I_{31}+I_{32}+I_{33}. \end{aligned} \tag{3.18}$$

*Combining Lemma 2.1, $(\mathcal{H}.3)$, and the Hölder inequality, yields*

$$\begin{aligned} I_{31} \leq\ & 9\mathbb{E}\| \int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_\alpha(t+h-s)\sigma(s)dB^H(s)\|^2 \\ &\leq 18Ht^{2H-1}\int_0^t\|[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]S_\alpha(t+h-s)\sigma(s)\|^2ds \\ &\leq \frac{18Ht^{2H-1}M^2}{\Gamma^2(\alpha)}\int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^2ds\int_0^t\|\sigma(s)\|^2_{L_2^0}ds \\ &\leq \frac{18Ht^{2H}M^2c}{\Gamma^2(\alpha)}\int_0^t[(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]^2ds. \end{aligned} \tag{3.19}$$

*Similarly,*

$$\begin{aligned} I_{32} &\leq 18Ht^{2H-1}\int_0^t \|(t-s)^{\alpha-1}[S_\alpha(t+h-s)-S_\alpha(t-s)]\sigma(s)\|^2 ds \\ &\leq 18Ht^{2H-1}\int_0^t (t-s)^{2\alpha-2}ds\int_0^t \|[S_\alpha(t+h-s)-S_\alpha(t-s)]\|^2\|\sigma(s)\|^2 ds \\ &\leq \frac{18T^{2\alpha}Ht^{2H-1}c}{2\alpha-1}(\sup_{s\in[0,t]}\|S_\alpha(t+h-s)-S_\alpha(t-s)\|)^2, \end{aligned} \tag{3.20}$$

*and*

$$\begin{aligned} I_{33} &\leq 18Ht^{2H-1}\int_t^{t+h}\|(t+h-s)^{\alpha-1}S_\alpha(t+h-s)\sigma(s)\|^2 ds \\ &\leq 18Ht^{2H-1}\int_t^{t+h}(t+h-s)^{2\alpha-2}ds\int_t^{t+h}\|S_\alpha(t+h-s)\sigma(s)\|^2 ds \\ &\leq \frac{18M^2Ht^{2H-1}c}{\Gamma^2(\alpha)(2\alpha-1)}h^{2\alpha+1}. \end{aligned} \tag{3.21}$$

*Therefore, for sufficiently small positive number $h$, it follows that*

$$\mathbb{E}\|\widehat{\Pi}_2^\lambda z)(t+h)-(\widehat{\Pi}_2^\lambda z)(t)\|^2 \longrightarrow 0 \ \ as\ h\longrightarrow 0.$$

*Thus, $\widehat{\Pi}_2^\lambda$ maps $\mathcal{B}_k$ into an equicontinuous family of functions.*

**Claim 2.** *$(\widehat{\Pi}_2^\lambda\mathcal{B}_k)(t)$ is a precompact set in $X$.*
*Let $0<t\leq T$ be fixed, and $0<\epsilon<t$. For $\delta>0$ and $z\in\mathcal{B}_k$, we define*

$$\begin{aligned} (\widehat{\Pi}_\epsilon^{\lambda,\delta}z)(t) &:= \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+y_s)d\theta ds \\ &+\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu_{z+y}^\lambda(s)d\theta ds \\ &+\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\sigma(s)d\theta dB^H(s) \\ &= S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,z_s+y_s)d\theta ds \\ &+S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)Bu_{z+y}^\lambda(s)d\theta ds \\ &+S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)\sigma(s)d\theta dB^H(s). \end{aligned}$$

*From the compactness of $S(t)$ $(t>0)$, we infer that the set $V_\epsilon^\delta(t)=\{(\widehat{\Pi}_\epsilon^{\lambda,\delta}z)(t):\ z\in\mathcal{B}_k\}$ is relative compact in $X$ for every $\epsilon$, $0<\epsilon<t$ and $\delta>0$. Moreover, for every $z\in\mathcal{B}_k$,*

$$\begin{aligned} \mathbb{E}\|\widehat{\Pi}^\lambda(z)(t)- & \widehat{\Pi}_\epsilon^{\lambda,\delta}(z)(t)\|^2 \leq 6\alpha^2\mathbb{E}\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+y_s)d\theta ds\|^2 \\ &+6\alpha^2\mathbb{E}\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+y_s)d\theta ds\|^2 \\ &+6\alpha^2\mathbb{E}\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu_{z+y}^\lambda(s)d\theta ds\|^2 \\ &+6\alpha^2\mathbb{E}\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu_{z+y}^\lambda(s)d\theta ds\|^2 \\ &+6\alpha^2\mathbb{E}\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\sigma(s)d\theta dB^H(s)\|^2 \\ &+6\alpha^2\mathbb{E}\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\sigma(s)d\theta dB^H(s)\|^2 \\ &= 6\sum_{i=1}^6 J_i. \end{aligned} \tag{3.22}$$

*A similar argument as before, shows that*

$$\begin{aligned} J_1 &\leq \alpha^2M^2T\mathbb{E}\int_0^t\|\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)f(s,z_s+y_s)d\theta\|^2 ds \\ &\leq \alpha^2M^2T\|\int_0^\delta \theta\eta_\alpha(\theta)d\theta\|^2\int_0^t(t-s)^{2\alpha-2}\mathbb{E}\|f(s,z_s+y_s)\|^2 ds \\ &\leq \alpha^2M^2TM_f(1+q')\|\int_0^\delta \theta\eta_\alpha(\theta)d\theta\|^2\int_0^t(t-s)^{2\alpha-2}ds. \end{aligned} \tag{3.23}$$

*For $J_2$, using (3.2), yields*

$$\begin{aligned} J_2 &\leq \alpha^2M^2TM_f(1+q')\|\int_0^\infty \theta\eta_\alpha(\theta)d\theta\|^2\int_{t-\epsilon}^t(t-s)^{2\alpha-2}ds \\ &\leq \frac{\alpha^2M^2TM_f(1+q')}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t(t-s)^{2\alpha-2}ds. \end{aligned} \tag{3.24}$$

*For $J_3$, using the Hölder inequality shows*

$$\begin{aligned} J_3 &\leq \alpha^2\mathbb{E}\left(\int_0^t\int_0^\delta \|\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)Bu_{z+y}^\lambda(s)\|d\theta ds\right)^2 \\ &\leq \alpha^2M^2M_bT\int_0^t(t-s)^{2\alpha-2}\mathbb{E}\|u_{z+y}^\lambda(s)\|^2 ds\|\int_0^\delta \theta\eta_\alpha(\theta)d\theta\|^2. \end{aligned} \tag{3.25}$$

*For $J_4$, (3.2), implies*

$$\begin{array}{ll} J_4 & \leq \alpha^2 M^2 \mathbb{E} \int_{t-\epsilon}^{t} \|(t-s)^{\alpha-1} B u_{z+y}^{\lambda}(s)\|^2 ds \int_{t-\epsilon}^{t} \| \int_0^{\infty} \theta \eta_\alpha(\theta) d\theta \|^2 ds \\ & \leq \frac{\epsilon \alpha^2 M^2 M_b}{\Gamma^2(\alpha+1)} \int_{t-\epsilon}^{t} (t-s)^{2\alpha-2} \mathbb{E}\|u_{z+y}^{\lambda}(s)\|^2 ds \end{array} \tag{3.26}$$

*Similarly, one can show that*

$$\begin{array}{ll} J_5 & \leq \alpha^2 \mathbb{E} \left( \int_0^t \int_0^{\delta} \|\theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) \sigma(s) d\theta dB^H(s) \right)^2 \\ & \leq \alpha^2 M^2 T 2HT^{2H-1} \int_0^t (t-s)^{2\alpha-2} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \| \int_0^{\delta} \theta \eta_\alpha(\theta) d\theta \|^2. \end{array} \tag{3.27}$$

*For $J_6$, using (3.2), yields*

$$\begin{array}{ll} J_6 & \leq \alpha^2 M^2 2HT^{2H-1} \int_{t-\epsilon}^{t} \|(t-s)^{\alpha-1} B u_{z+y}^{\lambda}(s)\|^2 ds \int_{t-\epsilon}^{t} \| \int_0^{\infty} \theta \eta_\alpha(\theta) d\theta \|^2 ds \\ & \leq \frac{\epsilon \alpha^2 M^2 2HT^{2H-1}}{\Gamma^2(\alpha+1)} \int_{t-\epsilon}^{t} (t-s)^{2\alpha-2} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \end{array} \tag{3.28}$$

*Using (3.23), (3.24), (3.25), (3.26), (3.27), (3.28) in (3.22) enable us to conclude that*

$$\mathbb{E}\|\widehat{\Pi}^{\lambda} z)(t) - \widehat{\Pi}_{\epsilon}^{\lambda,\delta} z)(t)\|^2 \longrightarrow 0, \qquad \text{as } \epsilon \longrightarrow 0^+,\ \delta \longrightarrow 0^+.$$

*Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\widehat{\Pi}^{\lambda} z)(t) :\ z \in B_k\}$. Hence, the set $V(t)$ is also precompact in $X$.*

*Thus, by Arzela-Ascoli theorem $\widehat{\Pi}^{\lambda}(B_k)$ is relatively compact. Consequently, we conclude that $\widehat{\Pi}^{\lambda}$ is completely continuous.*

**Step 2.** *The set*
$\Theta = \{z(t) + y(t) \in C([0,T], L_2(\Omega, X)) : z(t) + y(t) = \beta \widehat{\Pi}^{\lambda}(z(t) + y(t)), 0 < \beta < 1\}$ *is bounded.*

*Let be $z(t) + y(t) \in \Theta$. Observe that*

$$z(t) + y(t) = \beta(\widehat{\Pi}^{\lambda} x(z+y))(t) = \begin{array}{l} \beta \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds \\ +\beta \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u_{z+y}^{\lambda}(s) ds \\ +\beta \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s) dB^H(s). \end{array} \tag{3.29}$$

*Then*

$$\mathbb{E}\|z(t) + y(t)\|^2 \leq \begin{array}{l} 3\beta \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds\|^2 \\ +3\beta \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u_{z+y}^{\lambda}(s) ds\|^2 \\ +3\beta \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s) dB^H(s)\|^2 \\ \leq \frac{3\beta M^2}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} ds \int_0^T \mathbb{E}\|f(s, z_s + y_s)\|^2 ds \\ +\frac{3\beta M^2 M_b}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} ds \int_0^T \mathbb{E}\|u_{z+y}^{\lambda}(s)\|^2 ds \\ +\frac{3\beta M^2}{\Gamma^2(\alpha)} 2HT^{2H-1} \int_0^T (T-s)^{2\alpha-2} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\ \leq \frac{3\beta M^2}{\Gamma^2(\alpha)} \frac{T^{2\alpha-1}}{2\alpha-1} M_f \int_0^T\ (1 + \mathbb{E}\|z_s + y_s\|^2) ds \\ +\frac{3\beta M^2 M_b}{\Gamma^2(\alpha)} \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \mathbb{E}\|u_{z+y}^{\lambda}(s)\|^2 ds \\ +\frac{3\beta M^2}{\Gamma^2(\alpha)} \frac{T^{2\alpha-1}}{2\alpha-1} 2cHT^{2H-1} \end{array} \tag{3.30}$$

*where*

$$\mathbb{E}\|u_{z+y}^{\lambda}(t)\|^2 \leq \begin{array}{l} 5 \dfrac{M^2 M_b^2}{\lambda^2 \Gamma^2(\alpha)} \{\|E\bar{x}_T\|^2 + M^2 \mathbb{E}\|\varphi(0)\|^2 + 2HT^{2H-1} \int_0^T E\|\bar{\varphi}(s)\|_{L_0^2}^2 ds \\ +\frac{M^2}{\Gamma^2(\alpha)} M_f \int_0^T (T-s)^{2\alpha-2} ds \int_0^T\ (1 + \mathbb{E}\|z_s + y_s\|^2) ds \\ +2\frac{M^2}{\Gamma^2(\alpha)} HT^{2H-1} \int_0^T (T-s)^{(2\alpha-2)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} \\ \leq 5 \dfrac{M^2 M_b^2}{\lambda^2 \Gamma^2(\alpha)} \{\|E\bar{x}_T\|^2 + M^2 \mathbb{E}\|\varphi(0)\|^2 + \int_0^T E\|\bar{\varphi}(s)\|_{L_0^2}^2 ds \\ +\frac{M^2}{\Gamma^2(\alpha)} M_f \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T\ (1 + \mathbb{E}\|z_s + y_s\|^2) ds \\ +2\frac{M^2}{\Gamma^2(\alpha)} HT^{2H-1} c \frac{T^{2\alpha-1}}{2\alpha-1}\}. \end{array} \tag{3.31}$$

*Using (3.31) in (3.30) yields*

$$\mathbb{E}\|z(t)+y(t)\|^2 \le \mu + \nu \int_0^T \ \mathbb{E}\|z_s+y_s\|^2 ds \tag{3.32}$$

*where*

$$\begin{aligned}\mu = \ & \tfrac{3\beta M^2}{\Gamma^2(\alpha)}\tfrac{T^{2\alpha}}{2\alpha-1}M_f + \tfrac{3\beta M^2 M_b}{\Gamma^2(\alpha)}\tfrac{T^{2\alpha-1}}{2\alpha}5\frac{M^2M_b^2}{\lambda^2\Gamma^2(\alpha)}\{\|E\bar{x}_T\|^2 + M^2\mathbb{E}\|\varphi(0)\|^2 \\ & +2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds + \tfrac{M^2}{\Gamma^2(\alpha)}M_f\tfrac{T^{2\alpha-1}}{2\alpha} + 2\tfrac{M^2}{\Gamma^2(\alpha)}HT^{2H-1}c\tfrac{T^{2\alpha-1}}{2\alpha-1}\} \\ & +\tfrac{3\beta M^2}{\Gamma^2(\alpha)}\tfrac{T^{2\alpha-1}}{2\alpha-1}2cHT^{2H-1}\end{aligned} \tag{3.33}$$

*and*

$$\nu = \tfrac{3\beta M^2}{\Gamma^2(\alpha)}\tfrac{T^{2\alpha-1}}{2\alpha-1}M_f + \tfrac{15\beta M^2 M_b}{\Gamma^2(\alpha)}\tfrac{T^{2\alpha-1}}{2\alpha-1}\frac{M^2M_b^2}{\lambda^2\Gamma^2(\alpha)}\tfrac{M^2}{\Gamma^2(\alpha)}M_f\tfrac{T^{2\alpha}}{2\alpha-1}. \tag{3.34}$$

*Hence, by using Gronwall's lemma in (3.32), we deduce that $\Theta$ is bounded, as a consequence of Schaefer's fixed point theorem, we deduce that $\widehat{\Pi}^\lambda$ has a fixed point, so that $\Pi^\lambda$ has a fixed point. This conclude the proof.* □

***Theorem 3.3*** *Assume that $(\mathcal{H}.1)$- $(\mathcal{H}.3)$ are satisfied. If the functions $f$ is uniformly bounded, then the system (1.1) is approximately controllable on $[0,T]$.*

**Proof:** *Let $x_\lambda$ be a fixed point of $\Pi^\lambda$ . By using the stochastic Fubini theorem, it can easily be seen that*

$$\begin{aligned}& x_\lambda(T) = \bar{x}_T - \mathbb{E}\bar{x} + T_\alpha(T)\varphi(0) \\ & + \int_0^T (T-t)^{\alpha-1}S_\alpha(T-t)BB^*S_\alpha^*(T-t)(\lambda I+\Gamma_0^T)^{-1}\{E\bar{x}_T - T_\alpha(T)\varphi(0)\}dt \\ & - \int_0^T \bar{\varphi}(s)dB^H(s) + \int_0^T (T-t)^{\alpha-1}S_\alpha(T-t)BB^*S_\alpha^*(T-t)\int_0^T (\lambda I+\Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s)dt \\ & + \int_0^T (T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})ds \\ & - \int_0^T (T-t)^{\alpha-1}S_\alpha(T-t)BB^*S_\alpha^*(T-t)\int_0^T (\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})dsdt \\ & + \int_0^T (T-s)^{\alpha-1}S_\alpha(T-s)\sigma(s)dB^H(s) \\ & - \int_0^T (T-t)^{\alpha-1}S_\alpha(T-t)BB^*S_\alpha^*(T-t)\int_0^T (\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)\sigma(s)dB^H(s)dt \\ & = \bar{x_T} - (\lambda I+\Gamma_0^T)(\lambda I+\Gamma_0^T)^{-1}\{\mathbb{E}\bar{x} - T_\alpha(T)\varphi(0)\} + \Gamma(\lambda I+\Gamma_0^T)^{-1}\{E\bar{x}_T - T_\alpha(T)\varphi(0)\} \\ & -(\lambda I+\Gamma_0^T)(\lambda I+\Gamma_0^T)^{-1}\int_0^T \bar{\varphi}(s)dB^H(s) + \Gamma\int_0^T (\lambda I+\Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\ & +(\lambda I+\Gamma_0^T)(\lambda I+\Gamma_0^T)^{-1}\int_0^T (T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})ds \\ & -\Gamma\int_0^T (\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})ds \\ & +(\lambda I+\Gamma_0^T)(\lambda I+\Gamma_0^T)^{-1}\int_0^T \sigma(s)dB^H(s) - \Gamma\int_0^T (\lambda I+\Gamma_0^T)^{-1}\sigma(s)dB^H(s) \\ & = \bar{x}_T - \lambda(\lambda I+\Gamma_0^T)^{-1}\{\mathbb{E}\bar{x}_T - T_\alpha(T)\varphi(0)\} \\ & -\lambda\int_0^T (\lambda I+\Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\ & +\lambda\int_0^T (\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})ds \\ & +\lambda\int_0^T (\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)\sigma(s)dB^H(s), \ t\in[0,T].\end{aligned} \tag{3.35}$$

*It follows from the assumption on $f$ that there exists $\bar{D} > 0$ such that*

$$\|f(s,x_\lambda(s))\|^2 \le \bar{D} \tag{3.36}$$

*for all $(s,\omega) \in [0,T]\times\Omega$. Then there is a subsequence (which we still denote by $f(s,x_\lambda(s))$) which converges weakly to, say, $f(s)$ in $X$. The compactness of $S_\alpha(t)$, $t>0$ implies that*

$$S_\alpha(T-s)f(s,x_\lambda(s)) \longrightarrow S_\alpha(T-s)f(s). \tag{3.37}$$

*From the equation (3.35) , we have*

$$
\begin{aligned}
&E\|x_\lambda(T)-\bar{x}_T\|^2 \leq 4E\|\lambda(\lambda I+\Gamma_0^T)^{-1}\{E\bar{x}_T - T_\alpha(T)\varphi(0)\}\|^2\\
&+4E\|\int_0^T \lambda(\lambda I+\Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s)\|^2\\
&+4E\|\int_0^T \lambda(\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)f(s,x_{s,\lambda})ds\|^2\\
&+4E\|\int_0^T \lambda(\lambda I+\Gamma_0^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-s)\sigma(s)dB^H(s)\|^2,\ t\in[0,T].
\end{aligned}
$$

*On the other hand, by Lemma 3.3, the operator* $\lambda(\lambda I+\Gamma)^{-1}\longrightarrow 0$ *strongly as* $\lambda\to 0^+$ *for all* $0\leq s\leq T$*, and, moreover,* $\|\lambda(\lambda I+\Gamma)^{-1}\|\leq 1$*. Thus, by the Lebesgue dominated convergence theorem, the compactness of* $S_\alpha(t)$ *and the compactness of* $T_\alpha(t)$ *implies that* $E\|x_\lambda(T)-\bar{x}_T\|\longrightarrow 0$ *as* $\lambda\to 0^+$*. This shows the approximate controllability of (1.1). The proof is complete.*

□

## 4. Example

*To illustrate main result, we consider the following fractional neutral stochastic partial differential equation with infinite delay, driven by an fBm of the form*

$$
\begin{cases}
dJ_t^{1-\alpha}[v(t,\xi)-\varphi(0,\xi)] = [\frac{\partial^2}{\partial^2\xi}v(t,\xi)+c(\xi)u(t)+f(t,t-r,\xi)]dt\\
+\sigma(t)\frac{dB^H(t)}{dt}, \quad 0\leq t\leq T,\ r>0,\ 0\leq\xi\leq 1\\
v(t,0)=v(t,1)=0, \qquad 0\leq t\leq T,\\
v(s,\xi)=\varphi(s,\xi),\ \ -\infty<s\leq 0 \quad 0\leq\xi\leq 1,
\end{cases}
\tag{4.1}
$$

*where* $B^H(t)$ *is cylindrical fractional Brownian motion and* $\varphi:(-\infty,0]\times[0,1]\longrightarrow\mathbb{R}$ *is a given measurable function for which* $\|\varphi\|^2_{\mathcal{B}_h}<\infty$.

*We rewrite (4.1) in the abstract form (1.1). We take* $X=Y=U=L^2([0,1])$*. Define the operator* $A: D(A)\subset X\longrightarrow X$ *given by* $A=\frac{\partial^2}{\partial^2\xi}$ *with*

$$D(A)=\{y\in X:\ y' \text{ is absolutely continuous}, y''\in X, \quad y(0)=y(1)=0\}$$

*Observe that*

$$Ax=\sum_{n=1}^{\infty} n^2 <x,e_n>_X e_n, \quad x\in D(A),$$

*where* $e_n:=\sqrt{\frac{2}{\pi}}\sin nx,\ n=1,2,....$ *is an orthogonal set of eigenvectors of* $-A$.

*It is will known that A generates a compact analytic semigroup* $\{S(t)\}_{t\geq 0}$ *in X, and is given by*

$$S(t)x=\sum_{n=1}^{\infty} e^{-n^2t}<x,e_n>e_n,$$

*for* $x\in X$ *and* $t\geq 0$ *(see [5]). Since the semigroup* $\{S(t)\}_{t\geq 0}$ *is analytic, there exists a constant* $M>0$ *such that* $\|S(t)\|^2\leq M$ *for every* $t\geq 0$*. Hence, the condition* $(\mathcal{H}.1)$ *holds.*

*If we choose* $\alpha\in(\frac{3}{4},1)$,

$$S_\alpha(t)x=\int_0^\infty \alpha\theta\eta_\alpha(\theta)S(\theta t^\alpha)d\theta, \quad x\in X.$$

*Further, the operator* $B:\mathbb{R}\longrightarrow X$ *defined by*

$$Bu(t)(\xi)=c(\xi)u(t),\ 0\leq\xi\leq 1,\ c(\xi)\in L^2([0,1])$$

*is a bounded linear operator.*

*We choose the phase function* $h(s)=e^{2s},\ s<0$*, then* $l=\int_{-\infty}^0 h(s)ds=\frac{1}{2}<\infty$*, and the abstract phase space* $\mathcal{B}_h$ *is Banach space when equipped with the norm*

$$\|\varphi\|_{\mathcal{B}_h}=\int_{-\infty}^0 h(s)\sup_{\theta\in[s,0]}(\mathbb{E}\|\varphi(\theta)\|^2)^{\frac{1}{2}}ds.$$

*To rewrite the initial-boundary value problem (4.1) in the abstract form (1.1), we assume the following:*

*For* $(t,\varphi)\in[0,T]\times\mathcal{B}_h$*, where* $\varphi(\theta)(\xi)=\varphi(\theta,\xi),\ (\theta,\xi)\in(-\infty,0]\times[0,1]$*, we put* $v(t)(\xi)=v(t,\xi)$*. Define* $f:[0,T]\times\mathcal{B}_h\longrightarrow X$ *by*

$$f(t, z(t,\xi)) := \frac{e^{-t}|z(t,\xi)|}{(1+e^t)(1+|z(t,\xi)|)}.$$

*Clearly, we have*

$$\|f(t+z(t,\xi))\| \leq |z(t,\xi)|, \quad \|f(t+z(t,\xi))\| \leq \frac{1}{2}\frac{|z(t,\xi)|}{(1+|z(t,\xi)|)} \leq \frac{1}{2}$$

*and*

$$\begin{aligned}\|f(t+z_1(t))\xi - f(t+z_2(t))\xi\| &= \frac{e^{-t}|z_1(t,\xi)-z_2(t,\xi)|}{(1+e^t)(1+|z_1(t,\xi)|)(1+|z_2(t,\xi)|)} \\ &\leq \tfrac{e^{-t}}{1+e^{-t}}|z_1(t,\xi)-z_2(t,\xi)| \\ &\leq \tfrac{1}{2}|z_1(t,\xi)-z_2(t,\xi)|.\end{aligned} \tag{4.2}$$

*Then,* $(\mathcal{H}.2)$ *is satisfied.*

*In order to define the operator* $Q : Y := L^2([0,1],\mathbb{R}) \longrightarrow Y$, *we choose a sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, *set* $Qe_n = \lambda_n e_n$, *and assume that*

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

*Define the fractional Brownian motion in* $Y$ *by*

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta^H(t)e_n,$$

*where* $H \in (\frac{1}{2}, 1)$ *and* $\{\beta_n^H\}_{n\in\mathbb{N}}$ *is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the function* $\sigma : [0,+\infty) \to \mathcal{L}_2^0(L^2([0,1]), L^2([0,1]))$ *satisfies*

$$\sup_{0\leq s\leq T} \|\sigma(s)\|^2_{\mathcal{L}_2^0} < \infty.$$

*Then, all the assumptions of Theorem 3.3 are satisfied. Therefore, we conclude that the system (4.1) is approximately controllable on* $(-\infty, T]$.

## References


1. *A. A. Kilbas, H.M. Srivastava and J.J. Trujillo. Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).*
2. *A. M .Yaglom. Certain types of random fields in n-dimensional space similar to stationary stochastic processes. Teor. Veroyatnostei i ee Primen. 2(3), 292–338 (1957).*
3. *A. M .Yaglom. Correlation Theory of Stationary and Related Random Functions. Springer, New York. (1987).*
4. *A. N. Kolmogrov. Wienersche spiralen und einige andere interessante kurven in Hilberschen Raum. C. R. Dokl. Acad. USSS(NS) 26, 115–118 (1940).*
5. *A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York (1983).*
6. *A. Slama, A. Boudaoui. Approximate controllability of fractional impulsive neutral stochastic integrodifferential equations with nonlocal conditions and infinite delay. Annals 2, 001 (2015).*
7. *B. Boufoussi and S. Hajji. Neutral stochastic functional differential equation driven by a fractional Brownian motion in a Hilbert space, Statist. Probab. Lett., 82, 1549-1558 (2012).*
8. *B. Boufoussi, S. Hajji, and E. Lakhel. Functional differential equations in Hilbert spaces driven by a fractional Brownian motion. Afrika Matematika, 23 (2) , 173-194 (2012).*
9. B. *Mandelbrot and V. Ness. Fractional Brownian motion, fractional noises and applications. SIAM Reviews,* ***10****(4), 422-437 (1986).*
10. *E. Lakhel. Controllability of Neutral Stochastic Functional Integro-Differential Equations Driven By Fractional Brownian Motion. Stochastic Analysis and Applications, vol. 34 (3) 427,440 (2016).*
11. *E. Lakhel. Controllability of Neutral Functional Differential Equations Driven by Fractional Brownian Motion With Infinite Delay. Nonlinear Dynamics And Systems Theory, Volume 17, (3), 291-302 (2017).*
12. *E. Lakhel and M. A. McKibben. Controllability of Impulsive Neutral Stochastic Functional Integro-Differential Equations Driven by Fractional Brownian Motion. Chapter 8 In book : Brownian Motion: Elements, Dynamics, and Applications. Editors: M. A. McKibben & M. Webster. Nova Science Publishers, New York, pp. 131-148 (2015).*
13. *E. Lakhel and M. A. McKibben. Existence of solutions for fractional neutral functional differential equations driven by fBm with infinite delay. Stochastics 90, 313-329 (2018).*

14. F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic Calculus for Fractional Brownian Motion and Application.* Springer-Verlag, (2008).
15. F. Mainardi. *Fractional Calculus: Some basic problems in continuum and statistical mechanics. In: Fractals and Fractional Calculus in Continuum Mechanics (A. Carpinteri and F. Mainardi eds.). Springer-Verlag, New York, pp. 291–348 (1997).*
16. J. Cui and L. Yan. *Existence result for fractional neutral stochastic integrodifferential equations with infinite delay. Journal of Physics A: Mathematical and Theoretical 44 , 1-16 (2011).*
17. K. Li. *Stochastic delay fractional evolution equations driven by fractional brownian motion. Mathematical Methods in the Applied Sciences . DOI 10. 1002/mma. 3169 (2014).*
18. M. Pierri, D. ORegan. *Some comments on the paper: controllability of fractional neutral stochastic functional differential systems. Z. Angew. Math. Phys. 65(5), 941–959 (2014).*
19. N. Mahmudov. *Existence and approximate controllability of sobolev type fractional stochastic evolution equations. Bull. Pol. Acad. Sci. Tech. Sci. 62(2), 205–215 (2014).*
20. NT. Dung. *Stochstic Volterra integro-differential equations driven by by fractional Brownian motion in Hilbert space, Stochastics, 87 (1), 142-159 (2015).*
21. R. Triggiani. *A note on the lack of exact controllability for mild solutions in banach spaces. SIAM J. Control Optim. 15(3), 407–411 (1977).*
22. R. Sakthivel, R. Ganesh, S. Suganya. *Approximate controllability of fractional neutral stochastic system with infinite delay. Rep. Math. Phys. 70(3), 291–311 (2012).*
23. R. Sakthivel, R. Ganesh, Y. Ren, and S. M. Anthoni. *Approximate controllability of nonlinear fractional dynamical systems. Commun. Nonlinear Sci. Numer. Simul., 18, 3498-3508 (2013).*
24. R. Sakthivel, S. Suganya, S. M. Anthoni. *Approximate controllability of fractional stochastic evolution equations. Comput. Math. Appl. 63(3), 660–668 (2012).*
25. S. Farahi, T. Guendouzi. *Approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions. Results Math. 65(3–4), 501–521 (2014).*
26. T. Guendouzi, S. Farahi. *Approximate controllability of semilinear fractional stochastic dynamic systems with nonlocal conditions in Hilbert spaces. Mediterr. J. Math. 13(2), 637–656 (2016).*
27. V. Lakshmikantham and A. S. Vatsala (2008). *Basic theory of fractional differential equations. Nonlinear Analysis TMA 69:2677–2682.*
28. Y. Li and B. Liu. *Existence of solution of nonlinear neutral functional differential inclusion with infinite delay. Stoc. Anal. Appl. 25 , 397-415 (2007).*
29. Y. Zhou, J. Feng. *Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. 59 , 1063-1077 ( 2010).*


[1,2] *Cadi Ayyad University, National School of Applied Sciences, 46000 Safi, Morocco*
*E-mail address:* lahmoudi1992@gmail.com