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Exact Calculus for s-Hölder Subgradients in Banach Spaces

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ABSTRACT: We give in this paper some useful calculus results related to the s-Hölder subdifferential of extended-real valued functions defined on arbitrary real Banach spaces.

Key Words: Exact calculus, s-Hölder subdifferential, normal cones, coderive.

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1. Introduction

The s-Hölder subdifferential was first studied by J. M. Borwein and al in [2], followed by joint work with R. Girgensohn and X. Wang in [3] in order to derive some properties of lower semicontinuous functions with its lower subderivatives and to construct Lipschitz functions that satisfy some properties on a Banach spaces. These authors and others for example B. Mordukhovich, M. Nguyen, F. Bernicot, J. Venel, A. Ioffe didn't prove some useful calculus results in general Banach spaces for this subdifferential but they are usually used the so called proximal subdifferential, the case when s=1, or the Fréchet subdifferential, the case when s=0 for simplicity, see [1,2,3,4,5,8,11] for more details and informations.
In this paper, we aim to establish some interesting calculus rules related to the s-Hölder subdifferential for extended-real-valued functions defined on Banach spaces including difference rules, chain rules, scalarization formula, product and quotient rules. Most (but not all) of these results involve assumptions about Lipschitz continuity of some components in composition and the principal calculus result obtained in this paper, the difference rule, is expressed in form of inclusion without subdifferential regularity assumptions all of the components.
Let X be an arbitrary real Banach space, and let $\varphi : X \to [-\infty, +\infty]$ be finite at $\bar{x}$. Following J. Borwein and al in [2], we say that $x^* \in X^*$ is an s-Hölder subgradient of φ at $\bar{x}$ ($s \in]0,1]$) if there exists positive constants $\delta_{\bar{x}}$ and $C_{\bar{x}}$ such that

$$\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq -C_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s} \tag{1.1}$$

whenever $\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$. The set of all such x^* is called the s-Hölder subdifferential of φ at $\bar{x}$ and is denoted by $\partial_{H(s)}\varphi(\bar{x})$
Follows [3] φ is s-Hölder smooth at $\bar{x}$ if there exists positive constants $\delta_{\bar{x}}$ and $C_{\bar{x}}$ and $x^* \in X^*$ such that

$$\|\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle\| \leq C_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s} \quad whenever \quad \parallel x - \bar{x} \parallel < \delta_{\bar{x}}.$$

When s=1, we say that φ is Lipschitz smooth at $\bar{x}$ and if φ is s-Hölder smooth at $\bar{x}$ then φ is differentiable at $\bar{x}$ and $\partial_{H(s)}\varphi(\bar{x})$ is a singleton.
The rest of this paper is organized as follows. Section 2 deals with definitions and preliminary material.

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In section 3, we establish the main result of this paper related to difference rule for the s-Hölder subdifferential and in section 4 we prove some other calculus formulas for subdifferentials of scalarisation, chain rule, products and quotients rules.

2. Basic definitions and properties

Throughout this paper we use standard notation except special symbols introduced when they are defined. All spaces considered are real Banach whose norms are always denoted by $\| \, . \, \|$ and the canonical pairing $\langle ., . \rangle$ between X and its topologically dual space X^*; obviously $\langle ., . \rangle$ reduces to the standard inner product in the case of Hilbert spaces with $X = X^*$.
Considering a nonempty set Ω in X and its indicator function $\delta(., \Omega)$ equal 0 if $x \in \Omega$ and ∞ otherwise, we define the s-Hölder normal cone to Ω at $\bar{x} \in \Omega$ by

$$N_{H(s)}(\bar{x}; \Omega) = \partial_{H(s)} \delta(\bar{x}, \Omega) \tag{2.1}$$

We can derive directly follows (1) and (2) in the arbitrary Banach space setting: $x^* \in N_{H(s)}(\bar{x}; \Omega)$ if and only if there exists the positif numbers $\delta_{\bar{x}}$ and $C_{\bar{x}}$ such that

$$\langle x^*, x - \bar{x} \rangle \leq C_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s} \quad for\, any\, x \in \Omega \cap B_{\delta_{\bar{x}}}(\bar{x}) \tag{2.2}$$

where $B_{\delta_{\bar{x}}}(\bar{x})$ means the open ball centered at $\bar{x}$ with radius $\delta_{\bar{x}}$. On the other hand, one can observe based (3) that the representation

$$\partial_{H(s)} \varphi(\bar{x}) = \{x^* \in X^* / (x^*, -1) \in N_{H(s)}((\bar{x}; \varphi(\bar{x})); epi\varphi)\} \tag{2.3}$$

where $\varphi : X \to \bar{\mathbb{R}}$ is an extended-real-valued function and its usual epigraphe is defined by:

$$epi\varphi = \{(x, \mu) \in X \times IR / \mu \geq \varphi(x)\}.$$

In the contrast to the case of single-valued mappings $F : X \to Y$, the symbol $F : X \rightrightarrows Y$ stands for a multifunction from X into Y with graph

$$gphF = \{(x, y) \in X \times Y / y \in F(x)\}$$

and we define the s-Hölder coderivative of F at $(\bar{x}, \bar{y}) \in gphF$ by

$$D^*_{H(s)} F(\bar{x}, \bar{y})(y^*) = \{x^* \in Y / (x^*, -y^*) \in N_{H(s)}((\bar{x}; \bar{y}); gphF)\} \tag{2.4}$$

where $\bar{y}$ is omitted if $F = f : X \to Y$ is single-valued.

3. The main result

3.1. Difference rule for s-Hölder subgradients.

In this section we give the important result of difference rule for the s-Hölder subgradients, which holds in a rather general setting, in contrast to its sum rule counterpart.

Theorem 3.1. *Let $\varphi_i : X \to \bar{\mathbb{R}}$ be finite at $\bar{x}$ for i=1,2. Assume that $\partial_{H(s)} \varphi_2(\bar{x}) \neq \emptyset$. Then*

$$\partial_{H(s)}(\varphi_1 - \varphi_2)(\bar{x}) \subset \bigcap_{x^* \in \partial_{H(s)} \varphi_2(\bar{x})} [\partial_{H(s)} \varphi_1(\bar{x}) - x^*] \subset \partial_{H(s)} \varphi_1(\bar{x}) - \partial_{H(s)} \varphi_2(\bar{x}).$$

Proof. Fix arbitrary subgradients $x^* \in \partial_{H(s)}(\varphi_1 - \varphi_2)(\bar{x})$ and $x^*_2 \in \partial_{H(s)} \varphi_2(\bar{x})$. Then there exists the positive constants $C^1_{\bar{x}},\ C^2_{\bar{x}},\ \delta_{\bar{x}}$ such that:

$$\varphi_1(x) - \varphi_2(x) - (\varphi_1(\bar{x}) - \varphi_2(\bar{x})) - \langle x^*, x - \bar{x} \rangle \geq -C^1_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s}$$
$$and \quad \varphi_2(x) - \varphi_2(\bar{x}) - \langle x^*_2, x - \bar{x} \rangle \geq -C^2_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s}$$

whenever $\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$.
The above inequalities directly imply that

$$\varphi_1(x) - \varphi_1(\bar{x}) - \langle x^* + x^*_2, x - \bar{x} \rangle \geq -(C^2_{\bar{x}} + C^1_{\bar{x}}) \parallel x - \bar{x} \parallel^{1+s} \quad for\ all\ \ \parallel x - \bar{x} \parallel < \delta_{\bar{x}}.$$

which justifies $x^* + x^*_2 \in \partial_{H(s)} \varphi_1(\bar{x})$ that imply that $x^* \in \partial_{H(s)} \varphi_1(\bar{x}) - x^*_2$ with $x^*_2 \in \partial_{H(s)} \varphi_2(\bar{x})$ thus,we achieve the proof.

3.2. Chain rule and other calculus rules.

Analogously to section 3, we give other calculus rules for the s-Hölder subdifferential. Let us start by the scalarization formula that relates the s-Hölder coderivative of an arbitrary locally Lipschitzian mapping $f : X \to Y$ between Banach spaces with the s-Hölder subdifferential of its scalarization.

$$\langle y^*, f\rangle(x) = \langle y^*, f(x)\rangle \; x \in X, \;\; y^* \in Y^*$$

Theorem 3.2 (Scalarization formula)**.** *Let $f : X \to Y$ be Lipschitz continuous around $\bar{x}$. Then*

$$\partial_{H(s)}\langle y^*, f\rangle(\bar{x}) = D^*_{H(s)} f(\bar{x})(y^*) \;\; for\, all\, y^* \in Y^*.$$

Proof. Fix arbitrary subgradient $x^* \in \partial_{H(s)}\langle y^*, f\rangle(\bar{x})$. Then there exists the positive constants $C_{\bar{x}}, \;\delta_{\bar{x}}$ such that:

$$\langle y^*, f\rangle(x) - \langle y^*, f\rangle(\bar{x}) - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s}$$

whenever $\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$. Then

$$\langle y^*, f(x)\rangle - \langle y^*, f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}} \parallel x - \bar{x} \parallel^{1+s}$$

For any such x we have

$$\langle y^*, f(x) - f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}}(\parallel x - \bar{x} \parallel + \parallel f(x) - f(\bar{x}) \parallel)^{1+s}$$

which implies by 2.4 that $(x^*, -y^*) \in N_{H(s)}((\bar{x}; f(\bar{x})); gph f)$ and hence $x^* \in D^*_{H(s)} f(\bar{x})(y^*)$.
Let us prove the opposite inclusion holds if f is Lipschitz continuous around $\bar{x}$ with modulus $l \geq 0$.
Let an arbitrary $x^* \in D^*_{H(s)} f(\bar{x})(y^*)$. Then there are the positive constants $C_{\bar{x}}, \;\delta_{\bar{x}}$ such that:

$$\langle y^*, f(x) - f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}}(\parallel x - \bar{x} \parallel + \parallel f(x) - f(\bar{x}) \parallel)^{1+s}$$

for all $\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$ Applying the fact that f is Lipschitz continuous around $\bar{x}$ with modulus l, then

$$\langle y^*, f(x) - f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}}(l+1)^{1+s} \parallel x - \bar{x} \parallel)^{1+s} \; for\; all \;\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$$

This gives

$$\langle y^*, f\rangle(x) - \langle y^*, f\rangle(\bar{x}) - \langle x^*, x - \bar{x}\rangle \geq -C_{\bar{x}}(l+1)^{1+s} \parallel x - \bar{x} \parallel^{1+s} \;\; for\; all \;\parallel x - \bar{x} \parallel < \delta_{\bar{x}}$$

which yields $x^* \in \partial_{H(s)}\langle y^*, f\rangle(\bar{x})$ thus,we achieve the proof.

Now let us obtain chain rule for the s-Hölder subdifferential of composition in the case of Lipschitz continuous f and C^2 mapping φ between Banach spaces.

Theorem 3.3. *Let $f : X \longrightarrow Y$ a Lipschitz continuous single-valued mapping around $\bar{x}$ and let $\varphi : X \times Y \longrightarrow \bar{\mathbb{R}}$ be finite at $(\bar{x}, \bar{y})$ with $\bar{y} = f(\bar{x})$. Assume that $\varphi \in C^2$ around $(\bar{x}, \bar{y})$. Then*

$$\partial_{H(s)}(\varphi \circ f)(\bar{x}) = \nabla_x\varphi(\bar{x}, \bar{y}) + \partial_{H(s)}\langle \nabla_y\varphi(\bar{x}, \bar{y}), f\rangle(\bar{x}) = \nabla_x\varphi(\bar{x}, \bar{y}) + D^*_{H(s)} f(\bar{x})(\nabla_y\varphi(\bar{x}, \bar{y}))$$

where $\varphi \circ f$ is the function acting from X into $\bar{\mathbb{R}}$ defined by:

$$(\varphi \circ f)(x) = \varphi(x, f(x)).$$

Proof. Fix an arbitrary $x^* \in \nabla_x\varphi(\bar{x}, \bar{y}) + \partial_{H(s)}\langle \nabla_y\varphi(\bar{x}, \bar{y}), f\rangle(\bar{x})$, by inequality 1, we find the positive constants $C_{\bar{x}}, \;\delta_{\bar{x}}$ such that:

$$\langle \nabla_y\varphi(\bar{x}, \bar{y}), f(x)\rangle - \langle \nabla_y\varphi(\bar{x}, \bar{y}), f(\bar{x})\rangle - \langle x^* - \nabla_x\varphi(\bar{x}, \bar{y}), x - \bar{x}\rangle$$

$$\geq -C_{\bar{x}}(\parallel x - \bar{x} \parallel + \parallel f(x) - f(\bar{x}) \parallel)^{1+s} \; \forall x \in B_{\delta_{\bar{x}}}(\bar{x}).$$

By the Lipschitz continuity of f around $\bar{x}$, we have

$$\langle \nabla_y \varphi(\bar{x},\bar{y}), f(x)\rangle - \langle \nabla_y \varphi(\bar{x},\bar{y}), f(\bar{x})\rangle - \langle x^* - \nabla_x \varphi(\bar{x},\bar{y}), x-\bar{x}\rangle$$
$$\geq -C_{\bar{x}}(l+1)^{1+s} \parallel x-\bar{x} \parallel^{1+s} \ \ \forall x \in B_{\delta_{\bar{x}}}(\bar{x}) \ (\alpha_1).$$

Furthermore, the assumption on $\varphi \in C^2$ around $(\bar{x},\bar{y})$ with $\bar{y} = f(\bar{x})$ yields the existence of the positive constants $C'_{\bar{x}}$, $\delta'_{\bar{x}}$ such that:

$$\varphi(x,f(x)) - \varphi(\bar{x},f(\bar{x})) - \langle \nabla_x \varphi(\bar{x},\bar{y}), x-\bar{x}\rangle - \langle \nabla_y \varphi(\bar{x},\bar{y}), f(x)-f(\bar{x})\rangle$$
$$\geq -C'_{\bar{x}}(\parallel x-\bar{x} \parallel + \parallel y-\bar{y} \parallel) \ \ whenever \ \ \parallel x-\bar{x} \parallel + \parallel y-\bar{y} \parallel < \delta'_{\bar{x}}$$

This imply that:

$$\varphi(x,f(x)) - \varphi(\bar{x},f(\bar{x})) - \langle \nabla_x \varphi(\bar{x},\bar{y}), x-\bar{x}\rangle - \langle \nabla_y \varphi(\bar{x},\bar{y}), f(x)-f(\bar{x})\rangle$$
$$\geq -C'_{\bar{x}}(\parallel x-\bar{x} \parallel + \parallel y-\bar{y} \parallel)^{1+s} \ \ whenever \ \ \parallel x-\bar{x} \parallel + \parallel y-\bar{y} \parallel < \delta'_{\bar{x}}.$$

and by the Lipschitz continuity of f around $\bar{x}$, we have

$$\varphi(x,f(x)) - \varphi(\bar{x},f(\bar{x})) - \langle \nabla_x \varphi(\bar{x},\bar{y}), x-\bar{x}\rangle - \langle \nabla_y \varphi(\bar{x},\bar{y}), f(x)-f(\bar{x})\rangle$$
$$\geq -C'_{\bar{x}}(1+l)^{1+s} \parallel x-\bar{x} \parallel^{1+s} \ \ whenever \ \ \parallel x-\bar{x} \parallel + \parallel y-\bar{y} \parallel < \delta'_{\bar{x}} \ (\alpha_2).$$

Denoting $C = 2max\{C_{\bar{x}}(1+l)^{1+s}, C'_{\bar{x}}(1+l)^{1+s}\}$ and $\delta = min\{\frac{\delta_{\bar{x}}}{2}, \frac{\delta'_{\bar{x}}}{2(l+1)}\}$ Therefore, by virtu of (α_1) and (α_2) we have

$$\varphi(x,f(x)) - \varphi(\bar{x},f(\bar{x})) \geq \langle x^*, x-\bar{x}\rangle - C\|x-\bar{x}\|^{1+s} \ whenever \ \|x-\bar{x}\| < \delta.$$

Consequently, $x^* \in \partial_{H(s)}(\varphi \circ f)(\bar{x})$. This proves the inclusion $\nabla_x \varphi(\bar{x},\bar{y}) + \partial\langle \nabla_y \varphi(\bar{x},\bar{y}), f\rangle(\bar{x}) \subset \partial_{H(s)}(\varphi \circ f)(\bar{x})$. To establish the opposite inclusion we employ the similar arguments starting with a point $x^* \in \partial_{H(s)}(\varphi \circ f)(\bar{x})$. Thus, we achieve the proof.

At the end of this section we give some additional calculus formulas for the s-Hölder subdifferential. The first one is the product rule involving Lipschitzian functions.

Theorem 3.4. *Let $\varphi_i : X \longrightarrow \bar{\mathbb{R}}$, $i = 1,2$ be two Lipschitz continuous functions around $\bar{x}$. Assume that $\partial_{H(s)}(-\varphi_1(\bar{x})\varphi_2)(\bar{x}) \neq \emptyset$. Then we have*

$$\partial_{H(s)}(\varphi_1.\varphi_2)(\bar{x}) \subset \bigcap_{x^* \in \partial_{H(s)}(-\varphi_1(\bar{x})\varphi_2)(\bar{x})} [\partial_{H(s)}(\varphi_2(\bar{x})\varphi_1)(\bar{x}) - x^*].$$

Proof. Consider the functions $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $f : X \longrightarrow \mathbb{R}^2$ defined by:

$$f(x) = (\varphi_1(x), \varphi_2(x)) \, , \, \psi(y_1,y_2) = y_1.y_2.$$

Then $\varphi_1.\varphi_2 = \psi o f$. Applying the above chain rule Theorem to the composition, we deduce the

$$\partial_{H(s)}(\varphi_1.\varphi_2)(\bar{x}) = D^*_{H(s)}f(\bar{x})(\nabla\psi(f(\bar{x})) = D^*_{H(s)}f(\bar{x})(\varphi_2(\bar{x}), \varphi_1(\bar{x})) \tag{3.1}$$

Since $f(x) = f_1(x) - f_2(x)$ with $f_1(x) = (\varphi_1(x), 0)$ and $f_2(x) = (0, -\varphi_2(x))$, we derive from the coderivative difference rule

$$D^*_{H(s)}(f_1 - f_2)(\bar{x})(y^*) = \partial_{H(s)}\langle y^*, f_1 - f_2\rangle(\bar{x})$$
$$= \bigcap_{x^* \in \partial_{H(s)}\langle y^*, f_2\rangle(\bar{x})} [\partial_{H(s)}\langle y^*, f_1\rangle(\bar{x}) - x^*] = \bigcap_{x^* \in D^*_{H(s)} f_2(\bar{x})(y^*)} [D^*_{H(s)} f_1(\bar{x})(y^*) - x^*]$$

from Theorem 3.1 with the supposition $D^*_{H(s)} f_2(\bar{x})(y^*) \neq \emptyset$. Then

$$D^*_{H(s)}(f_1 - f_2)(\bar{x})(\varphi_2(\bar{x}), \varphi_1(\bar{x})) = \bigcap_{x^* \in D^*_{H(s)} f_2(\bar{x})(\varphi_2(\bar{x}), \varphi_1(\bar{x}))} [D^*_{H(s)} f_1(\bar{x})(\varphi_1(\bar{x}), \varphi_2(\bar{x})) - x^*] \tag{3.2}$$

By the scalarization formula from Theorem 3.2 and the obvious representation

$$N_{H(s)}((x_1,x_2);\Omega_1\times\Omega_2)=N_{H(s)}(x_1;\Omega_1)\times N_{H(s)}(x_2;\Omega_2)$$

of s-Hölder normals cones in product spaces, we have

$$D^*_{H(s)}f_1(\bar{x})(\varphi_2(\bar{x}),\varphi_1(\bar{x}))=D^*_{H(s)}\varphi_1(\bar{x})(\varphi_2(\bar{x}))=\partial_{H(s)}(\varphi_2(\bar{x})\varphi_1)(\bar{x})$$

and

$$D^*_{H(s)}f_2(\bar{x})(\varphi_2(\bar{x}),\varphi_1(\bar{x}))=D^*_{H(s)}(-\varphi_2)(\bar{x})(\varphi_1(\bar{x}))=\partial_{H(s)}(-\varphi_1(\bar{x})\varphi_2)(\bar{x})$$

Thus from 3.1 and 3.2, we achieve the proof.
Similarly to Theorem 3.4 we obtain the following quotient rules.

Theorem 3.5. *Let $\varphi_i : X \longrightarrow \mathbb{R}$, $i=1,2$ be Lipschitz continuous functions around $\bar{x}$. Assume that $\varphi_2(\bar{x})\neq 0$ and $\partial_{H(s)}(\varphi_1(\bar{x})\varphi_2)(\bar{x})\neq\emptyset$. Then we have*

$$\partial_{H(s)}(\frac{\varphi_1}{\varphi_2})(\bar{x})\subset\bigcap_{x^*\in\partial_{H(s)}(\varphi_1(\bar{x})\varphi_2)(\bar{x})}\frac{[\partial_{H(s)}(\varphi_2(\bar{x})\varphi_1)(\bar{x})-x^*]}{[\varphi(\bar{x})]^2}.$$

Proof. Similary to the proof of Theorem 3.4, consider the functions $f : X \longrightarrow \mathbb{R}^2$ and $\psi:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by:

$$f(x)=(\varphi_1(x),\varphi_2(x))\,,\ \psi(y_1,y_2)=\frac{y_1}{y_2}.$$

Applying the chain rule Theorem to the composition $(\psi o f)(x)=\left(\frac{\varphi_1}{\varphi_2}\right)(x)$ we have

$$\nabla\psi(f(\bar{x}))=(\frac{1}{\varphi_2(\bar{x})},-\frac{\varphi_1(\bar{x})}{(\varphi_2(\bar{x}))^2})$$

and using the representation: $f(x)=f_1(x)-f_2(x)$ with $f_1(x)=(\varphi_1(x),0)$ and $f_2(x)=(0,-\varphi_2(x))$. Then we have

$$D^*_{H(s)}f_2(\bar{x})(\frac{1}{\varphi_2(\bar{x})},-\frac{\varphi_1(\bar{x})}{(\varphi_2(\bar{x}))^2})=\frac{1}{(\varphi_2(\bar{x}))^2}\partial_{H(s)}(\varphi_1(\bar{x})\varphi_2)(\bar{x})\neq\emptyset$$

and

$$D^*_{H(s)}f_1(\bar{x})(\frac{1}{\varphi_2(\bar{x})},-\frac{\varphi_1(\bar{x})}{(\varphi_2(\bar{x}))^2})=\partial_{H(s)}(\frac{1}{\varphi_2(\bar{x})}\varphi_1)(\bar{x})$$

Employing Theorem 3.4, we have

$$\partial_{H(s)}(\frac{\varphi_1}{\varphi_2})(\bar{x})\subset\bigcap_{x^*\in\frac{1}{(\varphi_2(\bar{x}))^2}\partial_{H(s)}(\varphi_1(\bar{x})\varphi_2)(\bar{x})}[\partial_{H(s)}(\frac{1}{\varphi_2(\bar{x})}\varphi_1)(\bar{x})-x^*]$$

$$=\bigcap_{x^*\in\partial_{H(s)}(\varphi_1(\bar{x})\varphi_2)(\bar{x})}\frac{[\partial_{H(s)}(\varphi_2(\bar{x})\varphi_1)(\bar{x})-x^*]}{[\varphi(\bar{x})]^2}.$$

which completes the proof.

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