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Fixed Point Theorems in Bipolar Vector Metric Spaces

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ABSTRACT: This article introduces the definition of bipolar vector metric space and derives some of its properties. Additionally, for bipolar vector metric spaces, some fixed point results of covariant and contravariant maps satisfying Banach contraction and Kannan contraction conditions are demonstrated. Furthermore, a few examples are provided to demonstrate our main results.

Key Words: Riesz space, lattice, fixed point.

Contents

1. Introduction

On various generalized metric spaces, several authors have studied fixed point results. In [4], Cevic, v., and Altun, I., introduced the notion of vector metric spaces, some properties of vector metric spaces were derived, and fixed point results of contraction mappings were proved. There are many articles on fixed point theory in vector metric spaces; see [2,3,5,6].

Definition 1.1 *A partially ordered set* $(E, \leq)$ *is a lattice if each pair of elements has a supremum and an infimum.*

Definition 1.2 *A partially ordered vector space is a partially ordered set* $(E, \leq)$*, where* E *is a real vector space, such that*

(i) $\gamma \leq \delta$ *implies* $\gamma + \eta \leq \delta + \eta$, $\forall\ \gamma, \delta, \eta \in E$

(ii) $\gamma \leq \delta$ *implies* $a\gamma \leq a\delta$, $\forall\ \gamma, \delta \in E$ *and* $a > 0$,

Definition 1.3 *A partially ordered vector space* E *which is also a lattice is called a Riesz space. The cone in a Riesz space* E *is denoted by* E_+*, where* $E_+ = \{\gamma \in E : \gamma \geq 0\}$.

We refer to [1] for notations and additional information regarding Riesz spaces.

Definition 1.4 *[4] Let* Ψ *be a non empty set and* E *be a Riesz space. A vector metric is a mapping* $d : \Psi \times \Psi \to E_+$ *satisfying the following axioms.*

(i) $d(\varpi, \varrho) = 0$ *if and only if* $\varpi = \varrho$ *in* Ψ,

(ii) $d(\varpi, \varrho) = d(\varrho, \varpi)$, $\forall\ \varpi, \varrho \in \Psi$

(iii) $d(\varpi, \varrho) \leq d(\varpi, \rho) + d(\rho, \varrho)$, $\forall\ \varpi, \varrho, \rho \in \Psi$.

The triple (Ψ, d, E) *is called a vector metric space.*

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The notion of bipolar metric space has introduced by Mutlu, A., and Gurdal, U., [12], giving a new definition of distance measurement between the members of two separate sets. Bipolar metric space is a metric space generalization. Many articles are appearing for fixed point theory in bipolar metric spaces, see for example [8,9,11,13,14,15,16] and the references therein.

Definition 1.5 *[12] Let Ψ and Φ be two non empty sets. A bipolar metric is a mapping $D : \Psi \times \Phi \to [0, \infty)$ satisfying the following axioms.*

(I) $D(\varpi, \varrho) = 0 \Rightarrow \varpi = \varrho$, *whenever* $(\varpi, \varrho) \in \Psi \times \Phi$,

(II) $\varpi = \varrho \Rightarrow D(\varpi, \varrho) = 0$, *whenever* $(\varpi, \varrho) \in \Psi \times \Phi$,

(III) $D(\varpi, \varrho) = D(\varrho, \varpi)$, $\forall\ \varpi, \varrho \in \Psi \cap \Phi$,

(IV) $D(\varpi_1, \varrho_2) \leq D(\varpi_1, \varrho_1) + D(\varpi_2, \varrho_1) + D(\varpi_2, \varrho_2)$, $\forall\ \varpi_1, \varpi_2 \in \Psi$, *and* $\varrho_1, \varrho_2 \in \Phi$.

The triple (Ψ, Φ, D) is called a bipolar metric space.

In this paper, by extending the domain of vector metric to the Cartesian product of two non-empty sets, we present a new definition of bipolar vector metric space that generalizes the notion of vector metric space. We derive some properties of bipolar vector metric spaces. Also, we prove some fixed point results of covariant and contravariant maps satisfying Banach contraction and Kannan contraction conditions in a bipolar vector metric space. Moreover, we generalize the Banach contraction principle (see [10]), and the Kannan fixed point result (see [7]).

2. Bipolar Vector Metric Spaces

Definition 2.1 Let E be a Riesz space. If $(\gamma_n)_{n=1}^{\infty}$ is a decreasing sequence in E such that $inf \gamma_n = \gamma$, we write $\gamma_n \downarrow \gamma$. E is said to be Archimedean if $\frac{1}{n}\gamma \downarrow 0$ for every $\gamma \in E_+$.

Definition 2.2 *A sequence $(\varpi_n)_{n=1}^{\infty}$ in a Riesz space E is said to order convergent to ϖ, written as $\varpi_n \xrightarrow{o} \varpi$ if there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E satisfying $\gamma_n \downarrow 0$ and $|\varpi_n - \varpi| \leq \gamma_n$, $\forall n$, where $|\varpi| = \varpi \vee -\varpi$.*

Definition 2.3 *A sequence $(\varpi_n)_{n=1}^{\infty}$ in a Riesz space E is said to order Cauchy if there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E satisfying $\gamma_n \downarrow 0$ and $|\varpi_n - \varpi_{n+p}| \leq \gamma_n$, $\forall n$ and p.*

Definition 2.4 *Riesz space E is said to be order complete if every order Cauchy sequence is order convergent.*

Lemma 2.1 *[2] If E is a Riesz space and $\varpi \leq k\varpi$ where $\varpi \in E_+$, $k \in [0, 1)$, then $\varpi = 0$.*

Definition 2.5 *Let Ψ and Φ be two non empty sets. A bipolar vector metric is a mapping $d : \Psi \times \Phi \to E_+$ satisfying the following conditions.*

(I) $d(\varpi, \varrho) = 0 \Rightarrow \varpi = \varrho$, *whenever* $(\varpi, \varrho) \in \Psi \times \Phi$,

(II) $\varpi = \varrho \Rightarrow d(\varpi, \varrho) = 0$, *whenever* $(\varpi, \varrho) \in \Psi \times \Phi$,

(III) $d(\varpi, \varrho) = d(\varrho, \varpi)$, $\forall\ \varpi, \varrho \in \Psi \cap \Phi$,

(IV) $d(\varpi_1, \varrho_2) \leq d(\varpi_1, \varrho_1) + d(\varpi_2, \varrho_1) + d(\varpi_2, \varrho_2)$, $\forall\ \varpi_1, \varpi_2 \in \Psi$, *and* $\varrho_1, \varrho_2 \in \Phi$.

The quadruple (Ψ, Φ, d, E) is called a bipolar vector metric space(or, BVMS).

Remark 2.1 *Let (Ψ, Φ, d, E) be a BVMS. If $\Psi \cap \Phi = \emptyset$, then (Ψ, Φ, d, E) is called disjoint. The space (Ψ, Φ, d, E) is said to be a joint if $\Psi \cap \Phi \neq \emptyset$. The sets Φ and Ψ are called right pole and left pole of (Ψ, Φ, d, E), respectively.*

Example 2.1 *Let $\Psi = \{1,2,3,4\}$, $\Phi = \{1,2,4\}$ and $E = \mathbb{R}^2$ be a Riesz space with coordinate wise ordering $\leq$ defined by $(\varpi_1, \varpi_2) \leq (\varrho_1, \varrho_2)$ if and only if $\varpi_1 \leq \varrho_1$ and $\varpi_2 \leq \varrho_2$. Define $d : \Psi \times \Phi \to \mathbb{R}^2$ as $d(\varpi, \varrho) = (|\varpi - \varrho|, |\frac{1}{\varpi} - \frac{1}{\varrho}|)$ if $(\varpi, \varrho) \notin \{(2,4),(4,2)\}$, and $d(\varpi, \varrho) = (4,4)$ if $(\varpi, \varrho) \in \{(2,4),(4,2)\}$. Then (Ψ, Φ, d, E) is a joint BVMS.*

Example 2.2 *Let $\Psi = (1, \infty)$, $\Phi = (0, 1]$ and $E = \mathbb{R}^2$ be a Riesz space with coordinate wise ordering. Define $d : \Psi \times \Phi \to \mathbb{R}^2$ as $d(\varpi, \varrho) = (|\varpi^2 - \varrho^2|, a|\varpi^2 - \varrho^2|)$, whenever $(\varpi, \varrho) \in \Psi \times \Phi$, $a > 0$. Then (Ψ, Φ, d, E) is a disjoint BVMS.*

Remark 2.2 *Let (Ψ, d, E) be a vector metric space, then (Ψ, Ψ, d, E) is a BVMS. Conversely, if (Ψ, Φ, d, E) is a BVMS such that $\Psi = \Phi$, then (Ψ, d, E) is a vector metric space.*

Definition 2.6 *Let (Ψ, Φ, d, E) be a BVMS. Where points of the sets Φ, Ψ, and $\Psi \cap \Phi$ are called right, left, and central points respectively. A sequence that contains only right(or left, or central) points is called a right (or left, or central) sequence in (Ψ, Φ, d, E).*

Definition 2.7 *Let (Ψ, Φ, d, E) be a BVMS. A left sequence $(\varpi_n)_{n=1}^{\infty}$ vectorial converges to a right point ϱ(or $(\varpi_n)_{n=1}^{\infty} \to \varrho$) if and only if there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$ and $d(\varpi_n, \varrho) < \gamma_n$, $\forall\ n$. Also a right sequence $(\varrho_n)_{n=1}^{\infty}$ vectorial converges to a left point ϖ (or $(\varrho_n)_{n=1}^{\infty} \to \varpi$) if and only if there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$ and $d(\varpi, \varrho_n) < \gamma_n$, $\forall\ n$. When it is given $(\kappa_n)_{n=1}^{\infty} \to \iota$ for a BVMS (Ψ, Φ, d, E) without precise data about the sequence, this means that either $(\kappa_n)_{n=1}^{\infty}$ is a right sequence and ι is a left point, or $(\kappa_n)_{n=1}^{\infty}$ is a left sequence and ι is a right point.*

Lemma 2.2 *Let (Ψ, Φ, d, E) be a BVMS. Then a left sequence $(\varpi_n)_{n=1}^{\infty}$ vectorial converges to a right point ϱ if and only if $d(\varpi_n, \varrho) \to 0$ in E, and also a right sequence $(\varrho_n)_{n=1}^{\infty}$ vectorial converges to a left point ϖ if and only if $d(\varpi, \varrho_n) \to 0$ in E.*

Proof: It is easy to prove. □

Lemma 2.3 *Let (Ψ, Φ, d, E) be a BVMS. If a central point is a vectorial limit of a sequence, then it is the unique vectorial limit of the sequence.*

Proof: Let $(\varpi_n)_{n=1}^{\infty}$ be a left sequence, $(\varpi_n)_{n=1}^{\infty} \to \varpi \in \Psi \cap \Phi$, and $(\varpi_n)_{n=1}^{\infty} \to \varrho \in \Phi$. Since there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, and

$$d(\varpi, \varrho) \leq d(\varpi, \varpi) + d(\varpi_n, \varpi) + d(\varpi_n, \varrho) < 0 + \gamma_n + \gamma_n.$$

we have $d(\varpi, \varrho) = 0$ which implies $\varpi = \varrho$. □

Lemma 2.4 *Let (Ψ, Φ, d, E) be a BVMS. If a left sequence $(\varpi_n)_{n=1}^{\infty}$ vectorial converges to ϱ and a right sequence $(\varrho_n)_{n=1}^{\infty}$ vectorial converges to ϖ, then $d(\varpi_n, \varrho_n) \xrightarrow{o} d(\varpi, \varrho)$ as $n \to \infty$.*

Proof: Let $(\varpi_n)_{n=1}^{\infty} \to \varrho \in \Phi$, and $(\varrho_n)_{n=1}^{\infty} \to \varpi \in \Psi$. Since there exists two sequences $(\gamma_n)_{n=1}^{\infty}$, $(\delta_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, $\delta_n \downarrow 0$ we have $d(\varpi_n, \varrho) < \gamma_n$, and $d(\varpi, \varrho_n) < \delta_n$, then

$$d(\varpi, \varrho) \leq d(\varpi, \varrho_n) + d(\varpi_n, \varrho_n) + d(\varpi_n, \varrho)$$

implies

$$d(\varpi, \varrho) - d(\varpi_n, \varrho_n) \leq d(\varpi, \varrho_n) + d(\varpi_n, \varrho),$$

and also

$$|d(\varpi_n, \varrho_n) - d(\varpi, \varrho)| \leq d(\varpi, \varrho_n) + d(\varpi_n, \varrho) < \gamma_n + \delta_n, \forall n,$$

and hence $d(\varpi_n, \varrho_n) \xrightarrow{o} d(\varpi, \varrho)$ as $n \to \infty$ in E. □

Definition 2.8 Let (Ψ_1, Φ_1, E) and (Ψ_2, Φ_2, E) be two bipolar vector metric spaces(or,BVMSs), and $f : \Psi_1 \cup \Phi_1 \to \Psi_2 \cup \Phi_2$.

(i) If $f(\Psi_1) \subseteq \Psi_2$ and $f(\Phi_1) \subseteq \Phi_2$, then f is called a covariant map from (Ψ_1, Φ_1, E) to (Ψ_2, Φ_2, E), and we write $f : (\Psi_1, \Phi_1, E) \rightrightarrows (\Psi_2, \Phi_2, E)$.

(ii) If $f(\Psi_1) \subseteq \Phi_2$ and $f(\Phi_1) \subseteq \Psi_2$, then f is called a contravariant map from (Ψ_1, Φ_1, E) to (Ψ_2, Φ_2, E), and we write $f : (\Psi_1, \Phi_1, E) \rightleftarrows (\Psi_2, \Phi_2, E)$.

Remark 2.3 Suppose d_1, and d_2 be two bipolar vector metrics on (Ψ_1, Φ_1, E) and (Ψ_2, Φ_2, E) respectively. We can also use the symbols $f : (\Psi_1, \Phi_1, d_1, E) \rightrightarrows (\Psi_2, \Phi_2, d_2, E)$ and $f : (\Psi_1, \Phi_1, d_1, E) \rightleftarrows (\Psi_2, \Phi_2, d_2, E)$ in the place of $f : (\Psi_1, \Phi_1, E) \rightrightarrows (\Psi_2, \Phi_2, E)$ and $f : (\Psi_1, \Phi_1, E) \rightleftarrows (\Psi_2, \Phi_2, E)$.

Definition 2.9 Let (Ψ, Φ, d, E) be a BVMS.

(i) A sequence (ϖ_n, ϱ_n) on the set $\Psi \times \Phi$ is called a bisequence on (Ψ, Φ, d, E).

(ii) If both $(\varpi_n)_{n=1}^{\infty}$ and $(\varrho_n)_{n=1}^{\infty}$ vectorial converges, then the bisequence (ϖ_n, ϱ_n) is called vectorial convergent. If both $(\varpi_n)_{n=1}^{\infty}$ and $(\varrho_n)_{n=1}^{\infty}$ vectorial converges to a same point $\varpi \in \Psi \cap \Phi$, then the bisequence is called vectorial biconvergent.

(iii) A bisequence (ϖ_n, ϱ_n) on (Ψ, Φ, d, E) is called a vectorial Cauchy bisequence if there exists a sequence $(\gamma_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, and $d(\varpi_n, \varrho_{n+p}) < \gamma_n$, $\forall$ n and p.

Lemma 2.5 *Let (Ψ, Φ, d, E) be a BVMS. Then (ϖ_n, ϱ_n) is a vectorial Cauchy bisequence if and only if $d(\varpi_n, \varrho_{n+p}) \to 0$ as $n \to \infty$.*

Proof: It is easy to prove. □

Proposition 2.1 *Let (Ψ, Φ, d, E) be a BVMS. Then every vectorial biconvergent bisequence is a vectorial Cauchy bisequence.*

Proof: Let (ϖ_n, ϱ_n) be a bisequence, which is vectorial biconvergent to a point $\varpi \in \Psi \cap \Phi$. Since there exists two sequences $(\gamma_n)_{n=1}^{\infty}$, $(\delta_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, $\delta_n \downarrow 0$ and $d(\varpi_n, \varpi) < \gamma_n$, $d(\varpi, \varrho_{n+p}) < \delta_n$, then we have

$$d(\varpi_n, \varrho_{n+p}) \leq d(\varpi_n, \varpi) + d(\varpi, \varpi) + d(\varpi, \varrho_{n+p}) < \gamma_n + 0 + \delta_n, \forall n, p.$$

So (ϖ_n, ϱ_n) is a vectorial Cauchy bisequence. □

Proposition 2.2 *Let (Ψ, Φ, d, E) be a BVMS. Then every vectorial convergent vectorial Cauchy bisequence is vectorial biconvergent.*

Proof: Let (ϖ_n, ϱ_n) be a vectorial Cauchy bisequence such that $(\varpi_n)_{n=1}^{\infty}$ vectorial convergent to ϱ in Φ and $(\varrho_n)_{n=1}^{\infty}$ vectorial convergent to ϖ in Ψ. Then there exists three sequences $(\gamma_n)_{n=1}^{\infty}$, $(\delta_n)_{n=1}^{\infty}$, $(\eta_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, $\delta_n \downarrow 0$, $\eta_n \downarrow 0$ and $d(\varpi_n, \varrho) < \gamma_n$, $d(\varpi, \varrho_{n+p}) < \delta_n$, and $d(\varpi_n, \varrho_{n+p}) < \eta_n$, for all n, p. Then

$$d(\varpi, \varrho) \leq d(\varpi, \varrho_{n+p}) + d(\varpi_n, \varrho_{n+p}) + d(\varpi_n, \varrho) < \gamma_n + \delta_n + \eta_n, \forall n, p.$$

Therefore $d(\varpi, \varrho) = 0$ and so that $\varpi = \varrho$. Then (ϖ_n, ϱ_n) is vectorial biconvergent. □

Definition 2.10 A BVMS (Ψ, Φ, d, E) is called vectorial complete if every vectorial Cauchy bisequence is vectorial convergent, or equivalently, vectorial biconvergent.

3. Main Results

Theorem 3.1 *Let (Ψ, Φ, d, E) be a vectorial complete BVMS with E is an Archimedean. If a covariant map $f : (\Psi, \Phi, d, E) \rightrightarrows (\Psi, \Phi, d, E)$ satisfies $d(f(\varpi), f(\varrho)) \leq \lambda d(\varpi, \varrho)$, whenever $(\varpi, \varrho) \in S \times T$ and $\lambda \in (0, 1)$, then the function $f : \Psi \cup \Phi \to \Psi \cup \Phi$ has a unique fixed point(UFP).*

Proof: Let $\varpi_0 \in \Psi$, $\varrho_0 \in \Phi$ and $\varpi_{n+1} = f(\varpi_n)$ and $\varrho_{n+1} = f(\varrho_n)$, for all $n \in \mathbb{N}$. Then (ϖ_n, ϱ_n) is a bisequence on (Ψ, Φ, d, E). By using the contraction condition:

$$\begin{aligned} d(\varpi_n, \varrho_n) &= d(f(\varpi_{n-1}), f(\varrho_{n-1})) \\ &\leq \lambda d(\varpi_{n-1}, \varrho_{n-1}) \leq ... \leq \lambda^n d(\varpi_0, \varrho_0) \end{aligned}$$

$$\begin{aligned} d(\varpi_n, \varrho_{n+1}) &= d(f(\varpi_{n-1}), f(\varrho_n)) \\ &\leq \lambda d(\varpi_{n-1}, \varrho_n) \leq ... \leq \lambda^n d(\varpi_0, \varrho_1). \end{aligned}$$

For every $n, q \in \mathbb{N}$ and hence,

$$\begin{aligned} d(\varpi_{n+q}, \varrho_n) &\leq d(\varpi_{n+q}, \varrho_{n+1}) + d(\varpi_n, \varrho_{n+1}) + d(\varpi_n, \varrho_n) \\ &\leq d(\varpi_{n+q}, \varrho_{n+1}) + \lambda^n d(\varpi_0, \varrho_1) + \lambda^n d(\varpi_0, \varrho_0) \\ &= d(\varpi_{n+q}, \varrho_{n+1}) + \lambda^n M, \ (M = d(\varpi_0, \varrho_1) + d(\varpi_0, \varrho_0)) \\ &\leq d(\varpi_{n+q}, \varrho_{n+2}) + d(\varpi_{n+1}, \varrho_{n+2}) + d(\varpi_{n+1}, \varrho_{n+1}) + \lambda^n M \\ &\leq d(\varpi_{n+q}, \varrho_{n+2}) + (\lambda^{n+1} + \lambda^n)M \\ &\leq ... \\ &\leq d(\varpi_{n+q}, \varrho_{n+q}) + (\lambda^{n+q-1} + ... + \lambda^{n+1} + \lambda^n)M \\ &\leq (\lambda^{n+q} + ... + \lambda^{n+1} + \lambda^n)M \\ &\leq \lambda^n M \sum_{z=0}^{\infty} \lambda^z \\ &= \frac{\lambda^n}{1-\lambda} M = K_n, \end{aligned}$$

where $K_n = \frac{\lambda^n}{1-\lambda} M$. Similarly $d(\varpi_n, \varrho_{n+q}) \leq K_n$, for all $n, q \in \mathbb{N}$.
Since E is an Archimedean then $K_n \downarrow 0$ so that (ϖ_n, ϱ_n) is a vectorial Cauchy bisequence. Since (Ψ, Φ, d, E) is vectorial complete, then (ϖ_n, ϱ_n) vectorial converges, and vectorial biconverges to a point $k \in \Psi \cap \Phi$. Also, $f(\varrho_n) = \varrho_{n+1} \to \kappa \in \Psi \cap \Phi$ as $n \to \infty$. Then there exists two sequences $(\gamma_n)_{n=1}^{\infty}$, $(\delta_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$, $\delta_n \downarrow 0$ and $d(\varpi_n, \kappa) < \gamma_n$, $d(\kappa, \varrho_{n+1}) < \delta_n$. For all $n \in \mathbb{N}$,

$$\begin{aligned} d(\kappa, f(\kappa)) &\leq d(\kappa, f(\varrho_n)) + d(f(\varpi_n), f(\varrho_n)) + d(f(\varpi_n), f(\kappa)) \\ &\leq d(\kappa, \varrho_{n+1}) + \lambda d(\varpi_n, \varrho_n) + \lambda d(\varpi_n, \kappa) \\ &\leq d(\kappa, \varrho_{n+1}) + \lambda^{n+1} d(\varpi_0, \varrho_0) + \lambda d(\varpi_n, \kappa) \\ &\leq \delta_n + \lambda^{n+1} d(\varpi_0, \varrho_0) + \lambda \gamma_n \end{aligned}$$

Hence, $d(\kappa, f(\kappa)) = 0$ so that $f(\kappa) = \kappa$. Therefore, κ is a fixed point of f.
If ι is another fixed point of f, then $f(\iota) = \iota$ implies $\iota \in \Psi \cap \Phi$, and

$$d(\kappa, \iota) = d(f(\kappa), f(\iota)) \leq \lambda d(\kappa, \iota) \leq ... \leq \lambda^n d(\kappa, \iota), \text{ for every } n = 1, 2, 3, ...$$

Since Lemma 2.1 we have $d(\kappa, \iota) = 0$ so that $\kappa = \iota$, and hence f has a UFP and this completes the proof. □

The Theorem 3.1 is a generalization of the Banach contraction principle (see [10]).

Example 3.1 *Let $\Psi = \{0, \frac{1}{2}, 2\}$, $\Phi = \{0, \frac{1}{2}\}$, and $E = \mathbb{R}^2$ be a Archimedean Riesz space with coordinate wise ordering. Let $d(\varpi, \varrho) = (|\varpi - \varrho|, a|\varpi - \varrho|)$, where $(\varpi, \varrho) \in \Psi \times \Phi$, $a > 0$. Then (Ψ, Φ, d, E) is a vectorial complete BVMS. Define a covariant map $f : (\Psi, \Phi, d, E) \rightrightarrows (\Psi, \Phi, d, E)$ by $f(0) = 0$, $f(\frac{1}{2}) = 0$, and $f(2) = \frac{1}{2}$. Then, f satisfies the inequality $d(f(\varpi), f(\varrho)) \leq \lambda d(\varpi, \varrho)$ for $\lambda = \frac{1}{3}$. By Theorem 3.1, f has a UFP zero in $\Psi \cap \Phi$.*

Theorem 3.2 *Let (Ψ, Φ, d, E) be a vectorial complete BVMS with E is an Archimedean. If a contravariant map $f : (\Psi, \Phi, d, E) \rightleftarrows (\Psi, \Phi, d, E)$ satisfies $d(f(\varrho), f(\varpi)) \leq \lambda d(\varpi, \varrho)$, whenever $(\varpi, \varrho) \in \Psi \times \Phi$, where $\lambda \in (0, 1)$, then the function $f : \Psi \cup \Phi \to \Psi \cup \Phi$ has a UFP.*

Proof: Let $\varpi_0 \in \Psi$, $\varrho_0 = f(\varpi_0) \in \Phi$, and $\varpi_1 = f(\varrho_0)$, and let $\varrho_n = f(\varpi_n)$ and $\varpi_{n+1} = f(\varrho_n)$, for all $n \in \mathbb{N}$. Then (ϖ_n, ϱ_n) is a bisequence on (Ψ, Φ, d, E). Hence,

$$\begin{aligned}
d(\varpi_n, \varrho_n) &= d(f(\varrho_{n-1}), f(\varpi_n)) \\
&\leq \lambda d(\varpi_n, \varrho_{n-1}) \\
&= \lambda d(f(\varrho_{n-1}), f(\varpi_{n-1})) \\
&\leq \lambda^2 d(\varpi_{n-1}, \varrho_{n-1}) \leq ... \leq \lambda^{2n} d(\varpi_0, \varrho_0) = K_n(1-\lambda) \leq K_n, \ (K_n = d(\varpi_0, \varrho_0)(\frac{\lambda^{2n}}{1-\lambda})).
\end{aligned}$$

So

$$\begin{aligned}
d(\varpi_{n+1}, \varrho_n) &= d(f(\varrho_n), f(\varpi_n)) \\
&\leq \lambda d(\varpi_n, \varrho_n) \leq \lambda^{2n+1} d(\varpi_0, \varrho_0).
\end{aligned}$$

For all $n, q \in \mathbb{N}$, we have

$$\begin{aligned}
d(\varpi_{n+q}, \varrho_n) &\leq d(\varpi_{n+q}, \varrho_{n+1}) + d(\varpi_{n+1}, \varrho_{n+1}) + d(\varpi_{n+1}, \varrho_n) \\
&\leq d(\varpi_{n+q}, \varrho_{n+1}) + (\lambda^{2n+2} + \lambda^{2n+1}) d(\varpi_0, \varrho_0) \\
&\leq d(\varpi_{n+q}, \varrho_{n+2}) + d(\varpi_{n+2}, \varrho_{n+2}) + d(\varpi_{n+2}, \varrho_{n+1}) + (\lambda^{2n+2} + \lambda^{2n+1}) d(\varpi_0, \varrho_0) \\
&\leq d(\varpi_{n+q}, \varrho_{n+2}) + (\lambda^{2n+4} + \lambda^{2n+3} + \lambda^{2n+2} + \lambda^{2n+1}) d(\varpi_0, \varrho_0) \\
&\leq ... \\
&\leq d(\varpi_{n+q}, \varrho_{n+q-1}) + (\lambda^{2n+2q-2} + ... + \lambda^{2n+1}) d(\varpi_0, \varrho_0) \\
&\leq (\lambda^{2n+2q-1} + \lambda^{2n+2q-2} + ... + \lambda^{2n+1}) d(\varpi_0, \varrho_0) \\
&\leq \lambda^{2n+1} \sum_{z=0}^{\infty} \lambda^z d(\varpi_0, \varrho_0) \\
&= d(\varpi_0, \varrho_0) \frac{\lambda^{2n+1}}{1-\lambda} \\
&= {K_n}^{\lambda} < K_n, \ (K_n = d(\varpi_0, \varrho_0) \frac{\lambda^{2n}}{1-\lambda}).
\end{aligned}$$

Similarly

$$d(\varpi_n, \varrho_{n+q}) \leq K_n.$$

Since E is an Archimedean then $K_n \downarrow 0$ so that (ϖ_n, ϱ_n) is a vectorial Cauchy bisequence. Since (Ψ, Φ, d, E) is vectorial complete, then (ϖ_n, ϱ_n) vectorial converges, and vectorial biconverges to a point $\kappa \in \Psi \cap \Phi$. Also, $\varrho_n \to \kappa \in \Psi \cap \Phi$ as $n \to \infty$. Then there exists a sequences $(\gamma_n)_{n=1}^{\infty}$ in E such that $\gamma_n \downarrow 0$ and $d(\kappa, \varrho_n) < \gamma_n$. For all $n \in \mathbb{N}$,

$$\begin{aligned}
d(\kappa, f(\kappa)) &\leq d(\kappa, f(\varpi_n)) + d(f(\varrho_n), f(\varpi_n)) + d(f(\varrho_n), f(\kappa)) \\
&\leq d(\kappa, \varrho_n) + \lambda d(\varpi_n, \varrho_n) + \lambda d(\kappa, \varrho_n) \\
&\leq d(\kappa, \varrho_n) + \lambda^{2n+1} d(\varpi_0, \varrho_0) + \lambda d(\kappa, \varrho_n), \\
&\leq \gamma_n + \lambda^{2n+1} d(\varpi_0, \varrho_0) + \lambda \gamma_n.
\end{aligned}$$

Therefore $d(\kappa, f(\kappa)) = 0$ so that $f(\kappa) = \kappa$. Hence κ is a fixed point.
If ι is another fixed point of f, then $f(\iota) = \iota$, $\iota \in \Psi \cap \Phi$, and

$$d(\kappa, \iota) = d(f(\kappa), f(\iota)) \leq \lambda d(\kappa, \iota) \leq ... \leq \lambda^n d(\kappa, \iota), \text{ for every } n = 1, 2, 3, ...$$

Since Lemma 2.1 we have $d(\kappa, \iota) = 0$ so that $\kappa = \iota$. So f has a UFP and this completes the proof. □

Theorem 3.3 *Let (Ψ, Φ, d, E) be a vectorial complete BVMS with E is an Archimedean. If a contravariant map $f : (\Psi, \Phi, d, E) \rightleftarrows (\Psi, \Phi, d, E)$ satisfies $d(f(\varrho), f(\varpi)) \leq \lambda[d(\varpi, f(\varpi)) + d(f(\varrho), \varrho)]$, whenever $(\varpi, \varrho) \in \Psi \times \Phi$, for some $\lambda \in (0, \frac{1}{2})$, then the function $f : \Psi \cup \Phi \to \Psi \cup \Phi$ has a UFP.*

Proof: Let $\varpi_0 \in \Psi$, $\varrho_0 = f(\varpi_0) \in \Phi$, and $\varpi_1 = f(\varrho_0)$. Suppose, $\varrho_n = f(\varpi_n)$ and $\varpi_{n+1} = f(\varrho_n)$, for all $n \in \mathbb{N}$. Then (ϖ_n, ϱ_n) is a bisequence on (Ψ, Φ, d, E). For all $n \in \mathbb{N}$, from

$$\begin{aligned} d(\varpi_n, \varrho_n) &= d(f(\varrho_{n-1}), f(\varpi_n)) \\ &\leq \lambda[d(\varpi_n, f(\varpi_n)) + d(f(\varrho_{n-1}), \varrho_{n-1})] \\ &= \lambda[d(\varpi_n, \varrho_n) + d(\varpi_n, \varrho_{n-1})] \end{aligned}$$

we conclude that

$$d(\varpi_n, \varrho_n) \leq \frac{\lambda}{1-\lambda}[d(\varpi_n, \varrho_{n-1})],$$

and

$$\begin{aligned} d(\varpi_n, \varrho_{n-1}) &= d(f(\varrho_{n-1}), f(\varpi_{n-1})) \\ &\leq \lambda[d(\varpi_{n-1}, f(\varpi_{n-1})) + d(f(\varrho_{n-1}), \varrho_{n-1})] \\ &\leq \lambda[d(\varpi_{n-1}, \varrho_{n-1}) + d(\varpi_n, \varrho_{n-1})] \end{aligned}$$

so that

$$d(\varpi_n, \varrho_{n-1}) \leq \frac{\lambda}{1-\lambda}[d(\varpi_{n-1}, \varrho_{n-1})].$$

Therefore, by putting $\gamma = \frac{\lambda}{1-\lambda}$, we have

$$d(\varpi_n, \varrho_n) \leq \gamma^{2n}(d(\varpi_0, \varrho_0))$$

and

$$d(\varpi_n, \varrho_{n-1}) \leq \gamma^{2n-1}(d(\varpi_0, \varrho_0)).$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\varpi_n, \varrho_m) &\leq d(\varpi_n, \varrho_n) + d(\varpi_{n+1}, \varrho_n) + d(\varpi_{n+1}, \varrho_m) \\ &\leq (\gamma^{2n} + \gamma^{2n+1})d(\varpi_0, \varrho_0) + d(\varpi_{n+1}, \varrho_m) \\ &\leq ... \\ &\leq (\gamma^{2n} + \gamma^{2n+1} + ... + \gamma^{2m-1})d(\varpi_0, \varrho_0) + d(\varpi_m, \varrho_m) \\ &\leq (\gamma^{2n} + \gamma^{2n+1} + ... + \gamma^{2m})d(\varpi_0, \varrho_0), \text{ if } m > n, \end{aligned}$$

and similarly, if $m < n$, then

$$d(\varpi_n, \varrho_m) \leq (\gamma^{2m+1} + \gamma^{2m+2} + ... + \gamma^{2n+1})d(\varpi_0, \varrho_0).$$

Since E is an Archimedean then (ϖ_n, ϱ_n) is a vectorial Cauchy bisequence. Since (Ψ, Φ, d, E) is vectorial complete, then (ϖ_n, ϱ_n) vectorial converges, and vectorial biconverges to a point $\kappa \in \Psi \cap \Phi$. Hence,

$f(\varpi_n) = \varrho_n \to \kappa \in \Psi \cap \Phi$ as $n \to \infty$ implies $d(f(\kappa), f(\varpi_n)) \xrightarrow{o} d(f(\kappa), \kappa)$ as $n \to \infty$, by using Lemma 2.4. Also by taking the limit from

$$\begin{aligned} d(f(\kappa), f(\varpi_n)) &\leq \lambda[d(\varpi_n, f(\varpi_n)) + d(f(\kappa), \kappa)] \\ &= \lambda[d(\varpi_n, \varrho_n) + d(f(\kappa), \kappa)], \end{aligned}$$

as $n \to \infty$, we get $d(f(\kappa), \kappa) \leq \lambda(d(f(\kappa), \kappa))$. Since $0 < \lambda < \frac{1}{2}$, and Lemma 2.1 we have $d(f(\kappa), \kappa) = 0$, hence $f(\kappa) = \kappa$. Therefore κ is a fixed point of f.
If ι is another fixed point of f, then $f(\iota) = \iota$, $\iota \in \Psi \cap \Phi$, and hence,

$$d(\kappa, \iota) = d(f(\kappa), f(\iota)) \leq \lambda(d(\kappa, f(\kappa)) + d(f(\iota), \iota)) = \lambda(d(\kappa, \kappa) + d(\iota, \iota))$$

Therefore $d(\kappa, \iota) = 0$ so that $\kappa = \iota$. So f has a UFP, and this completes the proof. □

The Theorem 3.3 is the generalization of the Kannan fixed point theorem [7].

Example 3.2 *Let Ψ be the collection of all singleton subsets of $\mathbb{R}$, Φ be the collection of all compact subsets of $\mathbb{R}$ and $E = \mathbb{R}^2$ be a Archimedean Riesz space with coordinate wise ordering. Let $d(\varpi, B) = (|\varpi - \inf(B)|, |\varpi - \sup(B)|)$, where $(\varpi, B) \in \Psi \times \Phi$. Then (Ψ, Φ, d, E) is a vectorial complete BVMS. Define a contravariant map $f : (\Psi, \Phi, d, E) \rightleftarrows (\Psi, \Phi, d, E)$ by $f(B) = \frac{\inf(B)+\sup(B)+6}{8}$, for all $B \in \Psi \cup \Phi$. Then, f satisfies the inequality $d(f(\varrho), f(\varpi)) \leq \lambda[d(\varpi, f(\varpi)) + d(f(\varrho), \varrho)]$ for $\lambda = \frac{1}{3}$. By Theorem 3.3, f has a UFP $\{1\} \in \Psi \cap \Phi$.*

4. Conclusion

All fixed point theorems in bipolar vector metric spaces can be regarded as generalizations of fixed point theorems in bipolar metric spaces. Also, all fixed point theorems in bipolar metric spaces can be regarded as generalizations of fixed point theorems in metric spaces. Therefore, studies of fixed point outcomes in bipolar vector metric spaces are significant.

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