Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.66571

S-k-primary ideals of semirings

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ABSTRACT: Let R be a commutative ring and $S \subsetneq R$ be a multiplicatively closed set. Essebti Massaoud [9] defined a proper ideal Q of R disjoint from S to be S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in Q$ then $sa \in Q$ or $sb \in \mathrm{Rad}(Q)$. In this paper, we introduce the notion of S-k-primary ideal of a semiring. We present some analogous results of primary and k-primary ideals of a semiring that S-k-primary ideal enjoy. We also study the properties of $\mathrm{Rad}(Q)$ if Q is an S-k-primary ideal of a semiring. We further study the form of S-k-primary ideals in the amalgamation of semirings R_1 with R_2 along an ideal J of R_2 with respect to a morphism f, introduced and studied by D'Anna et al. [3] and extended by Essebti Massaoud [9] for S-primary ideal of a ring.

Key Words: Semiring; S-primary ideal; S-k-primary ideal; amalgamation of semirings.

Contents

1. Introduction

Semiring is an algebraic structure with many important applications in computer science, automata theory, control theory, quantum mechanics, and a variety of other fields. In different literature, a semiring is defined in different ways. Following the definition given by Hebisch et al. [8], in this article, we define a semiring as a non-empty set R with two binary operations '+' and '$\cdot$' such that (i) $(R, +)$ be a commutative semigroup; (ii) $(R, \cdot)$ be a semigroup and (iii) $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ for all $x, y, z \in R$.

Throughout this paper we consider semiring $(R, +, \cdot)$ with zero element 0 and nonzero identity 1.

Let R be a semiring. A (left) R-semimodule [6] is a commutative monoid $(M, +)$ with additive identity 0 for which we have a function $R \times M \to M$, denoted by $(r, m) \to rm$ and called scalar multiplication, which satisfies the following conditions for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$: (i) $(r_1r_2)m_1 = r_1(r_2m_1)$, (ii) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$, (iii) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$, (iv) $1m_1 = m_1$, (v) $r_10 = 0 = 0m_1$.

Golan [6] defined the prime ideal of a semiring. Recall that an ideal P of a semiring R is said to be a prime ideal if for any two ideals A and B with $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$. Golan [6] also defined the prime ideal in commutative semiring settings. The k-ideal is one of the basic ideals in semiring theory. Let I be an ideal of a semiring R. Then the k-closure of ideal I is denoted by $\overline{I}$ and is defined as $\overline{I} = \{x \in R | x + y = z \text{ for some } y, z \in I\}$. A left ideal I(respectively right ideal, ideal) of R is said to be a left k-ideal (respectively right k-ideal, k-ideal) if for any $a \in R$ and $b \in I$, $a + b \in I$ implies $a \in I$. Purkait, Dutta, and Kar [11] studied the k-prime ideals of a semiring extensively. A semiring R is called an additively idempotent semiring if $a + a = a$ for every $a \in R$. A semiring R is said to be a cancellative semiring if and only if $x + a = y + a$ in R implies $x = y$. We say that N is a subsemimodule of M, or an R-subsemimodule of M if and only if N is closed under addition and scalar multiplication (so $0 \in N$). A subsemimodule N of a R-semimodule M is a k-subsemimodule if $N = \overline{N}$. A semimodule is said to be a k-semimodule if all of its subsemimodules are k-subsemimodules. A non-empty subset S of a semiring R is said to be a multiplicatively closed set if (i) $1 \in S$ and (ii) for $a, b \in S$ implies $ab \in S$. If R_1 and R_2 are two semirings then a function $f : R_1 \longrightarrow R_2$ is a morphism of semirings if and only if

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Submitted January 06, 2023. Published July 11, 2025
2010 *Mathematics Subject Classification:* 13A15, 16Y60.

$(i) f(0) = 0$, $(ii) f(1) = 1$ and $(iii) f(x + y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$ for all $x, y \in R_1$. Hamed and Malek [7] introduced and studied the properties of S-prime ideal of a ring, generalizing the concept of the prime ideal of a ring. More precisely, an ideal P of a ring R disjoint with a multiplicatively closed set $S \subseteq R$ is an S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$ then $sa \in P$ or $sb \in P$. Bhowmick et al. [1] introduced and studied S-prime, S-k-prime, S-semiprime and S-k-semiprime ideals of a semiring. Motivated by Hamed and Malek [7], Essebti Massaoud [9] defined a proper ideal Q of a ring R disjoint from a multiplicatively closed subset S of R to be S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in Q$ then $sa \in Q$ or $sb \in \mathrm{Rad}(Q)$.

In this paper, we introduce the notions of S-primary and S-k-primary ideals of a semiring and investigate some analogous results of primary and k-primary ideals of a semiring that S-k-primary ideals also enjoy. We also study the properties of $\mathrm{Rad}(Q)$ if Q is an S-k-primary ideal of a semiring. We further study the form of S-k-primary ideals in the amalgamation of semirings R_1 with R_2 along an ideal J of R_2 with respect to a morphism f, introduced and studied by D'Anna et al. [3] and extended by Essebti Massaoud [9] for S-primary ideals of a ring.

For any unexplained terminologies, we will follow [5,6,8].

2. S-k-primary ideals of semirings

In this section, we introduce the notions of S-primary and S-k-primary ideals of a semiring and study their basic properties. We begin with the following definitions.

Definition 2.1: Let R be a semiring, S a multiplicatively closed subset of R and Q be a proper ideal of R disjoint with S. The ideal Q is said to be an S-primary ideal if there exists $s \in S$ such that for all ideals A,B of R, if $AB \subseteq Q$ then $sA \subseteq Q$ or $sB \subseteq \mathrm{Rad}(Q)$.

Definition 2.2: An S-primary ideal Q of a semiring R is said to be an S-k-primary ideal of R if $Q = \overline{Q}$.

Proposition 2.3: Let R be a semiring, $S \subseteq R$ a multiplicatively closed set and Q a k-ideal of R disjoint with S. Then Q is an S-k-primary ideal of R if and only if there exists $s \in S$ such that for any two k-ideals A, B of R with $AB \subseteq Q$ implies that $sA \subseteq Q$ or $sB \subseteq \mathrm{Rad}(Q)$.

Proof: Let us assume that Q is an S-k-primary ideal of R. Let A, B be two k-ideals of R such that $AB \subseteq Q$. Suppose for any $s \in S$ we have $sA \nsubseteq Q$ and $sB \nsubseteq \mathrm{Rad}(Q)$. It implies that there exists $x \in A$ and $y \in B$ such that $sx \notin Q$ and $sy \notin \mathrm{Rad}(Q)$, which is absurd since Q is an S-k-primary ideal with $xy \in AB \subseteq Q$. Therefore, there exists $s \in S$ such that $sA \subseteq Q$ or $sB \subseteq \mathrm{Rad}(Q)$.

For the converse part, let Q be a S-k-primary ideal of R satisfying the given condition. Let A, B be two ideals of R such that $AB \subseteq Q$. We have $\overline{A}\ \overline{B} \subseteq \overline{\overline{A}\ \overline{B}} \subseteq \overline{AB} \subseteq \overline{Q} = Q$. By our assumption, it yields $s\overline{A} \subseteq Q$ or $s\overline{B} \subseteq \mathrm{Rad}(Q)$. This implies that $sA \subseteq Q$ or $sB \subseteq \mathrm{Rad}(Q)$. Hence Q is an S-k-primary ideal of R.

Corollary 2.4: Let R be a semiring, $S \subseteq R$ a multiplicatively closed set and Q a k-ideal of R disjoint with S. Then Q is an S-k-primary ideal of R if and only if for all k-ideals J_i of R with $J_1 J_2 \cdots J_n \subseteq Q$, there exists $s \in S$ such that $sJ_i \subseteq Q$ or $sJ_j \subseteq \mathrm{Rad}(Q)$ for some $i, j \in \{1, 2, \ldots, n\}$.

Theorem 2.5: [12] The following statements are equivalent for an ideal Q of a semiring R:

(i) Q is a primary ideal of a semiring R.

(ii) For any $a, b \in R, aRb \subseteq Q$ if and only if $a \in Q$ or $b \in \mathrm{Rad}(Q)$.

Theorem 2.6: Let R be a semiring and Q be an ideal of R. If Q is a k-primary ideal of R then for any $a, b \in R$, if $aRb \subseteq \overline{Q}$ then $a \in Q$ or $b \in \mathrm{Rad}(Q)$.

Proof: Let Q be a k-primary ideal of a semiring R. Then Q is a primary ideal of R and Q is a k-ideal of R, so $Q = \overline{Q}$. The result follows from Theorem 2.5.

Theorem 2.7: Let R be a semiring, $S \subseteq R$ be a multiplicatively closed set and Q an ideal of R disjoint with S. Then the following statements are equivalent:

(i) Q is an S-primary ideal of a semiring R.

(ii) There exists $s \in S$ such that for all $a, b \in R$, if $aRb \subseteq Q$ then $sa \in Q$ or $sb \in \text{Rad}(Q)$.

Proof: (i) $\Rightarrow$ (ii): Let Q be an S-primary ideal of R. Consider $a, b \in R$ such that $aRb \subseteq Q$ and $A =< a >$ and $B =< b >$. Then A and B are ideals of R with $aRb \subseteq AB$. Also, AB is contained in any ideal which contains aRb. Thus $aRb \subseteq Q$ implies that $AB \subseteq Q$ and hence $sA \subseteq Q$ or $sB \subseteq \text{Rad}(Q)$ for some $s \in S$. Thus $sa \in Q$ or $sb \in \text{Rad}(Q)$.

(ii) $\Rightarrow$ (i): Let A and B be ideals of R such that $AB \subseteq Q$. Let us assume that $sA \nsubseteq Q$ and let $a \in A - Q$. Then for each $b \in B$ we have $aRb \subseteq AB \subseteq Q$ which implies that $sb \in \text{Rad}(Q)$ since $sa \notin Q$ and hence $sB \subseteq \text{Rad}(Q)$. So Q is an S-primary ideal of R.

Corollary 2.8: Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and Q an ideal of R disjoint with S. Then Q is an S-primary ideal of R if and only if there exists an $s \in S$ such that for all $a, b \in R$, if $ab \in Q$, then $sa \in Q$ or $sb \in \text{Rad}(Q)$.

Proof: In a commutative semiring R, we have $ab \in Q$ if and only if $arb \in Q$ for all $r \in R$. The result follows from Theorem 2.7.

Example 1: Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define, $Q = < 6^k >$ for $k \geq 1$. Then Q is a k-ideal of R. Also, $Q \cap S = \emptyset$. Now, $ab \in Q =< 6^k > \Rightarrow ab = 6^k m$, for some positive integer m. Then either a or b must be a multiple of 2^p for some positive integer p. So, there exists $s = 3 \in S$ such that $(sa)^k \in Q$ or $(sb)^k \in Q$. Hence, Q is an S-k-primary ideal. Moreover, $2^k.3^k \in < 6^k >$ but $2^k \notin \text{Rad}(< 6^k >)$ and $3^k \notin \text{Rad}(< 6^k >)$ which implies that Q is not a k-primary ideal of $\mathbb{Z}_0^+$.

Now we define an ideal $I : s$ of a commutative semiring R for a given ideal I of R and $s \in R$ by $I : s = \{x \in R | sx \in I\}$. We get the following characterization of S-k-primary ideal of R.

Proposition 2.9: Let R be a commutative semiring, $S \subseteq R$ be a multiplicatively closed set consisting of nonzero divisors and Q be a k-ideal of R disjoint with S. Then Q is an S-k-primary ideal of R if and only if $Q : s$ is a k-primary ideal of R for some $s \in S$.

Proof: Let Q be an S-k-primary ideal of R. Let $a, b \in R$ with $ab \in Q : s$ for some $s \in S$. This implies that $sab \in Q$ so we get $s^2 a \in Q$ or $sb \in \text{Rad}(Q)$. Thus $sa \in Q$ or $(sb)^n \in Q$ for some $n \in \mathbb{N}$. If $sa \in Q$ then $a \in Q : s$. Also if $(sb)^n \in Q$ then $s^{n+1} \in \text{Rad}(Q)$ or $sb^n \in Q$. Since $Q \cap S = \emptyset$ so $s^{n+1} \notin \text{Rad}(Q)$. Thus $sb^n \in Q$, it yields $b^n \in Q : s$ and hence $b \in \text{Rad}(Q : s)$. Therefore $Q : s$ is a primary ideal of R.

Now, it is clear from the definition that $Q : s \subseteq \overline{Q : s}$. Consider $x \in \overline{Q : s}$. It follows that $x \in R, x + y \in Q : s$ for some $y \in Q : s$. This implies $x \in R, s(x + y) \in Q$ for some $sy \in Q$. Thus, $sx \in R, sx + sy \in Q$ for some $sy \in Q$. Since Q is a k-ideal, we have $sx \in Q$ and hence $x \in Q : s$. So, $Q : s = \overline{Q : s}$.
Thus, $Q : s$ is a k-ideal of R, and hence $Q : s$ is a k-primary ideal of R.

Conversely, let us assume that $Q : s$ is a k-primary ideal of R for some $s \in S$. Let $a, b \in R$ with $ab \in Q$, then we have $sab \in Q$ and so $ab \in Q : s$. Since $Q : s$ is a k-primary ideal of R, so we have $a \in Q : s$ or $b \in \text{Rad}(Q : s)$. If $a \in Q : s$ then $sa \in Q$. Otherwise if $b \in \text{Rad}(Q : s)$ then we have $b^m \in Q : s$ for some $m \in \mathbb{N}$. So $sb^m \in Q$ and it follows that $s^m b^m = (sb)^m \in Q$. Equivalently $sb \in \text{Rad}(Q)$.
Therefore, Q is an S-k-primary ideal of R since Q is a k-ideal of R.

Example 2: Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define, $Q = < 6^k >$ for $k \geq 1$. Then Q is a S-k-primary ideal of R. Let

$s = 3^k$. We see that $Q : s$ is a set of all multiples of positive integers in the form of 2^k. If $xy \in Q : s$ then either x or y must be a multiple of integers of the form 2^p where $p \leq k$. Then x^k or y^k must belong to $Q : s$. Hence, $Q : s$ is a k-primary ideal of R.

Proposition 2.10: Let R be a semiring and $S \subseteq R$ be a multiplicatively closed set . Let $R \subseteq T$ be an extension of a semiring. If Q is an S-k-primary ideal of T then $Q \cap R$ is an S-k-primary ideal of R.

Proof: Let us assume that Q is an S-k-primary ideal of T. Let $x, y \in R \subseteq T$ such that $xy \in Q \cap R$. Then $xy \in Q$ and it implies that there exists $s \in S$ such that $sx \in Q$ or $sy \in \text{Rad}(Q)$. If $sx \in Q$ then $sx \in Q \cap R$. If $sy \in \text{Rad}(Q)$ then $(sy)^n \in Q$ for some $n \in \mathbb{N}$. It follows that $(sy)^n \in Q \cap R$. Thus $sy \in \text{Rad}(Q \cap R)$. Therefore $Q \cap R$ is an S-primary ideal of R.

From the definition, we have $Q \cap R \subseteq \overline{Q \cap R}$. Let us consider $a \in \overline{Q \cap R}$ then $a \in R, a + b \in Q \cap R$ for some $b \in Q \cap R$. This implies that $a \in T, a + b \in Q$ for some $b \in Q$. Since Q is a k-ideal of T so $a \in Q$ and thus $a \in Q \cap R$. Therefore, $Q \cap R = \overline{Q \cap R}$ shows that $Q \cap R$ is k-primary ideal of R.
Hence $Q \cap R$ is S-k-primary ideal of R.

Definition 2.11: Let R be a commutative semiring, S a multiplicatively closed subset of R and I be an ideal of R disjoint with S. Let $s \in S$, we denote by $\hat{s}$ the equivalent class of s in R/I. Let $\hat{S} = \{\hat{s} | s \in S\}$, then $\hat{S}$ is a multiplicatively closed subset of R/I.

Proposition 2.12: Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and I a k-ideal of R disjoint with S. Let Q be a proper k-ideal of R containing I such that $Q/I \cap \hat{S} = \emptyset$. Then Q is an S-k-primary ideal of R if and only if Q/I is an $\hat{S}$-k-primary ideal of R/I.

Proof: Let Q be an S-k-primary ideal of R. For all $a, b \in R$ with $ab \in Q$ then there exists $s \in S$ such that $sa \in Q$ or $sb \in \text{Rad}(Q)$ and $Q = \overline{Q}$. Let $\hat{a}, \hat{b} \in R/I$ such that $\hat{a}\hat{b} \in Q/I$, then $\widehat{ab} \in Q/I$. Since Q is a k-ideal so $ab \in Q$ and thus $sa \in Q$ or $sb \in \text{Rad}(Q)$. If $sa \in Q$ then $\widehat{sa} \in Q/I$. Otherwise if $sb \in \text{Rad}(Q)$ then $(sb)^n \in Q$ for some $n \in \mathbb{N}$ and it implies $s^n b^n \in Q$. We get $\widehat{sb}^n \in Q/I$ and thus $\widehat{sb} \in \text{Rad}(Q/I)$. Since $Q/I \subseteq \overline{Q/I}$, consider that $\hat{x} \in \overline{Q/I}$ which implies that $\hat{x} \in R/I, \hat{x} + \hat{y} \in Q/I$ for some $\hat{y} \in Q/I$. Then $x \in R, x + y \in Q$ for some $y \in Q$ and so $x \in Q$. Thus we get $\hat{x} \in Q/I$. Therefore Q/I is an $\hat{S}$-k-primary ideal of R/I.

Conversely, if $Q/I \cap \hat{S} = \emptyset$ then Q must be disjoint with S. Let Q/I be an $\hat{S}$-k-primary ideal of R/I. For all $\hat{a}, \hat{b} \in R/I$ with $\widehat{ab} \in Q/I$ then there exists $\hat{s} \in \hat{S}$ such that $\widehat{sa} \in Q/I$ or $\widehat{sb} \in \text{Rad}(Q/I)$. Let $a, b \in Q$ with $ab \in Q$ then we have $\widehat{ab} \in Q/I$. Thus $\hat{s}\hat{a} \in Q/I$ or $\hat{s}\hat{b} \in \text{Rad}(Q/I)$ and hence $sa \in Q$ or $\widehat{sb}^m \in Q/I$ for some $m \in \mathbb{N}$. So $sa \in Q$ or $sb \in \text{Rad}(Q)$. Since $Q \subseteq \overline{Q}$, consider that $x \in \overline{Q}$ which implies that $x \in R, x + y \in Q$ for some $y \in Q$. Then $\hat{x} \in R/I, \widehat{x + y} \in Q/I$ for some $\hat{y} \in Q/I$ and so $\hat{x} \in Q/I$. Thus we get $x \in Q$. Therefore Q is an S-k-primary ideal of R.

The following result is established by E. Massaoud [9] for a commutative ring. It is easy to verify this for a semiring.

Proposition 2.13: Let R be a commutative semiring, $S \subseteq R$ be a multiplicatively closed set, Q be an ideal of R such that $Q \cap S = \emptyset$. If Q is an S-primary ideal of R then $\text{Rad}(Q)$ is an S-prime ideal of R.

Remark. For a commutative semiring R, let S be a multiplicatively closed subset of R and Q be an ideal of R disjoint with S. If Q is an S-k-primary ideal of R then $\text{Rad}(Q)$ may not be an S-k-prime ideal of R.

Lemma 2.14: [10] Let R be an additively idempotent semiring satisfying $a + ab = a$. If I is a k-ideal of R then $\text{Rad}(I)$ is also a k-ideal of R.

Lemma 2.15: [4] Let R be a cancellative semiring. If I is a k-ideal of R then $\text{Rad}(I)$ is also a k-ideal of R.
Now we give the above Proposition 2.13 in k-ideal form.

Proposition 2.16: Let R be a commutative semiring and $S \subseteq R$ a multiplicatively closed set of R, Q be an ideal of R disjoint with S. If Q is an S-k-primary ideal of R then the following hold:

(i) If R is an additively idempotent semiring satisfying $a + ab = a$ then $\mathrm{Rad}(Q)$ is an S-k-prime ideal of R.

(ii) If R is a cancellative semiring then $\mathrm{Rad}(Q)$ is an S-k-prime ideal of R.

Proof:

(i) Let R be an additively idempotent semiring satisfying $a+ab = a$. Let Q be an S-k-primary ideal of the semiring R. Then by Proposition 2.13 and Lemma 2.14, it follows that $\mathrm{Rad}(Q)$ is an S-k-prime ideal of R.

(ii) Let R be a cancellative semiring and Q be an S-k-primary ideal of R. Then by Proposition 2.13 and Lemma 2.15, it follows that $\mathrm{Rad}(Q)$ is an S-k-prime ideal of R.

Definition 2.17: [12] Let R be a semiring. Let M and N be two sets such that $\emptyset \neq M \subseteq N \subseteq R$. Then we say that N is an m-system with respect to M if, for any $a \in N, b \in M$, there exists $r \in R$ such that $arb \in N$.

Lemma 2.18: [12] Let R be a semiring and Q be a proper ideal of R. Then Q is a primary ideal of R if and only if Q^c is m-system with respect to $\mathrm{Rad}(Q)^c$.

Definition 2.19: Let R be a commutative semiring. Let M and N be two sets such that $\emptyset \neq M \subseteq N \subseteq R$ and S be a multiplicatively closed subset of R. Then we say that N is an S-m-system with respect to M if for any $sa \in N, sb \in M$, there exists $r \in R$ such that $arb \in N$.

Proposition 2.20: Let R be a commutative semiring, S be a multiplicatively closed subset of R and Q be an ideal of R. Then Q is S-primary ideal of R if and only if Q^c is S-m-system with respect to $\mathrm{Rad}(Q)^c$.

Proof: Let Q be an S-primary ideal of R if and only if for any $x, y \in R$ if $xRy \subseteq Q$ then there exists $s \in S$ such that $sx \in Q$ or $sy \in \mathrm{Rad}(Q)$ if and only if $sx \in Q^c$ and $sy \in \mathrm{Rad}(Q)^c$ then there exists $r \in R$ such that $xry \notin Q$ and so $xry \in Q^c$ if and only if Q^c is an S-m-system.

Proposition 2.21: Let R_1, R_2 be two commutative semirings and $f : R_1 \longrightarrow R_2$ be a morphism of commutative semirings. Let S be a multiplicatively closed subset of R_1, Q be a k-ideal of R_2 such that $f^{-1}(Q) \cap S = \emptyset$. If Q is an $f(S)$-k-primary ideal of R_2 then $f^{-1}(Q)$ is an S-k-primary ideal of R_1.

Proof: Let us consider $f : R_1 \longrightarrow R_2$ to be a morphism of commutative semirings. We have $f(S)$ as a multiplicatively closed subset of R_2 if S is a multiplicatively closed subset of R_1. Since Q is an $f(S)-k-$primary ideal of R_2 so there exists $s \in S$ such that for $a, b \in R_2$ with $ab \in Q$ then we must have $f(s)a \in Q$ or $f(s)b \in \mathrm{Rad}(Q)$. Now let $x, y \in R_1$ with $xy \in f^{-1}(Q)$. This implies that $f(xy) \in Q$ and so $f(x)f(y) \in Q$. As Q is an $f(S)$-k-primary ideal of R_2 so there exists $s \in S$ such that $f(s)f(x) \in Q$ or $f(s)f(y) \in \mathrm{Rad}(Q)$. If $f(s)f(x) \in Q$ then $f(sx) \in Q$ and so $sx \in f^{-1}(Q)$. Again if $f(s)f(y) \in \mathrm{Rad}(Q)$ then $f(sy) \in \mathrm{Rad}(Q)$ which implies that there exists $n \in \mathbb{N}$ such that $(f(sy))^n = f((sy)^n) \in Q$. Therefore $(sy)^n \in f^{-1}(Q)$ and thus $sy \in \mathrm{Rad}(f^{-1}(Q))$. Hence $f^{-1}(Q)$ is an S-primary ideal of R_1.

We have $f^{-1}(Q) \subseteq \overline{f^{-1}(Q)}$. Let $x \in \overline{f^{-1}(Q)}$, then $x \in R_1, x + y \in f^{-1}(Q)$ for some $y \in f^{-1}(Q)$. It follows that $f(x) \in R_2, f(x + y) \in Q$ for some $f(y) \in Q$ and thus $f(x) \in R_2, f(x) + f(y) \in Q$ for some $f(y) \in Q$. As Q is a k-ideal of R_2 so we get $f(x) \in Q$ and hence $x \in f^{-1}(Q)$. Therefore $f^{-1}(Q)$ is a k-ideal of R_1. Thus $f^{-1}(Q)$ is an S-k-primary ideal of R_1.

Let R be a commutative semiring. The semiring R is said to be decomposable if there exist commutative semirings R_1 and R_2 such that $R = R_1 \times R_2$. If I_1 and I_2 are ideals of R_1, R_2 respectively then $I_1 \times I_2, I_1 \times R_2$ and $R_1 \times I_2$ are ideals of $R_1 \times R_2$. Moreover $\mathrm{Rad}(I_1 \times R_2) = \mathrm{Rad}(I_1) \times R_2$ and $\mathrm{Rad}(R_1 \times I_2) = R_1 \times \mathrm{Rad}(I_2)$. The following result is established by Massaoud [9] for the case of a commutative ring.

Theorem 2.22: [9] Let R_1 and R_2 be commutative rings and let S_1 and S_2 be multiplicative closed subsets of R_1 and R_2 respectively. Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then the following holds:

(i) Q_1 is an S_1-primary ideal of R_1 if and only if $Q_1 \times R_2$ is an S-primary ideal of R.

(ii) Q_2 is an S_2-primary ideal of R_2 if and only if $R_1 \times Q_2$ is an S-primary ideal of R.

Now, we establish the above result in the k-ideal version for a semiring.

Theorem 2.23: Let R_1 and R_2 be commutative semirings with identity and let S_1 and S_2 be multiplicatively closed subsets of R_1 and R_2 respectively. Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then the following holds:

(i) Q_1 is an S_1-k-primary ideal of R_1 if and only if $Q_1 \times R_2$ is an S-k-primary ideal of R.

(ii) Q_2 is an S_2-k-primary ideal of R_2 if and only if $R_1 \times Q_2$ is an S-k-primary ideal of R.

Proof: It is clear that $Q_1 \cap S_1 = \emptyset$ if and only if $(Q_1 \times R_2) \cap S = \emptyset$, and $Q_2 \cap S_2 = \emptyset$ if and only if $(R_1 \times Q_2) \cap S = \emptyset$.

(i) Let Q_1 be an S_1-k-primary ideal of R_1. Let us consider $(x, y), (u, v) \in R$ such that $(x, y)(u, v) \in Q_1 \times R_2$. It follows that $(xu, yv) \in Q_1 \times R_2$. As Q_1 is an S_1-k-primary ideal so there exists $s_1 \in S_1$ such that $s_1x \in Q_1$ or $s_2u \in \mathrm{Rad}(Q_1)$. Hence, we have $(s_1, 1) \in S$ so that $(s_1, 1)(x, y) \in Q_1 \times R_2$ or $(s_1, 1)(u, v) \in \mathrm{Rad}(Q_1) \times R_2 = \mathrm{Rad}(Q_1 \times R_2)$. Therefore, $Q_1 \times R_2$ is an S-primary ideal of R. Now we show that $Q_1 \times R_2$ is a k-ideal of R.

We know $Q_1 \times R_2 \subseteq \overline{Q_1 \times R_2}$. For the reverse inclusion, we suppose $(a, b) \in \overline{Q_1 \times R_2}$ then $(a, b) \in R, (a, b) + (c, d) \in Q_1 \times R_2$ for some $(c, d) \in Q_1 \times R_2$. From here we obtain $a \in R_1, a + c \in Q_1$ for some $c \in Q_1$. Since Q_1 is a k-ideal of R_1 so we get $a \in Q_1$ and hence $(a, b) \in Q_1 \times R_2$. So $Q_1 \times R_2 = \overline{Q_1 \times R_2}$. Therefore, $Q_1 \times R_2$ is an S-k-primary ideal of R

For the converse part, we consider $Q_1 \times R_2$ be an S-k-primary ideal of R. Let $x, y \in R_1$ such that $xy \in Q_1$. Clearly $(xy, 1) \in Q_1 \times R_2$ and it follows that $(x, 1)(y, 1) \in Q_1 \times R_2$. Since $Q_1 \times R_2$ is an S-k-primary ideal there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(x, 1) \in Q_1 \times R_2$ or $(s_1, s_2)(y, 1) \in \mathrm{Rad}(Q_1 \times R_2) = \mathrm{Rad}(Q_1) \times R_2$ which yields $s_1x \in Q_1$ or $s_1y \in \mathrm{Rad}(Q_1)$. Therefore, Q_1 is an S-primary ideal of R_1.

We know $Q_1 \subseteq \overline{Q_1}$. For the reverse inclusion, we consider $a \in \overline{Q_1}$ which gives $a \in R_1, a + b \in Q_1$ for some $b \in Q_1$. Which implies $(a, 1) \in R, (a + b, 1 + 0) \in Q_1 \times R_2$ for some $(b, 0) \in Q_1 \times R_2$. Since $Q_1 \times R_2$ is a k-ideal so we get $(a, 1) \in Q_1 \times R_2$ and hence $a \in Q_1$. So, $Q_1 = \overline{Q_1}$. Therefore, Q_1 is an S-k-primary ideal of R_1.

(ii) The proof is similar to the proof of (i).

3. S-k-primary ideals in semiring amalgamations

In this section, we define the idealization of a semimodule in a semiring. We further define and study the properties of amalgamation of a semiring along an ideal.

For this purpose of studying idealization, we consider R to be a commutative semiring with identity and M to be an R-semimodule. We define, $R(+)M = \{(r, m)|r \in R, m \in M\}$. This set is a commutative semiring under usual addition and the multiplication defined as

$$(r, m)(r', m') = (rr', rm' + r'm)$$

for all $(r, m), (r', m') \in R(+)M$. This $R(+)M$ is called the idealization of semimodule M in R.

Let S be a multiplicatively closed subset of R, then it is clear that $S(+)M = \{(s, m)|s \in S, m \in M\}$ is a multiplicatively closed subset of $R(+)M$.

Lemma 3.1: Let R be a semiring, M be a unitary k-semimodule over the semiring R and I be an ideal of R. Then $I(+)M$ is a k-ideal of $R(+)M$ if and only if I is a k-ideal of R.

Proof: Let us consider I to be a k-ideal of R. It is clear that $I(+)M \subseteq \overline{I(+)M}$. For the reverse inclusion, we consider $(a,b) \in \overline{I(+)M}$. This implies that $(a,b) \in R(+)M, (a,b)+(c,d) \in I(+)M$ for some $(c,d) \in I(+)M$ and it follows $(a,b) \in R(+)M, (a+c, b+d) \in I(+)M$ for some $(c,d) \in I(+)M$. It implies that $a \in R, a+c \in I$ for some $c \in I$ and $b \in M, b+d \in M$ for some $d \in M$. Since I and M are k-ideal of R and k-semimodule respectively, therefore we get $a \in I$ and $b \in M$ and hence $(a,b) \in I(+)M$. Thus, $I(+)M$ is a k-ideal of $R(+)M$.

For the converse part, we consider $I(+)M$ to be a k-ideal of $R(+)M$. Since $I \subseteq \overline{I}$, we consider $x \in \overline{I}$ to prove the reverse inclusion. We have $x \in R, x+y \in I$ for some $y \in I$. Which implies that $(x, 1_M) \in R(+)M, (x,1_M)+(y,0_M) \in I(+)M$ for some $(y,0_M) \in I(+)M$. As $I(+)M$ is a k-ideal of $R(+)M$ so we get that $(x,1_M) \in I(+)M$ and hence $x \in I$. Therefore, I is a k-ideal of R.

Proposition 3.2: Let R be a commutative semiring and M be a unitary R-semimodule. Let S be a multiplicatively closed subset of R and Q be an ideal of R disjoint with S. Then, $Q(+)M$ is an $S(+)M$-k-primary ideal of $R(+)M$ if and only if Q is an S-k-primary ideal of R.

Proof: It is clear that $Q \cap S = \emptyset$ if and only if $Q(+)M \cap S(+)M = \emptyset$.

Consider Q to be an S-k-primary ideal of R. Let $(a,b),(c,d) \in R(+)M$ such that $(a,b)(c,d) \in Q(+)M$. This implies that $(ac, ad+cb) \in Q(+)M$, It follows that $ac \in Q$. Since Q is an S-k-primary ideal of R, there exists $s \in S$ such that $sa \in Q$ or $sb \in \operatorname{Rad}(Q)$. Therefore $sa \in Q$ or $(sb)^n = s^n b^n \in Q$ for some $n \in \mathbb{N}$. Thus there exists $(s,0_M) \in S(+)M$ such that $(s,0_M)(a,b) \in Q(+)M$ or $((s,0_M)(c,d))^n = (s^n c^n, ns^n c^{n-1} d) \in Q(+)M$. Therefore $(s,0_M)(a,b) \in Q(+)M$ or $(s,0_M)(c,d) \in \operatorname{Rad}(Q(+)M)$. Hence $Q(+)M$ is an $S(+)M$-primary ideal of $R(+)M$. By Lemma 3.1, we further prove that $Q(+)M$ is an $S(+)M$-k-primary ideal of $R(+)M$.

For the converse part, we consider $Q(+)M$ to be an $S(+)M$-k-primary ideal of $R(+)M$. Let $a,b \in R$ such that $ab \in Q$. It follows that $(a,0_M)(b,0_M) \in Q(+)M$. Since $Q(+)M$ is an $S(+)M-k-$primary ideal, there exists $(s,m) \in S(+)M$ such that $(s,m)(a,0_M) \in Q(+)M$ or $(s,m)(b,0_M) \in \operatorname{Rad}(Q(+)M)$. Therefore $(sa, am) \in Q(+)M$ or $(sb, bm)^n = (s^n b^n, ns^{n-1} m b^n) \in Q(+)M$ for some $n \in \mathbb{N}$. Hence, $sa \in Q$ or $sb \in \operatorname{Rad}(Q)$. So, Q is an S-primary ideal of R. Further, by Lemma 3.1, we get Q is an S-k-primary ideal of R.

Let R_1 and R_2 be two commutative semirings with identity, and consider J to be an ideal of R_2. Let $f : R_1 \longrightarrow R_2$ be a semiring morphism. We define the following set

$$R_1 \bowtie^f J = \{(a, f(a)+j) | a \in R_1, j \in J\}$$

We can check that $R_1 \bowtie^f J$ is a subsemiring of $R_1 \times R_2$, and it is called the amalgamation of R_1 with R_2 along the ideal J with respect to f. This type of construction was studied by D'Anna et al [3] in the ring theoretic settings.

As defined by D'Anna et al. [3] for ring theoretic settings, we define two corresponding ideals of $R_1 \bowtie^f J$ respectively as $I \bowtie^f J = \{(i, f(i)+j) | i \in I, j \in J\}$ and $K^f = \{(r, f(r)+j) | r \in R_1, j \in J, f(r)+j \in K\}$, where I and K are ideals of semirings R_1 and $f(R_1)+J$ respectively.

We see that $S \bowtie^f J = \{(s, f(s)+j) | s \in S, j \in J\}$ and $S' = \{(s, f(s)) | s \in S\}$ are multiplicatively closed subsets of $R_1 \bowtie^f J$ when S is a multiplicatively closed subset of R_1. Moreover, $f(S)$ is a multiplicatively closed subset of R_2 whenever zero is not contained in $f(S)$.

Lemma 3.3: [2] Let I and J be two ideals of the semiring R_1 and R_2 respectively. Then $\operatorname{Rad}(I) \bowtie^f J =$

$\mathrm{Rad}(I \bowtie^f J)$.

Lemma 3.4: Let R_1 and R_2 be two semirings, I be an ideal of R_1 and J be a k-ideal of R_2. Then $I \bowtie^f J$ is a k-ideal of $R_1 \bowtie^f J$ if and only if I is a k-ideal of R_1.

Proof: Let I and J be k-ideals of R_1 and R_2 respectively. It is clear that $I \bowtie^f J \subseteq \overline{I \bowtie^f J}$. For the reverse inclusion, we consider $(i, f(i)+j) \in \overline{I \bowtie^f J}$. This implies that $(i, f(i)+j) \in R_1 \bowtie^f J, (i, f(i)+j)+(i_0, f(i_0)+j_0) \in I \bowtie^f J$ for some $(i_0, f(i_0)+j_0) \in I \bowtie^f J$ and it follows $(i, f(i)+j) \in R_1 \bowtie^f J, (i+i_0, f(i+i_0)+j+j_0) \in I \bowtie^f J$ for some $(i_0, f(i_0)+j_0) \in I \bowtie^f J$. From here, we obtain that $i \in R_1, i+i_0 \in I$ for some $i_0 \in I$ and $j \in J$. Since I is a k-ideal of R_1, so $i \in I$ and $j \in J$ and hence $(i, f(i)+j) \in I \bowtie^f J$. Thus, $I \bowtie^f J$ is a k-ideal of R.

For the converse part, we consider $I \bowtie^f J$ to be a k-ideal of $R_1 \bowtie^f J$. Since $I \subseteq \overline{I}$, we consider $x \in \overline{I}$ to prove the reverse inclusion. We have $x \in R_1, x+y \in I$ for some $y \in I$. Let $u, v \in J$, then we have $(x, f(x)+u) \in R_1 \bowtie^f J, (x+y, f(x+y)+u+v) \in I \bowtie^f J$ for some $(y, f(y)+v) \in I \bowtie^f J$. This implies that $(x, f(x)+u) \in R_1 \bowtie^f J, (x, f(x)+u)+(y, f(y)+v) \in I \bowtie^f J$ for some $(y, f(y)+v) \in I \bowtie^f J$. Since $I \bowtie^f J$ is a k-ideal of $R_1 \bowtie^f J$ so we get $(x, f(x)+u) \in I \bowtie^f J$ and hence $x \in I$. Therefore I is a k-ideal of R_1.

Proposition 3.5: Consider the amalgamation of semirings R_1 and R_2 along the k-ideal J of R_2 with respect to a semiring morphism f. Let S be a multiplicatively closed subset of R_1 and I be an ideal of R_1, the following statements hold :

(i) $I \bowtie^f J$ is an $(S \bowtie^f J)$-k-primary ideal of $R_1 \bowtie^f J$ if and only if I is an S-k-primary ideal of R_1.

(ii) $I \bowtie^f J$ is an S'-k-primary ideal of $R_1 \bowtie^f J$ if and only if I is an S-k-primary ideal of R_1.

(iii) If $f(S)$ does not contain zero, then K^f is an S'-k-primary ideal of $R_1 \bowtie^f J$ if and only if K is an $f(S)$-k-primary ideal of $f(R_1)+J$.

Proof:

(i) It is clear that $S \cap I = \emptyset$ if and only if $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$.
Let I be an S-k-primary ideal of R_1. Consider $(a, f(a)+i), (b, f(b)+j) \in R_1 \bowtie^f J$ such that $(a, f(a)+i)(b, f(b)+j) \in I \bowtie^f J$. It implies that $(ab, f(ab)+if(b)+jf(a)+ij) \in I \bowtie^f J$ and we get $ab \in I$. There exists $s \in S$ such that $sa \in I$ or $sb \in \mathrm{Rad}(I)$. We have $(s, f(s)+j') \in S \bowtie^f J$ such that,

$$(s, f(s)+j')(a, f(a)+j) = (sa, f(sa)+j'f(a)+jf(s)+jj') \in I \bowtie^f J$$

and

$$(s, f(s)+j')((b, f(b)+j)) = (sb, f(sb)+j'f(b)+jf(s)+jj')$$

Therefore,
$(sb, f(sb)+j'f(b)+jf(s)+jj')^n = (s^n b^n, (f(sb)+j'f(b)+jf(s)+jj')^n) \in \mathrm{Rad}(I) \bowtie^f J = \mathrm{Rad}(I \bowtie^f J)$. [By Lemma 3.3.]

Hence, $I \bowtie^f J$ is an $(S \bowtie^f J)$-primary ideal of $R_1 \bowtie^f J$. By Lemma 3.4, we get $I \bowtie^f J$ is an $(S \bowtie^f J)$-k-primary ideal of $R_1 \bowtie^f J$.

For the converse part, we consider that $I \bowtie^f J$ be an $(S \bowtie^f J)$-k-primary ideal of $R_1 \bowtie^f J$. Let $a, b \in R_1$ with $ab \in I$. It implies that $(ab, f(ab)) \in I \bowtie^f J$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$. Therefore either $(s, f(s)+j)(a, f(a)) \in I \bowtie^f J$ or $(s, f(s)+j)(b, f(b)) \in \mathrm{Rad}(R) \bowtie^f J$. So either $sa \in I$ or $sb \in \mathrm{Rad}(I)$. Hence I is an S-primary ideal of R_1. Moreover, by Lemma 3.4 we get I is an S-k-primary ideal of R_1.

(ii) It is clear that $S \cap I = \emptyset$ if and only if $S' \cap (I \bowtie^f J) = \emptyset$.
Let I be an S-k-primary ideal of R_1. Consider $(a, f(a)+i), (b, f(b)+j) \in R_1 \bowtie^f J$ such that $(a, f(a)+i)(b, f(b)+j) \in I \bowtie^f J$. It implies that $(ab, f(ab)+if(b)+jf(a)+ij) \in I \bowtie^f J$ and we get $ab \in I$. There exists $s \in S$ such that $sa \in I$ or $sb \in \mathrm{Rad}(I)$. We have $(s, f(s)) \in S'$ such that,

$$(s, f(s))(a, f(a)+j) = (sa, f(sa)+jf(s)) \in I \bowtie^f J$$

and

$$(s, f(s))((b, f(b)+j)) = (sb, f(sb)+jf(s))$$

Therefore,
$(sb, f(sb)+jf(s))^n = (s^n b^n, (f(sb)+jf(s))^n) \in \mathrm{Rad}(I) \bowtie^f J = \mathrm{Rad}(I \bowtie^f J)$. [By Lemma 3.3.]

Hence, $I \bowtie^f J$ is an S'-primary ideal of $R_1 \bowtie^f J$. By Lemma 3.4 we get $I \bowtie^f J$ is an $(S \bowtie^f J)$-k-primary ideal of $R_1 \bowtie^f J$.

For the converse part, we consider that $I \bowtie^f J$ be an S'-k-primary ideal of $R_1 \bowtie^f J$. Let $a, b \in R_1$ with $ab \in I$. It implies that $(ab, f(ab)) \in I \bowtie^f J$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$. Therefore either $(s, f(s))(a, f(a)) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)) \in \mathrm{Rad}(I) \bowtie^f J$. So either $sa \in I$ or $sb \in \mathrm{Rad}(I)$. Hence I is an S-primary ideal of R_1. Moreover, by Lemma 3.4 we get I is an S-k-primary ideal of R_1.

(iii) It is clear that $S' \cap K^f = \emptyset$ if and only if $f(S) \cap K = \emptyset$.

Let us assume that K^f be an S'-k-primary ideal of $R_1 \bowtie^f J$. Let $f(a)+j, f(b)+j' \in f(R_1)+J$ with $(f(a)+j)(f(b)+j') \in K$. Then, $(a, f(a)+j)(b, f(b)+j) = (ab, f(ab)+jf(b)+j'f(a)+jj') \in K^f$. Since, K^f is an S'-k-primary ideal of $R_1 \bowtie^f J$, so there exists $(s, f(s)) \in S'$ such that $(s, f(s))(a, f(a)+j) \in K^f$ or $(s, f(s))(b, f(b)+j) \in \mathrm{Rad}(K^f)$.

If $(s, f(s))(a, f(a)+j) \in K^f$, then it implies $f(s)(f(a)+j) \in K$. Else if, $(s, f(s))(b, f(b)+j) \in \mathrm{Rad}(K^f)$, it implies $(s, f(s^n))(b, (f(b)+j')^n) \in K^f$ for some $n \in \mathbb{N}$, and thus $f(s^n)(f(b)+j')^n \in K$. Hence, K is an $f(S)$-primary ideal of $f(R_1)+J$.

Suppose, $f(a)+j \in \overline{K}$. This implies $f(a)+j \in f(R_1)+J$, $f(a)+j+f(b)+j' \in K$ for some $f(b)+j' \in K$. Then $(a, f(a)+j) \in R_1 \bowtie^f J, (a+b, f(a+b)+j+j') \in K^f$ for some $(b, f(b)+j') \in K^f$. Since K^f is a k-ideal of $R_1 \bowtie^f J$, so $(a, f(a)+j) \in K^f$ and thus $f(a)+j \in K$. It follows $\overline{K} \subseteq K$, and by definition, we have $K = \overline{K}$. Hence, K is a k-ideal of $f(R_1)+J$.
Therefore, K is an $f(S)$-k-primary ideal of $f(R_1)+J$.

For the converse part, let us assume that K is an $f(S)$-k-primary ideal of $f(R_1)+J$. Let $(a, f(a)+j), (b, f(b)+j') \in R_1 \bowtie^f J$ such that $(a, f(a)+j)(b, f(b)+j') \in K^f$. This implies $(ab, f(ab)+jf(b)+j'f(a)+jj') \in K^f$ and we have $f(ab)+jf(b)+j'f(a)+jj' = (f(a)+j)(f(b)+j') \in K$. Since K is a $f(S)$-k-primary ideal of $f(R_1)+J$, so there exists $f(s) \in f(S)$ such that $f(s)(f(a)+j) \in K$ or $f(s)(f(b)+j') \in \mathrm{Rad}(K)$.

If $f(s)(f(a)+j) \in K$, then it implies $f(sa)+jf(s) \in K$. It yields $(sa, f(sa)+jf(s)) \in K^f$ and thus $(s, f(s))(f(a)+j) \in K^f$. Otherwise, we have $f(s^n)(f(b)+j')^n \in K$ for some $n \in \mathbb{N}$. It implies that $(s, f(s))^n(f(b)+j')^n \in K^f$. Thus $(s, f(s))(f(b)+j') \in \mathrm{Rad}(K^f)$. Hence, K^f is an S'-primary ideal of $R_1 \bowtie^f J$.

Suppose, $(a, f(a)+j) \in \overline{K^f}$. This implies $(a, f(a)+j) \in R_1 \bowtie^f J, (a, f(a)+j)+(b, f(b)+j') \in K^f$ for some $(b, f(b)+j') \in K^f$.So, $(a, f(a)+j) \in R_1 \bowtie^f J, (a+b, f(a+b)+j+j') \in K^f$ for some $(b, f(b)+j') \in K^f$. It follows $f(a)+j \in f(R_1)+J, f(a+b)+j+j' = f(a)+j+f(b)+j' \in K$ for some $f(b)+j' \in K$. Since K is a k-ideal of $f(R_1)+J$, so $f(a)+j \in K$ and thus $(a, f(a)+j) \in K^f$. It follows $\overline{K^f} \subseteq K^f$, and by definition, $K^f = \overline{K^f}$. Hence, K^f is a k-ideal of $R_1 \bowtie^f J$.
Therefore, K^f is an S'-k-primary ideal of $R_1 \bowtie^f J$.

Acknowledgments

The authors would like to express their deep gratitude to the referee for a very careful reading of the article, and many valuable suggestions to improve the article.

Conflict of interest:
The authors declare that they have no conflict of interest.

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