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Double 3–dimensional Riordan arrays and their applications

Ömer Duran, Neşe Ömür and Sibel Koparal

ABSTRACT: In this paper, we give the group of double 3−dimensional Riordan arrays. Also we examine new sums involving Fibonacci numbers and special numbers, using combinatorial identities and the double 3−dimensional Riordan arrays. For instance, for non-negative integer n,

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n+l}\frac{(-1)^{l+j}\,d_{n+l-j}^{k+l}d_{j}^{l+k}}{(n-j+l)!j!}=F_n \text{ and } \sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-k-l}\frac{(-1)^{j}\,d_l d_j^k H_{n-k-l-j}^{k+2}}{l!j!}=H_n^2,$$

where d_n^r, F_n and H_n^r are $r-$derangement number, Fibonacci number and hyperharmonic number of order r, respectively.

Key Words: Double 3−dimensional Riordan arrays, generating functions, sums.

Contents

1. Introduction

Let $\mathcal{F}=\{f \mid f=f_0+f_1t+f_2t^2+\cdots, f_n\in\mathbb{C}\}$. For $f\in\mathcal{F}$, if $m\in\mathbb{N}$ is the smallest number such that $f_m\neq 0$, then the order of f is m. We denote $\mathcal{F}_m$ for the set of functions of order m i.e. $\mathcal{F}_m=\{f \mid f=f_mt^m+f_{m+1}t^{m+1}+\cdots, f_m\neq 0\}\subset\mathcal{F}$. Let $r_{n,k}$ be the coefficient of t^n in gf^k for $g\in\mathcal{F}_0$ and $f\in\mathcal{F}_1$. Then $R=(g,f)=(r_{n,k})_{n,k\geq 0}$ is a Riordan matrix (or array). Notice that the Riordan matrices are lower triangular, infinite matrices. For example, taking $tg(t)=f(t)=\frac{t}{1-t}$, the Pascal matrix is as following:

$$\left(\frac{1}{1-t},\frac{t}{1-t}\right)=\begin{bmatrix}1&&&&&\\1&1&&&&\\1&2&1&&&\\1&3&3&1&&\\1&4&6&4&1&\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

Let the set of all Riordan matrices be represented as $\mathcal{R}$. Then $\mathcal{R}$ is a group under the binary operation

$$(g,f)(u,v)=(gu(f),v(f)),$$

and it is named as Riordan group [6,7]. The Riordan group has many important subgroups [7]. For example, the checkerboard subgroup is

$$\{(g,f) \mid g \text{ is an even function and } f \text{ is an odd function}\}.$$

Recently, Solo [8] defined multi-dimensional matrices and their operations which are addition, multiplication and multiplication by a scalar. Let A and B be infinite 3−dimensional arrays such that

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$A=(a_{i,j,k})_{i,j,k\geq 0}$ and $B=(b_{i,j,k})_{i,j,k\geq 0}$. Then the usual matrix multiplication ($(2,1)-$multiplication) of A and B is defined by

$$AB=(c_{i,j,k})_{i,j,k\geq 0},$$

where $c_{i,j,k}=\sum_{x\geq 0} a_{i,x,k}b_{x,j,k}$.

In [2], Cheon and Jin wrote the set of all $3-$dimensional Riordan arrays denoted as

$$\mathcal{R}^{\langle 3\rangle}=\{(g,f,h)\ \mid g,h\in\mathcal{F}_0, f\in\mathcal{F}_1\}.$$

Also $\mathcal{R}^{\langle 3\rangle}$ formed a group under the binary operation

$$(g,f,h)\,(u,v,w)=(gu(f),v(f),hw(f)).$$

$\mathcal{R}^{\langle 3\rangle}$ is called the $3-$dimensional (or $3-$D) Riordan group. The $3-$dimensional Riordan group has many important and interesting subgroups [9]. For example, the checkerboard subgroup is

$$\{(g,f,h)\mid g,h \text{ are even functions and } f \text{ is an odd function}\}.$$

In [3], Davenport, Shapiro and Woodson defined the double Riordan group as follows: Let $g(t)=\sum_{n\geq 0} g_{2n}t^{2n}$, $f_1(t)=\sum_{n\geq 0} f_{1,2n+1}t^{2n+1}$, $f_2(t)=\sum_{n\geq 0} f_{2,2n+1}t^{2n+1}$ such that g_0, $f_{1,1}$, $f_{2,1}\neq 0$. Then the double Riordan array (or matrix) of $g(t)$, $f_1(t)$ and $f_2(t)$ is given by

$$(g;f_1,f_2)=\begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ g & gf_1 & gf_1f_2 & gf_1^2f_2 & gf_1^2f_2^2 & gf_1^3f_2^2 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \end{bmatrix}.$$

The set of all aerated double Riordan matrices (or arrays) is denoted as $\mathcal{DR}$. $\langle\mathcal{DR},*\rangle$ is double Riordan group under the binary operation $*$ as follows:

$$(g;f_1,f_2)*(u;v_1,v_2)=\left(gu(\sqrt{f_1f_2});\sqrt{f_1/f_2}v_1(\sqrt{f_1f_2}),\sqrt{f_2/f_1}v_2(\sqrt{f_1f_2})\right).$$

In [9], He defined the array called the double quasi-Riordan array, which is the compression array of the double Riordan array. For double Riordan array $(r_{n,k})_{n,k\geq 0}=(g;f_1,f_2)$, the double quasi-Riordan array associated with $(r_{n,k})_{n,k\geq 0}$, which is denoted by $(\hat{r}_{n,k})_{n,k\geq 0}$, is given by

$$\hat{r}_{n,k}:=r_{2n-k,k},\ n\geq k\geq 0. \tag{1.1}$$

Also the author studied the structure of double quasi-Riordan array $(\hat{r}_{n,k})_{n,k\geq 0}$ given by

$$\hat{r}_{n,k}=\begin{cases} [t^n]\,\hat{g}\left(\hat{f}_1\hat{f}_2\right)^{k/2}, & \text{if } k \text{ is even,}\\ [t^n]\,\hat{g}\hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^{(k-1)/2}, & \text{if } k \text{ is odd,}\end{cases} \tag{1.2}$$

where $\hat{g}(t)=\sum_{n\geq 0} g_{2n}t^n$, $\hat{f}_1=\sum_{n\geq 0} f_{1,2n+1}t^{n+1}$ and $\hat{f}_2=\sum_{n\geq 0} f_{2,2n+1}t^{n+1}$.

Harmonic numbers and their generalizations have important role in combinatorics, number theory, and there are a lot of works involving these numbers. The harmonic numbers H_n are defined by

$$H_n=\sum_{i=1}^{n}\frac{1}{i}\ \text{ for } n\geq 1$$

and $H_0=0$. The generating function of H_n is

$$\sum_{n=0}^{\infty} H_nt^n=-\frac{\log(1-t)}{1-t}.$$

The generalized harmonic numbers of rank r, $H(n,r)$ [5,4] are defined by

$$H(n,r)=\sum_{1\leq k_0+k_1+\cdots+k_r\leq n}\frac{1}{k_0k_1\cdots k_r},$$

for $n\geq 1$, $r\geq 0$. Notice that, when $r=0$, $H(n,0)=H_n$ and generating function of $H(n,r)$ is

$$\sum_{n=0}^{\infty}H\left(n,r\right)t^n=\frac{\left(-\log\left(1-t\right)\right)^{r+1}}{1-t}.$$

The hyperharmonic numbers of order r, H_n^r [1] are defined by

$$H_n^0=\frac{1}{n},\ \ H_n^r=\sum_{i=1}^{n}H_i^{r-1},\quad r,n\geq 1$$

where for $n\leq 0$ or $r<0$, $H_n^r=0$ and the generating function of H_n^r is

$$\sum_{n=0}^{\infty}H_n^rt^n=-\frac{\log\left(1-t\right)}{\left(1-t\right)^r}. \tag{1.3}$$

The Catalan numbers, denoted by C_n, are defined as

$$C_n=\frac{1}{n+1}\binom{2n}{n},$$

for non-negative integer n and the generating function of C_n is

$$\sum_{n=0}^{\infty}C_nt^n=\frac{1-\sqrt{1-4t}}{2t}.$$

The Cauchy numbers of order r, denoted by $\mathcal{C}_n^r$, are defined via their generating function as follows:

$$\sum_{n=0}^{\infty}\mathcal{C}_n^r\frac{t^n}{n!}=\left(\frac{t}{\log\left(1+t\right)}\right)^r.$$

The Daehee polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}=\left(\frac{\log\left(1+t\right)}{t}\right)(1+t)^x.$$

When $x=0$, $D_n(0)=D_n$ are called the Daehee numbers.

The $r-$derangement numbers d_n^r satisfy recursive formulas given by

$$d_n^r=rd_{n-1}^{r-1}+(n-1+r)d_{n-1}^r+(n-1)d_{n-2}^r,\ \ n>2 \text{ and } r>0,$$

with initial conditions

$$d_n^1=d_{n+1},\ \ d_r^r=r!\ (r\geq 1)\ \text{ and }\ d_{r+1}^r=r(r+1)!\ (r\geq 2).$$

The generating function of the $r-$derangement numbers d_n^r is given by

$$\sum_{n=r}^{\infty}\frac{d_n^r}{n!}t^n=\frac{t^re^{-t}}{\left(1-t\right)^{r+1}}.$$

In [10], the closed form for d_n^r is also given by

$$d_n^r=\sum_{k=r}^{n}(-1)^{n-k}\frac{n!}{(n-k)!}\binom{k}{r}\quad (n\geq r\geq 0).$$

For $r=0$, $d_n^0=d_n$, where d_n is called derangement number.

The generalized geometric series are given by for non-negative integer l,

$$\sum_{n=0}^{\infty}\binom{n+l}{l}t^n=\frac{1}{(1-t)^{l+1}}. \tag{1.4}$$

The Lambert W function is defined as the solution of the implicit equation $we^w=t$ with $t\in\mathbb{R}$. It is immediate to see that for $t\geq 0$, there is a single solution, denoted $W_0(t)$, with a power series expansion given by Lagrange inversion

$$W_0(t)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}t^n.$$

Note that for $t\geq -1$, $W_0(te^t)=t$.

The Laguerre polynomials are defined recursively as for $n\geq 1$,

$$L_{n+1}(x)=\frac{(2n+1-x)\,L_n(x)-nL_{n-1}(x)}{n+1},$$

where $L_0(x)=1, L_1(x)=1-x$. The closed form for $L_n(x)$ is

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}x^k.$$

The generating function of these polynomials is given by

$$\sum_{n=0}^{\infty}L_n(x)t^n=\frac{e^{-xt/(1-t)}}{1-t}.$$

The generalized second order sequence $\{U_n\}$ is defined for positive integer n and non-zero integer numbers p,q by

$$U_{n+1}=pU_n+qU_{n-1}$$

in which $U_0=0$, $U_1=1$. For $p=q=1$, $U_n=F_n$ (the nth Fibonacci number). If α and β are the roots of equation $x^2-px-q=0$, the Binet formula of the sequence $\{U_n\}$ has the form

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}.$$

The generating function of U_n is given by

$$\sum_{n=0}^{\infty}U_nt^n=\frac{t}{1-pt-qt^2}. \tag{1.5}$$

It is hard to seen that there are sums involving Fibonacci numbers, generalized harmonic numbers, $r-$derangement numbers etc. in literature. In this paper, using double $3-$dimensional Riordan arrays, we have nice sums involving Fibonacci numbers and some special numbers. For instance,

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{i=0}^{n+l}\frac{(-1)^{l+i}d_{n+l-i}^{k+l}d_i^{l+k}}{(n+l-i)!i!}=F_n\quad\text{and}\quad\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{i=0}^{n-k-l}\frac{(-1)^i d_l d_i^k H_{n-k-l-i}^{k+2}}{l!i!}=H_n^2.$$

2. Double 3-dimensional Riordan arrays

In this section, we will define double 3−dimensional Riordan arrays and give double 3−dimensional Riordan group. Also, we will show some applications of them using special numbers and their generating functions given in Section 1.

Throughout this paper, we will take the set

$$\mathcal{F}_r^2 = \left\{ f \mid f = f_r t^r + f_{r+2} t^{r+2} + f_{r+4} t^{r+4} + \cdots,\ f_i \in \mathbb{C},\ f_r \neq 0 \right\}.$$

Theorem 2.1 *Let $(r_{n,k})_{n,k\geq 0} = (g; f_1, f_2)$ be a double Riordan array. Then*

$$\sum_{k=0}^{n} \hat{r}_{n,2k} a_k = [t^n] \hat{g} A(\hat{f}_1 \hat{f}_2), \tag{2.1}$$

$$\sum_{k=0}^{n} \hat{r}_{n,2k+1} a_k = [t^n] \hat{g} \hat{f}_1 A(\hat{f}_1 \hat{f}_2), \tag{2.2}$$

where $\hat{g}(t) = \sum_{n\geq 0} g_{2n} t^n$, $\hat{f}_1(t) = \sum_{n\geq 0} f_{1,2n+1} t^{n+1}$, $\hat{f}_2(t) = \sum_{n\geq 0} f_{2,2n+1} t^{n+1}$ and $A(t) = \sum_{n\geq 0} a_n t^n$.

Proof: By (1.1) and (1.2), we have $\hat{r}_{n,2k} = [t^n]\, \hat{g}(\hat{f}_1 \hat{f}_2)^k$. Since Riordan matrix is lower triangular, then

$$\sum_{k=0}^{n} \hat{r}_{n,2k} a_k = \sum_{k=0}^{n} [t^n]\, \hat{g}(\hat{f}_1 \hat{f}_2)^k a_k = [t^n]\, \hat{g} \sum_{k=0}^{n} a_k (\hat{f}_1 \hat{f}_2)^k = [t^n]\, \hat{g} A(\hat{f}_1 \hat{f}_2),$$

as claimed. Similarly, second identity can be shown. □

Let $g, h \in \mathcal{F}_0^2$ and $f_1, f_2 \in \mathcal{F}_1^2$. The double 3−dimensional Riordan matrix (or array) $(g; f_1, f_2; h)$ is defined by

$$(g; f_1, f_2; h) = \begin{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \\ g & gf_1 & gf_1 f_2 & gf_1^2 f_2 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{bmatrix} \\ \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \\ gh & gf_1 h & gf_1 f_2 h & gf_1^2 f_2 h & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{bmatrix} \\ \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \\ gh^2 & gf_1 h^2 & gf_1 f_2 h^2 & gf_1^2 f_2 h^2 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{bmatrix} \\ \vdots \end{bmatrix}.$$

We will show $(g; f_1, f_2; h)$ by $R = (r_{n,k,l})_{n,k,l\geq 0}$. The set of all double 3−dimensional Riordan matrices is shown by $\mathcal{DR}^{\langle 3 \rangle}$. Let $L_l(R)$ denote lth layer matrix of the double 3−dimensional Riordan array $R = (g; f_1, f_2; h)$. It follows that $L_l(R)$ is an proper double Riordan array given by $(gh^l; f_1, f_2)$ for fixed l. With the help of layer matrices, the double 3−dimensional Riordan array R may be denoted by $R = (L_0(R), L_1(R), L_2(R), \cdots)$.

Definition 2.1 *For $(g; f_1, f_2; h)$, $(u; v_1, v_2; w) \in \mathcal{DR}^{\langle 3 \rangle}$, binary operation $\triangle$ on $\mathcal{DR}^{\langle 3 \rangle}$ is given by*

$$(g; f_1, f_2; h) \triangle (u; v_1, v_2; w) = \left(gu(\sqrt{f_1 f_2}); \sqrt{\frac{f_1}{f_2}} v_1(\sqrt{f_1 f_2}), \sqrt{\frac{f_2}{f_1}} v_2(\sqrt{f_1 f_2}); hw(\sqrt{f_1 f_2}) \right).$$

Theorem 2.2 $\left\langle \mathcal{DR}^{\langle 3 \rangle}, \triangle \right\rangle$ *is a group.*

Proof: Matrix multiplication is associative. For any $(g; f_1, f_2; h) \in \mathcal{DR}^{\langle 3\rangle}$, we have

$$(g; f_1, f_2; h) \triangle (1; t, t; 1) = \left(g; \sqrt{\frac{f_1}{f_2}} \sqrt{f_1 f_2}, \sqrt{\frac{f_2}{f_1}} \sqrt{f_1 f_2}; h \right) = (g; f_1, f_2; h) .$$

Similarly,$(1; t, t; 1) \triangle (g; f_1, f_2; h) = (g; f_1, f_2; h)$. Thus, the identity element is the matrix $(1; t, t; 1)$. For $(g; f_1, f_2; h) \in \mathcal{DR}^{\langle 3\rangle}$, if $(g; f_1, f_2; h) \triangle (u; v_1, v_2; w) = (1; t, t; 1)$, then

$$u \left(\sqrt{f_1 f_2} \right) = \frac{1}{g}, \ hw \left(\sqrt{f_1 f_2} \right) = 1, \ \sqrt{\frac{f_1}{f_2}} v_1 \left(\sqrt{f_1 f_2} \right) = t, \ \sqrt{\frac{f_2}{f_1}} v_2 \left(\sqrt{f_1 f_2} \right) = t.$$

From here, taking $\varphi = \sqrt{f_1 f_2}$, we write

$$u = \frac{1}{g(\overline{\varphi})}, \ v_1 = \frac{t\overline{\varphi}}{f_1(\overline{\varphi})}, \ v_2 = \frac{t\overline{\varphi}}{f_2(\overline{\varphi})}, \ \ w = \frac{1}{h(\overline{\varphi})},$$

where $\overline{\varphi}$ is the compositional inverse of φ. Similarly, $(u; v_1, v_2; w) \triangle (g; f_1, f_2; h) = (1; t, t; 1)$. So,

$$(g; f_1, f_2; h)^{-1} = \left(\frac{1}{g(\overline{\varphi})}; \frac{t\overline{\varphi}}{f_1(\overline{\varphi})}, \frac{t\overline{\varphi}}{f_2(\overline{\varphi})}; \frac{1}{h(\overline{\varphi})} \right) \in \mathcal{DR}^{\langle 3\rangle}.$$

Thus, we have the proof. □

$\left\langle \mathcal{DR}^{\langle 3\rangle}, \triangle \right\rangle$ is called double 3−dimensional Riordan group. Here is a list of some special subgroups of the group $\mathcal{DR}^{\langle 3\rangle}$:

1. $\left\{ (g; t, t; h) \ | \ g, h \in \mathcal{F}_0^2 \right\}, \left\{ (1; t, t; h) \ | \ h \in \mathcal{F}_0^2 \right\}.$
2. $\left\{ (1; f_1, f_2; h) \ | \ h \in \mathcal{F}_0^2, \ f_i \in \mathcal{F}_1^2 \right\}, \left\{ (g; f_1, f_2; 1) \ | \ g \in \mathcal{F}_0^2, \ f_i \in \mathcal{F}_1^2 \right\}.$
3. $\left\{ (g; tg, f_2; h) \ | \ g, h \in \mathcal{F}_0^2, \ f_2 \in \mathcal{F}_1^2 \right\}, \left\{ (g; f_1, tg; h) \ | \ g, h \in \mathcal{F}_0^2, \ f_1 \in \mathcal{F}_1^2 \right\}.$
4. $\left\{ \left(\frac{f_1}{t}; f_1, f_2; h \right) \middle| h \in \mathcal{F}_0^2, \ f_i \in \mathcal{F}_1^2 \right\}, \left\{ \left(\frac{f_2}{t}; f_1, f_2; h \right) \middle| h \in \mathcal{F}_0^2, f_i \in \mathcal{F}_1^2 \right\}.$
5. $\left\{ (gh; f_1, tg; h) \,|\, g, h \in \mathcal{F}_0^2, \ f_1 \in \mathcal{F}_1^2 \right\}, \left\{ (gh; tg, f_2; h) \ |g, h \in \mathcal{F}_0^2, f_2 \in \mathcal{F}_1^2 \right\}.$
6. $\left\{ (g; f_1, t; 1) \ | \ g \in \mathcal{F}_0^2, \ f_1, \in \mathcal{F}_1^2 \right\}, \ \left\{ (g; t, f_2; 1) \ | \ g \in \mathcal{F}_0^2 \ , f_2 \in \mathcal{F}_1^2 \right\}.$
7. $\left\{ \left(g; f_1, f_2; \frac{f_1}{t} \right) \middle| \ g \in \mathcal{F}_0^2, \ f_i \in \mathcal{F}_1^2 \right\}, \left\{ \left(g; f_1, f_2; \frac{f_2}{t} \right) \middle| \ g \in \mathcal{F}_0^2, f_i \in \mathcal{F}_1^2 \right\}.$

Definition 2.2 *Let $(r_{n,k,l})_{n,k,l\geq 0} = (g; f_1, f_2; h)$ be double 3−dimensional Riordan array. Then the double 3−dimensional quasi-Riordan array associated with $(r_{n,k,l})_{n,k,l\geq 0}$, which is denoted by $(\hat{r}_{n,k,l})_{n,k,l\geq 0}$, is defined by*

$$\hat{r}_{n,k,l} := r_{2n-k,k,l}, \ n \geq k \geq 0 \text{ and } l \geq 0. \tag{2.3}$$

Theorem 2.3 *Let $(r_{n,k,l})_{n,k,l\geq 0} = (g; f_1, f_2; h)$ be a double 3−dimensional Riordan array and the double 3−dimensional quasi-Riordan array associated with $(r_{n,k,l})_{n,k,l\geq 0}$ be $(\hat{r}_{n,k,l})_{n,k,l\geq 0}$. Then*

$$\hat{r}_{n,k,l} = \begin{cases} [t^n] \, \hat{g} \left(\hat{f}_1 \hat{f}_2 \right)^{k/2} \hat{h}^l, & \text{if } k \text{ is even}, \\ [t^n] \, \hat{g} \hat{f}_1 \left(\hat{f}_1 \hat{f}_2 \right)^{(k-1)/2} \hat{h}^l, & \text{if } k \text{ is odd}, \end{cases} \tag{2.4}$$

where $\hat{g}(t) = \sum_{n\geq 0} g_{2n} t^n$, $\hat{h}(t) = \sum_{n\geq 0} h_{2n} t^n$, $\hat{f}_1(t) = \sum_{n\geq 0} f_{1,2n+1} t^{n+1}$, and $\hat{f}_2(t) = \sum_{n\geq 0} f_{2,2n+1} t^{n+1}$.

Proof: For $k = 2m$, from (2.3), we have

$$\hat{r}_{n,2m,l} = r_{2n-2m,2m,l} = [t^{2n-2m}] g (f_1 f_2)^m h^l = [t^{2n}] g (t^2 f_1 f_2)^m h^l.$$

In that case, taking $t^2 \to t$, we get $g(t^{1/2}) = \hat{g}(t)$, $t^{1/2} f_1(t^{1/2}) = \hat{f}_1(t)$, $t^{1/2} f_2(t^{1/2}) = \hat{f}_2(t)$ and $h(t^{1/2}) = \hat{h}(t)$. Then

$$\hat{r}_{n,k,l} = [t^n] \hat{g} \left(\hat{f}_1 \hat{f}_2 \right)^m \hat{h}^l = [t^n] \hat{g} \left(\hat{f}_1 \hat{f}_2 \right)^{k/2} \hat{h}^l.$$

The case of $k = 2m + 1$ can be proven similar way. Thus, we have the proof. □

Definition 2.3 *Let* $R = (r_{n,k,l})_{n,k,l\geq 0} = (g; f_1, f_2; h)$ *be double* $3-$*dimensional Riordan array and the double* $3-$*dimensional quasi-Riordan array associated with* R *be* $\hat{R} = (\hat{r}_{n,k,l})_{n,k,l\geq 0} = (\hat{g}; \hat{f}_1, \hat{f}_2; \hat{h})$*, where* $\hat{g}(t) = \sum_{n\geq 0} g_{2n}t^n$*,* $\hat{h}(t) = \sum_{n\geq 0} h_{2n}t^n$*,* $\hat{f}_1(t) = \sum_{n\geq 0} f_{1,2n+1}t^{n+1}$*, and* $\hat{f}_2(t) = \sum_{n\geq 0} f_{2,2n+1}t^{n+1}$*. Then the* $3-$*dimensional matrix* $R_s = (r^s_{n,k,l})_{n,k,l\geq 0} = (g; f_1, f_2; t^2h)_s$ *is called shifted double* $3-$*dimensional Riordan array of* R *by*

$$\begin{aligned} r^s_{n,k,l} &:= \begin{cases} \left[t^{n-2l}\right] \hat{g}\left(\hat{f}_1\hat{f}_2\right)^{k/2}\hat{h}^l & \text{if } k \text{ is even,} \\ \left[t^{n-2l}\right] \hat{g}\hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^{(k-1)/2}\hat{h}^l & \text{if } k \text{ is odd,} \end{cases} \\ &= r_{n-2l,k,l}, \end{aligned}$$

and the shifted double $3-$*dimensional quasi-Riordan array associated with* R*, denoted by* $\hat{R}_s = (\hat{r}^s_{n,k,l})_{n,k,l\geq 0} = (\hat{g}; \hat{f}_1, \hat{f}_2; t\hat{h})_s$ *is defined by*

$$\begin{aligned} \hat{r}^s_{n,k,l} &:= [t^n]\begin{cases} \left[t^{n-l}\right] \hat{g}\left(\hat{f}_1\hat{f}_2\right)^{k/2}\hat{h}^l & \text{if } k \text{ is even,} \\ \left[t^{n-l}\right] \hat{g}\hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^{(k-1)/2}\hat{h}^l & \text{if } k \text{ is odd,} \end{cases} \\ &= \hat{r}_{n-l,k,l}. \end{aligned} \tag{2.5}$$

For example, choosing $3-$dimensional double Riordan array as $R = \left(\frac{1}{1-t^2}; t, \frac{t}{1-t^2}; \frac{1}{1-t^2}\right)$, we have the following matrices:

$R=\left(\frac{1}{1-t^2};t,\frac{t}{1-t^2};\frac{1}{1-t^2}\right)$	$\hat{R}=\left(\frac{1}{1-t};t,\frac{t}{1-t};\frac{1}{1-t}\right)$
$\begin{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & 0 \\ 2 & 0 & 1 & & \\ 0 & 2 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & 0 \\ 3 & 0 & 1 & & \\ 0 & 3 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \vdots \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & 0 \\ 1 & 1 & 1 & & \\ 1 & 1 & 2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & 0 \\ 3 & 2 & 1 & & \\ 4 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & 0 \\ 6 & 3 & 1 & & \\ 10 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \vdots \end{bmatrix}$
$R_s=\left(\frac{1}{1-t^2};t,\frac{t}{1-t^2};\frac{t^2}{1-t^2}\right)_s$	$\hat{R}_s=\left(\frac{1}{1-t};t,\frac{t}{1-t};\frac{t}{1-t}\right)_s$
$\begin{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & 0 \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \vdots \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & 0 \\ 1 & 1 & 1 & & \\ 1 & 1 & 2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & 0 \\ 2 & 1 & 0 & & \\ 3 & 2 & 1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & 0 \\ 1 & 0 & 0 & & \\ 3 & 1 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ \vdots \end{bmatrix}$

Theorem 2.4 *Let $(\hat{r}^s_{n,k,l})_{n,k,l\geq 0} = (\hat{g};\hat{f}_1,\hat{f}_2;\hat{h})_s$ be shifted double 3−dimensional quasi-Riordan array where $\hat{g}\in\mathcal{F}_0$, $\hat{f}_1,\hat{f}_2,\hat{h}\in\mathcal{F}_1$. Then*

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\hat{r}^s_{n,2k,l}a_kb_l = [t^n]\hat{g}A\left(\hat{f}_1\hat{f}_2\right)B(\hat{h}), \tag{2.6}$$

and

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\hat{r}^s_{n,2k+1,l}a_kb_l = [t^n]\hat{g}\hat{f}_1A\left(\hat{f}_1\hat{f}_2\right)B(\hat{h}), \tag{2.7}$$

where $A(t)=\sum_{n\geq 0}a_nt^n$ and $B(t)=\sum_{n\geq 0}b_nt^n$.

Proof: We take double quasi-Riordan array as $(\hat{g}(\hat{h}/t)^l;\hat{f}_1,\hat{f}_2) = (\hat{c}_{n,k})_{n,k\geq 0}$. For first identity, by (2.1), we write

$$\hat{c}_{n,2k} = [t^n]\hat{g}\left(\frac{\hat{h}}{t}\right)^l\left(\hat{f}_1\hat{f}_2\right)^k = \left[t^{n+l}\right]\hat{g}\hat{h}^l\left(\hat{f}_1\hat{f}_2\right)^k. \tag{2.8}$$

From here, by replacing $n \longrightarrow (n-l)$ in (2.8), by (2.4) and Definition 2.3, we have

$$\hat{c}_{n-l,2k} = [t^n]\hat{g}\hat{h}^l \left(\hat{f}_1\hat{f}_2\right)^k = \hat{r}^s_{n,2k,l}.$$

With the help of (2.1), we get

$$\sum_{k=0}^{n} \hat{c}_{n,2k}a_k = [t^n]\hat{g}\frac{\hat{h}^l}{t^l}A\left(\hat{f}_1\hat{f}_2\right) = \left[t^{n+l}\right]\hat{g}\hat{h}^l A\left(\hat{f}_1\hat{f}_2\right),$$

and then

$$\sum_{k=0}^{n-l} \hat{r}^s_{n,2k,l}a_k = [t^n]\hat{g}\hat{h}^l A\left(\hat{f}_1\hat{f}_2\right).$$

Let $(\hat{g}A(\hat{f}_1\hat{f}_2); t, \hat{h}/t) = (\hat{u}_{n,k})_{n,k\geq 0}$ be double quasi-Riordan array. From (2.1), we have

$$\sum_{l=0}^{n}\left(\sum_{k=0}^{n-l} \hat{r}^s_{n,2k,l}a_k\right)b_l = \sum_{l=0}^{n}\left([t^n]\hat{g}A\left(\hat{f}_1\hat{f}_2\right)t^l\left(\frac{\hat{h}}{t}\right)^l\right)b_l = \sum_{l=0}^{n}\hat{u}_{n,2l}b_l = [t^n]\hat{g}A\left(\hat{f}_1\hat{f}_2\right)B(\hat{h}),$$

as claimed. For second identity, we will use same way. By (2.2), we write

$$\hat{c}_{n,2k+1} = [t^n]\hat{g}\left(\frac{\hat{h}}{t}\right)^l \hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^k = \left[t^{n+l}\right]\hat{g}\hat{h}^l\hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^k. \tag{2.9}$$

From here, by replacing $n \longrightarrow (n-l)$ in (2.9) and by (2.4) we have

$$\hat{c}_{n-l,2k+1} = [t^n]\hat{g}\hat{h}^l\hat{f}_1\left(\hat{f}_1\hat{f}_2\right)^k = \hat{r}^s_{n,2k+1,l}.$$

With the help of (2.2), we get

$$\sum_{k=0}^{n} \hat{c}_{n,2k+1}a_k = [t^n]\hat{g}\frac{\hat{h}^l}{t^l}\hat{f}_1 A\left(\hat{f}_1\hat{f}_2\right) = \left[t^{n+l}\right]\hat{g}\hat{h}^l\hat{f}_1 A\left(\hat{f}_1\hat{f}_2\right),$$

and then

$$\sum_{k=0}^{n-l} \hat{r}^s_{n,2k+1,l}a_k = [t^n]\hat{g}\hat{h}^l\hat{f}_1 A\left(\hat{f}_1\hat{f}_2\right).$$

Let $(\hat{g}A(\hat{f}_1\hat{f}_2)\hat{f}_1/t; t, \hat{h}/t) = (\hat{v}_{n,k})_{n,k\geq 0}$ be double quasi-Riordan array. From (2.2), we get

$$\begin{aligned}
\sum_{l=0}^{n}\left(\sum_{k=0}^{n-l} \hat{r}^s_{n,2k,l}a_k\right)b_l &= \sum_{l=0}^{n}\left([t^n]\hat{g}A\left(\hat{f}_1\hat{f}_2\right)\frac{\hat{f}_1}{t}t^{l+1}\left(\frac{\hat{h}}{t}\right)^l\right)b_l \\
&= \sum_{l=0}^{n}\hat{v}_{n,2l+1}b_l \\
&= [t^n]\left(\hat{g}A\left(\hat{f}_1\hat{f}_2\right)\frac{\hat{f}_1}{t}\right)tB(\hat{h}) \\
&= [t^n]\hat{g}\hat{f}_1 A\left(\hat{f}_1\hat{f}_2\right)B(\hat{h}).
\end{aligned}$$

So, the proof is complete. □

3. Applications of double 3-dimensional Riordan arrays

In this section, we will give some applications of double $3-$dimensional Riordan arrays.

Theorem 3.1 *Let n be positive integer. Then following identities hold:*

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{l}H_{n-2k-l+1}^{k+l+1}=\sum_{i=0}^{n}\frac{F_{i-1}}{n-i+1}, \tag{3.1}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{l}H_{n-2k-l}^{k+l+2}=\sum_{i=0}^{n}\frac{F_{i}}{n-i+1}. \tag{3.2}$$

Proof: Let $R=(r_{n,k,l})_{n,k,l\geq 0}=\left(\frac{-\log(1-t^2)}{t^2(1-t^2)};\frac{t}{1-t^2},t;\frac{1}{1-t^2}\right)$ be double $3-$dimensional Riordan array. Then we have shifted double $3-$dimensional Riordan array of R as $\left(\frac{-\log(1-t^2)}{t^2(1-t^2)};\frac{t}{1-t^2},t;\frac{t^2}{1-t^2}\right)_s$ and double $3-$dimensional quasi-Riordan array associated with R as $\left(\frac{-\log(1-t)}{t(1-t)};\frac{t}{1-t},t;\frac{1}{1-t}\right)$. Then the shifted double $3-$dimensional quasi-Riordan array associated with R is $(\hat{r}^s_{n,k,l})_{n,k,l\geq 0}=\left(\frac{-\log(1-t)}{t(1-t)};\frac{t}{1-t},t;\frac{t}{1-t}\right)_s$. From here, by (1.3), (2.4) and (2.5), we write

$$\begin{aligned}\hat{r}^s_{n,2k,l}=\hat{r}_{n-l,2k,l}&=[t^{n-l}]\frac{-\log(1-t)}{t(1-t)}\left(\frac{t}{1-t}\right)^k t^k\left(\frac{1}{1-t}\right)^l=\left[t^{n-2k-l+1}\right]\frac{-\log(1-t)}{(1-t)^{k+l+1}}\\&=\left[t^{n-2k-l+1}\right]\sum_{n=0}^{\infty}H_n^{k+l+1}t^n=H_{n-2k-l+1}^{k+l+1}.\end{aligned}$$

Thus, choosing $A(t)=\frac{1}{1-t}$ and $B(t)=\frac{1}{1+t}$ in (2.6), by (1.5) , then

$$\begin{aligned}\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{l}H_{n-2k-l+1}^{k+l+1}&=[t^n]\frac{-\log(1-t)}{t(1-t)}\frac{1}{1-\frac{t^2}{1-t}}\frac{1}{1+\frac{t}{1-t}}\\&=\left[t^{n+1}\right](-\log(1-t))\frac{1-t}{1-t-t^2}\\&=\left[t^{n+1}\right]\sum_{n=1}^{\infty}\frac{t^n}{n}\sum_{n=0}^{\infty}F_{n-1}t^n\\&=\sum_{i=0}^{n}\frac{F_{i-1}}{n-i+1}.\end{aligned}$$

So, (3.1) holds. For second identity, using functions $A(t)$, $B(t)$ and (2.7), we have

$$\begin{aligned}\hat{r}_{n,2k+1,l}&=[t^{n-l}]\frac{-\log(1-t)}{(1-t)t}\left(\frac{t}{1-t}\right)^{k+1}t^k\left(\frac{1}{1-t}\right)^l\\&=\left[t^{n-2k-l}\right]\frac{-\log(1-t)}{(1-t)^{k+l+2}}=\left[t^{n-2k-l}\right]\sum_{n=0}^{\infty}H_n^{k+l+1}t^n\\&=H_{n-2k-l}^{k+l+2},\end{aligned}$$

and

$$\begin{aligned}\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^l H_{n-2k-l}^{k+l+2} &= [t^n]\frac{-\log(1-t)}{t(1-t)}\frac{t}{1-t}\frac{1}{1-\frac{t^2}{1-t}}\frac{1}{1+\frac{t}{1-t}}\\ &= \left[t^{n+1}\right](-\log(1-t))\frac{t}{1-t-t^2}\\ &= \left[t^{n+1}\right]\sum_{n=1}^{\infty}\frac{t^n}{n}\sum_{n=0}^{\infty}F_n t^n\\ &= \sum_{i=0}^{n}\frac{F_i}{n-i+1},\end{aligned}$$

as claimed. So, we have identity (3.2). □

By Table 1, using same method in the proof of Theorem 3.1, identities in Table 2 can be find.

Table 1: $\hat{R}_s$ arrays and generating functions $A(t), B(t)$ for identities in Table 2.

Identities	$\hat{R}_s$	$A(t)$	$B(t)$
(3.3)	$\left(\frac{1}{1-t};\frac{t}{1-t},t;\frac{t}{1-t}\right)$	$\frac{1}{1-t}$	$\frac{1}{1+t}$
(3.4)	$\left(\frac{-\log(1-t)}{t(1-t)(1-pt)};\frac{qt}{1-pt},t;\frac{t}{1-t}\right)$	$\frac{1}{1-t}$	$\frac{1}{1+t}$
(3.5)	$\left(\frac{-\log(1+t)}{t\,(1+t)};\frac{t}{1+t},t;t\right)$	$\frac{1}{1-t}$	$\frac{t}{\log(1+t)}$
(3.6)	$\left(\frac{1}{1-t^2};\frac{t}{1-t},\frac{t}{1+t};\frac{t}{1-t^2}\right)$	$\frac{1}{1+t}$	$\frac{1}{1-t}$
(3.7)	$\left(\frac{e^{-t}}{1-t};\frac{t}{1-t},t;\frac{t}{1-t}\right)$	$\frac{1}{1-t}$	$\frac{1}{1+t}$
(3.8)	$\left(\frac{1}{1+t};\frac{t}{1+t},t;\frac{t}{1+t}\right)$	$\frac{1}{1+t}$	$\frac{1}{1-t}$
(3.9)	$\left(\frac{\log(1+t)}{t\,(1+t)};\frac{t}{1+t},t;t(1+t)\right)$	$\frac{1}{1-t}$	$\frac{-1+\sqrt{1+4t}}{2t}$
(3.10)	$\left(\frac{-\log(1-t)}{t(1-t)^2};\frac{t}{1-t},t;\frac{-\log(1-t)}{1-t}\right)$	$\frac{1}{1-t}$	$W_0(t)$
(3.11)	$\left(\frac{1}{1-t^2};\frac{t}{1+t},\frac{t}{1-t};\frac{t}{1+t}\right)$	$\frac{1}{1+t}$	$-\log(1-t)$
(3.12)	$\left(\frac{1}{1-t^2};\frac{t}{1+t},\frac{t}{1-t};\frac{t}{1-t}\right)$	$\frac{1}{1+t}$	$\log(1+t)$
(3.13)	$\left(\frac{1}{(1-t)^2};\frac{t}{1-t},t;\frac{-xt}{1-t}e^{-xt/(1-t)}\right)$	$\frac{1}{1-t}$	$W_0(t)$
(3.14)	$\left(\frac{1}{(1-t)^2};\frac{t}{1-t},t;\frac{-\log(1-t)}{1-t}\right)$	$\frac{1}{1-t}$	$W_0(t)$

(3.15)	$\left(\frac{-\log(1+t)e^{-t}}{t(1-t^2)}; \frac{t}{1-t}, \frac{t}{1+t}; t\right)$	$\frac{1}{1+t}$	$\frac{e^t}{1+t}$

In Table 2, $\left(\frac{\cdot}{3}\right)$ is the Legendre symbol and $I_n = \sum_{i=1}^{n} \frac{(-1)^i}{i}$ is the alternating harmonic number.

Table 2: Identities are obtained by using 3−dimensional double Riordan arrays and generating functions in Table 1.

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^l\binom{n-k}{k+l} = F_{n-1}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^l\binom{n-k}{k+l+1} = F_n \tag{3.3}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-l-2k+1}(-1)^l q^k p^j\binom{j+k}{j}H_{n-l-2k-j+1}^{l+1} = \sum_{j=0}^{n}\frac{U_{j+1}}{n-j+1}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-l-2k}(-1)^l q^k p^j\binom{j+k+1}{j}H_{n-l-2k-j}^{l+1} = \sum_{j=0}^{n}\sum_{i=0}^{j}\frac{p^i U_{j-i}}{n-j+1} \tag{3.4}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\frac{(-1)^l H_{n-2k-l+1}^{k+1}\mathcal{C}_l}{l!} = F_{n+1}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\frac{(-1)^l H_{n-2k-l}^{k+2}\mathcal{C}_l}{l!} = F_{n+2}-1 \tag{3.5}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n+l}\frac{(-1)^{l+j} d_{n+l-j}^{k+l} d_j^{l+k}}{(n+l-j)!j!} = F_n$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n+l}\frac{(-1)^{l+j} d_{n+l-j}^{k+l+1} d_j^{l+k}}{(n+l-j)!j!} = F_{n+1} \tag{3.6}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n+l}\frac{(-1)^{l+j+1} d_{n+l-j}^{k+l} d_j^{l+k+1}}{(n+l-j)!j!} = F_{n-2}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{l}\,\frac{d_{n-k}^{l+k}}{(n-k)!}=\sum_{j=0}^{n}\frac{(-1)^{j}\,F_{n-j-1}}{j!}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{l}\,\frac{d_{n-k}^{l+k+1}}{(n-k)!}=\sum_{j=0}^{n}\frac{(-1)^{j}\,F_{n-j}}{j!} \tag{3.7}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{n-l-k-1}\binom{n-k}{l+k}=\left(\frac{n+2}{3}\right)$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}(-1)^{n-l-k-1}\binom{n-k}{l+k+1}=\left(\frac{n}{3}\right) \tag{3.8}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k-l}(-1)^{n+l+j}\,\frac{\binom{j+k}{j}C_{l}D_{n-2k-l-j}\,(l)}{(n-2k-l-j)!}=\sum_{j=0}^{n+1}H_{n-j+1}F_{j+1}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k-l-1}(-1)^{n+l+j+1}\,\frac{\binom{j+k+1}{j}C_{l}D_{n-2k-l-j-1}\,(l)}{(n-2k-l-j-1)!}=\sum_{j=0}^{n}H_{n-j}^{2}F_{j+1} \tag{3.9}$$

$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k}\frac{(-l)^{l-1}\binom{j+k+l}{j}}{l!}H\left(n-2k-j+1,l\right)=\sum_{j=0}^{n+1}H\left(n-j+1,1\right)F_{j+1}$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k}\frac{(-l)^{l-1}\binom{j+k+l+1}{j}}{l!}H\left(n-2k-j,l\right)=\sum_{j=0}^{n}\sum_{i=0}^{j}H(j-i,1)F_{i+1} \tag{3.10}$$

$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k-l}\frac{(-1)^{k+j}\binom{n-k-l-j}{k}\binom{j+k+l}{j}}{l}=\frac{(-1)^{n+1}}{n}$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k-l-1}\frac{(-1)^{k+j}\binom{n-k-l-j-1}{k}\binom{j+k+l+1}{j}}{l}=(-1)^{n}H_{n-1} \tag{3.11}$$

$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n}\frac{(-1)^{l+j+1}\,d_{n-j}^{l+k}d_j^k}{j!(n-j)!l}=\frac{1}{n}$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n}\frac{(-1)^{n+l+j+1}\,d_{n-j}^{l+k}d_j^{k+1}}{j!(n-j)!l}=I_{n-1} \tag{3.12}$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n}\frac{(-1)^{l+j+1}\,d_{n-j}^{l+k+1}d_j^k}{j!(n-j)!l}=H_{n-1}$$

$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-l-2k}\frac{\binom{j+k+l}{j}(lx)^{l-1}L_{n-l-j-2k}(lx)}{l!}=F_{n+4}-n-3$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-l-2k-1}\frac{\binom{j+k+l+1}{j}(lx)^{l-1}L_{n-l-j-2k-1}(lx)}{l!}=F_{n+5}-\frac{n^2+5n+10}{2} \tag{3.13}$$

$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k}\frac{(-l)^{l-1}\binom{j+k+l}{j}}{l!}H(n-2k-j,l-1)=\sum_{j=0}^{n}H_{n-j}F_{j+1}$$
$$\sum_{l=1}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-2k-1}\frac{(-l)^{l-1}\binom{j+k+l+1}{j}}{l!}H(n-2k-j-1,l-1)=\sum_{j=0}^{n-1}H_{n-j-1}^2F_{j+1} \tag{3.14}$$

$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-k-l+1}\frac{(-1)^j\,d_ld_j^kH_{n-k-l-j+1}^{k+1}}{l!j!}=H_{n+1}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{j=0}^{n-k-l+1}\frac{(-1)^{n+j+1}\,d_ld_j^{k+1}H_{n-k-l-j+1}^{k+1}}{l!j!}=\sum_{j=0}^{n}(-1)^jH_j \tag{3.15}$$
$$\sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{i=0}^{n-k-l}\frac{(-1)^j\,d_ld_j^kH_{n-k-l-j}^{k+2}}{l!j!}=H_n^2$$

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Ömer Duran,
Department of Mathematics,
Kocaeli University,
41380 İzmit, Kocaeli, Turkey.
E-mail address: omer20841@gmail.com

and

Neşe Ömür,
Department of Mathematics,
Kocaeli University,
41380 İzmit, Kocaeli, Turkey.
E-mail address: neseomur@gmail.com

and

Sibel Koparal,
Department of Mathematics,
Bursa Uludağ University,
16059 Nilüfer, Bursa, Turkey.
E-mail address: sibelkoparal1@gmail.com