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Pseudospectrum and Essential Pseudospectrum of Operator Pencils on Ultrametric Banach Spaces

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ABSTRACT: In this paper, we introduce and study the bounded linear operator pencils, the pseudospectrum and the essential pseudospectrum of bounded linear operator pencils on ultrametric Banach spaces. We prove that the essential pseudospectrum of a bounded linear operator pencil is invariant under perturbation of compact operators on ultrametric Banach spaces over the field of p-adic numbers $\mathbb{Q}_p$. Finally, we give a characterization of the essential pseudospectrum of bounded linear operator pencils by means of the spectra of all perturbed compact operators in ultrametric Banach spaces.

Key Words: Ultrametric Banach spaces, operator pencils, pseudospectrum, essential pseudospectrum.

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1. Introduction

L. N. Trefethen and his co-workers [13] introduced and studied the pseudospectra of matrices and operators. For a given $N \times N$ matrix A and a given $\varepsilon > 0$, the ε-pseudospectrum $\sigma_\varepsilon(A)$ of the matrix A is a set in the complex plane $\mathbb{C}$ given by

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} : \|(\lambda I - A)^{-1}\| > \varepsilon^{-1}\}, \tag{1.1}$$

with the convention $\|(\lambda I - A)^{-1}\| = \infty$ whenever λ is an eigenvalue of A where I is the identity matrix. The definition of the ε-pseudospectrum of the matrix A (1.1) is equivalent to

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} : \exists C \in \mathbb{C}^{N\times N} \text{ with } \|C\| < \varepsilon \text{ such that } \lambda \in \sigma(A + C)\} \tag{1.2}$$

where $\sigma(A)$ is the spectrum (set of eigenvalues) of A which explains the name pseudospectrum (see [13]). When the matrix A is normal, its behavior in numerical processes is determined by the location of the eigenvalues. For non-normal matrices, however, this does not need to be true predictions based on eigenvalues may lead to wrong results. These phenomena occur in the convergence behavior of iterative methods for linear equations or in the stability behavior of time-stepping methods for ordinary and partial differential equations. The definitions (1.1) and (1.2) are immediately connected to the eigenvalue problem $Av = \lambda v$. However, in many applications one has to deal with the generalized eigenvalue problem

$$Av = \lambda Bv, \tag{1.3}$$

where B is a given $N \times N$ matrix which may be singular (see [4]).
In [4], J. L. M. van Dorsselaer introduced the ε-pseudospectra of matrix pencils, so that there is a connection with the generalized eigenvalue problem (1.3). For a given $\varepsilon > 0$, the ε-pseudospectrum of a matrix pencil (A, B) of the form $A - \lambda B$ is given by

$$\Lambda_\varepsilon(A, B) = \{\lambda \in \mathbb{C} : \|(A - \lambda B)^{-1}\| \geq \varepsilon^{-1}\}, \tag{1.4}$$

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with the convention $\|(A-\lambda B)^{-1}\| = \infty$ whenever λ is an eigenvalue of (A, B). Definition (1.4) is equivalent to

$$\Lambda_\varepsilon(A, B) = \{\lambda \in \mathbb{C} : \exists C \in \mathbb{C}^{N\times N} \text{ with } \|C\| \leq \varepsilon \text{ such that } \lambda \in \sigma(A + C, B)\},$$

where $\sigma(A, B)$ is the spectrum of the matrix pencil (A, B) (see [4]).

Throughout this paper, E and F are ultrametric Banach spaces over a non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $\mathbb{Q}_p$ is the field of p-adic numbers, I is the identity operator on E and $\mathcal{L}(E, F)$ denotes the set of all bounded linear operators from E into F. If $E = F$, we put $\mathcal{L}(E, E) = \mathcal{L}(E)$. If $F = \mathbb{K}$, $E^* = \mathcal{L}(E, \mathbb{K})$ is the dual space of E. For $A \in \mathcal{L}(E)$, $N(A)$ and $R(A)$ denote the kernel and the range of A respectively. For more details, see [3] and [12]. Recall that, an unbounded linear operator $S : D(S) \subseteq E \to F$ is closed if for any $(w_n)_{n\in\mathbb{N}} \subset D(S)$ such that $\|w_n - w\| \to 0$ and $\|Sw_n - y\| \to 0$ as $n \to \infty$, for some $w \in E$ and $y \in F$, then $w \in D(S)$ and $y = Sw$. The collection of closed linear operators from E into F is denoted by $\mathcal{C}(E, F)$. When $E = F$, we set $\mathcal{C}(E, E) = \mathcal{C}(E)$. If $A \in \mathcal{L}(E)$ and S is an unbounded linear operator, then $A + S$ is closed if and only if S is closed (see [3]).

2. Preliminaries

We start with some preliminaries.

Definition 2.1 *[3] Let $\omega = (\omega_i)_i$ be a sequence of $\mathbb{K}\backslash\{0\}$. We define $\mathbb{E}_\omega$ by*

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N},\ x_i \in \mathbb{K},\ \textit{and} \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathbb{E}_\omega : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

Remark 2.1 [3]

(i) The space $(\mathbb{E}_\omega, \|\cdot\|)$ is an ultrametric Banach space.

(ii) If

$$\begin{aligned} \langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega &\longrightarrow \mathbb{K} \\ (x, y) &\mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i, \end{aligned}$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then the space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is called an ultrametric Hilbert space.

(iii) The orthogonal basis $\{e_i, i \in \mathbb{N}\}$ is called the canonical basis of $\mathbb{E}_\omega$ where for all $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 2.2 *[11] Let $A \in \mathcal{L}(E, F)$. Then A is called an upper semi-Fredholm operator if*

$$\alpha(A) = \dim N(A) \textit{ is finite and } R(A) \textit{ is closed.}$$

The set of all upper semi-Fredholm operators from E into F is denoted by $\Phi_+(E, F)$.

Definition 2.3 *[11] Let $A \in \mathcal{L}(E, F)$. Then A is said to be a lower semi-Fredholm operator if $\beta(A) = \dim(F/R(A))$ is finite. The set of all lower semi-Fredholm operators is denoted by $\Phi_-(E, F)$.*

The set of all Fredholm operators from E into F is defined by

$$\Phi(E, F) = \Phi_+(E, F) \cap \Phi_-(E, F).$$

Let $A \in \Phi(E, F)$, the index $ind(A)$ of A is defined by $ind(A) = \alpha(A) - \beta(A)$. We have the following:

Definition 2.4 *[12] Let E and F be two ultrametric Banach spaces over $\mathbb{K}$. A bounded linear map $A : E \to F$ is compact if $A(B_E)$ is compactoid in F, where $B_E = \{x \in E : \|x\| \leq 1\}$.*

The set of all compact operators from E into F is denoted $\mathcal{K}(E, F)$.

Definition 2.5 *[12] Let $A \in \mathcal{L}(E, F)$. Then A is called an operator of finite rank if $\dim R(A)$ is finite.*

Theorem 2.1 *[12] Let $A \in \mathcal{L}(E, F)$. Then A is compact if, and only if, for every $\varepsilon > 0$, there exists $B \in \mathcal{L}(E, F)$ such that $R(B)$ is finite-dimensional and $\|A - B\| < \varepsilon$.*

Definition 2.6 *[3] Let E be an ultrametric Banach space and $A \in \mathcal{L}(E)$. A is said to be completely continuous if, there exists a sequence of finite rank linear operators $(A_n)_{n\in\mathbb{N}}$ such that $\|A_n - A\| \to 0$ as $n \to \infty$.*

The collection of completely continuous linear operators on E is denoted by $\mathcal{C}_c(E)$.

Remark 2.2 [12]

(i) In the ultrametric Banach space E, we do not have the relationship between $\mathcal{C}_c(E)$ and $\mathcal{K}(E)$ as a classical case. J. P. Serre has proved that those concepts coincide, when $\mathbb{K}$ is locally compact.

(ii) If $\mathbb{K}$ is locally compact. Then all completely continuous linear operators on E are compact.

(iii) If $\mathbb{K}$ is locally compact. Then A is compact if, and only if, $A(B_E)$ has compact closure.

Theorem 2.2 *[2] Suppose that $\mathbb{K}$ is spherically complete. Then, for each $A \in \Phi(E, F)$ and $K \in \mathcal{K}(E, F)$, $A + K \in \Phi(E, F)$ and $ind(A + K) = ind(A)$.*

Theorem 2.3 *[12] Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. For each $x \in E\backslash\{0\}$, there exists $x^* \in E^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.*

Theorem 2.4 *[7] Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $A, B \in \mathcal{L}(E)$. Then*

$$\sigma_e(A, B) = \bigcap_{K\in\mathcal{C}_c(E)} \sigma(A + K, B).$$

3. Bounded linear operator pencils on ultrametric Banach spaces

We introduce the following definition.

Definition 3.1 *[5] Let E be an ultrametric Banach space over $\mathbb{K}$. For a pair (A, B) of operators in $\mathcal{L}(E)$, the spectrum $\sigma(A, B)$ of a bounded linear operator pencil (A, B) is defined by*

$$\begin{aligned} \sigma(A, B) &= \{\lambda \in \mathbb{K} : A - \lambda B \quad \text{is not invertible in } \mathcal{L}(E)\} \\ &= \{\lambda \in \mathbb{K} : \ 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

The resolvent set $\rho(A, B)$ of the bounded linear operator pencil (A, B) is the complement of $\sigma(A, B)$ in $\mathbb{K}$ given by

$$\rho(A, B) = \{\lambda \in \mathbb{K} : \ R_\lambda(A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(E)\}.$$

$R_\lambda(A, B)$ is called the resolvent of the bounded linear operator pencil (A, B).

Remark 3.1 Let $A, B \in \mathcal{L}(E)$, we have:

(i) If $B = I$, then the spectrum of the bounded linear operator pencil (A, I) is the spectrum of A.

(ii) If $\dim E < \infty$, then the spectrum $\sigma(A, B)$ coincides with $\mathbb{K}$ or contains no more than n points.

Example 3.1 Let $\mathbb{K} = \mathbb{Q}_p$ and $a, b, c, d \in \mathbb{Q}_p^*$.

(i) If

$$A=\begin{pmatrix}0&0\\0&1\end{pmatrix} \text{ and } B=\begin{pmatrix}a&0\\0&0\end{pmatrix}.$$

Then it is easy to see that, for all $\lambda\in\mathbb{Q}_p$, $\det(A-\lambda B)=-\lambda a$, hence $\sigma(A,B)=\{0\}$.

(ii) If

$$A=\begin{pmatrix}a&b\\0&c\end{pmatrix} \text{ and } B=\begin{pmatrix}0&d\\0&0\end{pmatrix}.$$

Then we have for all $\lambda\in\mathbb{Q}_p$, $\det(A-\lambda B)=ac$, hence $\sigma(A,B)=\emptyset$.

(iii) If

$$A=\begin{pmatrix}a&b\\0&0\end{pmatrix} \text{ and } B=\begin{pmatrix}c&0\\0&0\end{pmatrix}.$$

Then, for all $\lambda\in\mathbb{Q}_p$, $\det(A-\lambda B)=0$, thus $\sigma(A,B)=\mathbb{Q}_p$.

(iv) If

$$A=\begin{pmatrix}a&c\\0&b\end{pmatrix} \text{ and } B=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Then, for all $\lambda\in\mathbb{Q}_p$, $\det(A-\lambda B)=(a-\lambda)(b-\lambda)$, thus $\sigma(A,B)=\{a,b\}$.

We have the following propositions.

Proposition 3.1 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A,B\in\mathcal{L}(E)$. Then*

$$\text{for all } \lambda,\mu\in\rho(A,B), R_\lambda(A,B)-R_\mu(A,B)=(\lambda-\mu)R_\lambda(A,B)BR_\mu(A,B).$$

Proof: For all $\lambda,\ \mu\in\rho(A,\ B)$, we have

$$\begin{aligned} R_\lambda(A,B)-R_\mu(A,B) &= R_\lambda(A,B)[(A-\mu B)-(A-\lambda B)]R_\mu(A,B)\\ &= (\lambda-\mu)R_\lambda(A,B)BR_\mu(A,B). \end{aligned}$$

□

Proposition 3.2 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A,B\in\mathcal{L}(E)$ such that $A^{-1}\in\mathcal{L}(E)$ and $\|A^{-1}B\|<1$, then $A-B$ is invertible. Furthermore,*

$$(A-B)^{-1}=\sum_{n=0}^{\infty}(A^{-1}B)^nA^{-1}$$

and

$$\|(A-B)^{-1}\| \leq \|A^{-1}\|. \tag{3.1}$$

Proof: Let $A,B\in\mathcal{L}(E)$ such that $A^{-1}\in\mathcal{L}(E)$ and $\|A^{-1}B\|<1$, hence $\lim\limits_{n\to\infty}\|A^{-1}B\|^n=0$, then $(I-A^{-1}B)^{-1}$ exists. Thus $(A-B)^{-1}$ exists.

Furthermore $(A-B)^{-1}=(I-A^{-1}B)^{-1}A^{-1}=\sum\limits_{n=0}^{\infty}(A^{-1}B)^nA^{-1}$. Moreover

$$\begin{aligned} \|(A-B)^{-1}\| &\leq \|\sum_{n=0}^{\infty}(A^{-1}B)^nA^{-1}\|\\ &\leq \max_{n\in\mathbb{N}}\|A^{-1}B\|^n\|A^{-1}\|\\ &= \|A^{-1}\|. \end{aligned}$$

□

Let $r > 0$, set $B(0, r) = \{\lambda \in \mathbb{K} : |\lambda| < r\}$. Assume that $A^{-1}, B \in \mathcal{L}(E)$ such that $\|A^{-1}B\| \neq 0$, we have the following proposition.

Proposition 3.3 *Let $A, B \in \mathcal{L}(E)$ such that $A^{-1} \in \mathcal{L}(E)$, then for all $\lambda \in B(0, \frac{1}{\|A^{-1}B\|})$, $R_\lambda(A, B)$ exists. Furthermore, for each $\lambda \in B(0, \frac{1}{\|A^{-1}B\|})$,*

$$R_\lambda(A, B) = \sum_{n=0}^{\infty} (\lambda A^{-1}B)^n A^{-1}$$

and

$$\|R_\lambda(A, B)\| \leq \|A^{-1}\|.$$

Proof: Let $A, B \in \mathcal{L}(E)$ and $\lambda \in B(0, \frac{1}{\|A^{-1}B\|})$, then $|\lambda|\|A^{-1}B\| < 1$, it suffices to apply Proposition 3.2. □

Proposition 3.4 *Let $A, B \in \mathcal{L}(E)$ such that $AB = BA$ and $B^{-1} \in \mathcal{L}(E)$, then*

$$\sigma(A, B) = \sigma(B^{-1}A).$$

Proof: Let $\lambda \in \rho(B^{-1}A)$, then $B^{-1}A - \lambda I$ is invertible in $\mathcal{L}(E)$. Hence

$$\begin{aligned} A - \lambda B &= B(B^{-1}A - \lambda I) \\ &= (B^{-1}A - \lambda I)B. \end{aligned}$$

Thus $\lambda \in \rho(A, B)$. Then $\sigma(A, B) \subseteq \sigma(B^{-1}A)$. Similarly, we obtain $\sigma(B^{-1}A) \subseteq \sigma(A, B)$. Consequently $\sigma(A, B) = \sigma(B^{-1}A)$. □

In the next proposition, we characterize the spectrum of the pencil of two diagonals operators on $\mathbb{E}_\omega$ over $\mathbb{Q}_p$.

Proposition 3.5 *Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A, B \in \mathcal{L}(\mathbb{E}_\omega)$ be two diagonals operators such that for all $i \in \mathbb{N}$, $Ae_i = a_i e_i$ and $Be_i = b_i e_i$ where $a_i, b_i \in \mathbb{Q}_p$ such that $\lim\limits_{i\to\infty} |a_i| \neq 0$, $\lim\limits_{i\to\infty} |b_i| \neq 0$, $\sup\limits_{i\in\mathbb{N}} |a_i|$ and $\sup\limits_{i\in\mathbb{N}} |b_i|$ are finite. Then $\sigma(A, B) = \{\lambda \in \mathbb{Q}_p : \inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| = 0\}$.*

Proof: It is sufficient to prove that $\lambda \in \rho(A, B)$ if and only if $\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| > 0$. Assume that $\lambda \in \mathbb{Q}_p$ such that $\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| > 0$. Let C be the linear operator defined on $\mathbb{E}_\omega$ by

$$\text{for all } u \in \mathbb{E}_\omega : \ u = (u_i)_{i\in\mathbb{N}}, \ Cu = \sum_{i=0}^{\infty} \frac{u_i e_i}{a_i - \lambda b_i}.$$

It is easy to check that the operator C is well-defined. From

$$\|C\| = \sup_{i\in\mathbb{N}} \frac{\|Ce_i\|}{\|e_i\|} = \sup_{i\in\mathbb{N}} \left| \frac{1}{a_i - \lambda b_i} \right| = \frac{1}{\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i|}$$

is finite, we get $C \in \mathcal{L}(\mathbb{E}_\omega)$. It is easy to see that for $\lambda \in \mathbb{Q}_p$ such that $\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| > 0$, $(A - \lambda B)C = C(A - \lambda B) = I$, that is, $C = (A - \lambda B)^{-1}$ and therefore $\lambda \in \rho(A, B)$.

Conversely, suppose that $\lambda \in \rho(A,B)$ with $\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| = 0$. Clearly, there exists $(\lambda_{j_k})_{k\in\mathbb{N}}$ a subsequence of $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\|(A-\lambda B)e_{j_k}\|}{|\omega_{j_k}|^{\frac{1}{2}}} = \lim_{k\to\infty} |a_k - \lambda b_k| = 0. \tag{3.2}$$

Now using the fact that $(A-\lambda B)C = C(A-\lambda B) = I$, it follows that for all $k \in \mathbb{N}$,

$$\frac{\|(A-\lambda B)e_{j_k}\|}{|\omega_{j_k}|^{\frac{1}{2}}} \geq \frac{1}{\|(A-\lambda B)^{-1}\|} > 0. \tag{3.3}$$

From (3.2) and (3.3), we obtain a contradiction. Then

$$\lambda \in \{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda b_i| > 0\}.$$

Consequently $\sigma(A,B) = \{\lambda \in \mathbb{Q}_p : \inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i| = 0\}$. □

We have the following propositions.

Proposition 3.6 *Let E be an ultrametric Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(E)$ such that $A \neq B$ and B be a non-null operator and $AB = BA$. Then*

(i) For all $\lambda \in \rho(A,B)$,

$$\begin{aligned} AR_\lambda(A,B) &= R_\lambda(A,B)A, \\ BR_\lambda(A,B) &= R_\lambda(A,B)B. \end{aligned}$$

(ii) For all $\lambda, \mu \in \rho(A,B)$, $R_\lambda(A,B)R_\mu(A,B) = R_\mu(A,B)R_\lambda(A,B)$.

Proof:

(i) Since by hypothesis, B commute with A, then the last two operators commute with $A - \lambda B$. In addition, since $\lambda \in \rho(A,B)$, we obtain that B and A commute with $R_\lambda(A,B)$.

(ii) Let $\lambda, \mu \in \rho(A,B)$, we have

$$\begin{aligned} R_\lambda(A,B)R_\mu(A,B) &= [(A-\mu B)(A-\lambda B)]^{-1} \\ &= [-\lambda B(A-\mu B) + A(A-\mu B)]^{-1} \\ &= [(A-\lambda B)(A-\mu B)]^{-1} \\ &= R_\mu(A,B)R_\lambda(A,B). \end{aligned}$$

□

Proposition 3.7 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A \in \mathcal{C}(E)$ and $B \in \mathcal{L}(E)$ such that $A \neq B$ and B be a non-null operator. Then $\rho(A,B)$ is open.*

Proof: If $\rho(A,B) = \emptyset$, then it's open. If $\rho(A,B) \neq \emptyset$, let $\lambda_0 \in \rho(A,B)$. It suffices to find $\varepsilon > 0$ such that

$$B(\lambda_0, \varepsilon) \subset \rho(A,B)$$

where $B(\lambda_0, \varepsilon)$ is the open ball centered at λ_0 with radius ε. Let $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} A - \lambda B &= A - \lambda_0 B + (\lambda_0 - \lambda)B \\ &= (A - \lambda_0 B)[I + (\lambda_0 - \lambda)R_{\lambda_0}(A,\ B)B]. \end{aligned}$$

If $|\lambda_0 - \lambda|\|R_{\lambda_0}(A,\ B)B\| < 1$, then

$$I + (\lambda_0 - \lambda)R_{\lambda_0}(A, B)B, \tag{3.4}$$

is invertible, this is equivalent to say that if

$$|\lambda_0 - \lambda| < \frac{1}{\|R_{\lambda_0}(A,\ B)B\|}, \tag{3.5}$$

then $\lambda \in \rho(A, B)$. Consequently, it suffices to take $\varepsilon = \frac{1}{\|R_{\lambda_0}(A,B)B\|}$. □

We have the following definitions.

Definition 3.2 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A, B \in \mathcal{L}(E)$ such that $A \neq B$ and B be a non-null operator. The Fredholm spectrum $\sigma_F(A, B)$ of (A, B) is defined by*

$$\sigma_F(A, B) = \{\lambda \in \mathbb{K} : A - \lambda B \notin \Phi(E)\}. \tag{3.6}$$

The Fredholm resolvent $\rho_F(A, B)$ of (A, B) is defined by $\rho_F(A, B) = \mathbb{K}\backslash\sigma_F(A, B)$.

Definition 3.3 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A, B \in \mathcal{L}(E)$ such that $A \neq B$ and B be a non-null operator. The essential spectrum $\sigma_e(A, B)$ of a bounded linear operator pencil (A, B) is defined by*

$$\sigma_e(A, B) = \{\lambda \in \mathbb{K} : A - \lambda B \quad \text{is not a Fredholm operator of index } 0\}.$$

Theorem 3.1 *Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $\lambda \in \mathbb{Q}_p$ such that $A - \lambda B \in \Phi(E)$ and $K \in \mathcal{K}(E)$, then $A + K - \lambda B \in \Phi(E)$.*

Proof: It suffices to apply Theorem 2.2. □

We have the following definition.

Definition 3.4 *[5] Let E be an ultrametric Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(E)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(A, B)$ of a bounded linear operator pencil (A, B) on E is defined by*

$$\sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(A, B)$ of a bounded linear operator pencil (A, B) is defined by

$$\rho_\varepsilon(A, B) = \rho(A, B) \cap \{\lambda \in \mathbb{K} :\ \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

We have the following results.

Proposition 3.8 *Let $A, B \in \mathcal{L}(E)$ and $\varepsilon > 0$, we have*

(i) $\sigma(A, B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(A, B)$.

(ii) For all ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(A, B) \subset \sigma_{\varepsilon_1}(A, B) \subset \sigma_{\varepsilon_2}(A, B)$.

Proof:

(i) By Definition 3.4, we have for all $\varepsilon > 0$, $\sigma(A, B) \subset \sigma_\varepsilon(A, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon > 0} \sigma_\varepsilon(A, B)$, then for all $\varepsilon > 0$, $\lambda \in \sigma_\varepsilon(A, B)$. If $\lambda \notin \sigma(A, B)$, then $\lambda \in \{\lambda \in \mathbb{K} :\ \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|(A - \lambda B)^{-1}\| = \infty$. Thus $\lambda \in \sigma(A, B)$.

(ii) For ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A, B)$, then $\|(A - \lambda B)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A, B)$.

$\square$

Theorem 3.2 *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$, let $A, B \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(A, B) = \bigcup_{C \in \mathcal{L}(E): \|C\| < \varepsilon} \sigma(A + C, B).$$

Proof: Let $A, B, C \in \mathcal{L}(E)$ and $\varepsilon > 0$, let $\lambda \in \bigcup_{C \in \mathcal{L}(E): \|C\| < \varepsilon} \sigma(A + C, B)$. We argue by contradiction. Suppose that $\lambda \in \rho(A, B)$ and $\|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}$.
Consider D defined on E by

$$D = \sum_{n=0}^{\infty} (A - \lambda B)^{-1} \Big(- C(A - \lambda B)^{-1} \Big)^n.$$

It is easy to see that D can be written as follows $D = (A - \lambda B)^{-1}(I + C(A - \lambda B)^{-1})^{-1}$. For all $x \in E$, $D(I + C(A - \lambda B)^{-1})x = (A - \lambda B)^{-1}x$.
Let $y = (A - \lambda B)^{-1}x$, then for all $y \in E$, $D(A - \lambda B + C)y = y$. Moreover, we have

$$\text{for all} \quad x \in E, (A - \lambda B + C)Dx = x.$$

Hence, we conclude that $A - \lambda B + C$ is invertible which is a contradiction. Thus $\lambda \in \sigma_\varepsilon(A, B)$.
Conversely, let $A, B, C \in \mathcal{L}(E)$ and $\varepsilon > 0$, suppose that $\lambda \in \sigma_\varepsilon(A, B)$. We discuss two cases.

First case: If $\lambda \in \sigma(A, B)$, we may put $C = 0$.

Second case: Assume that $\lambda \in \sigma_\varepsilon(A, B)$ and $\lambda \notin \sigma(A, B)$. Then, there exists $y \in E \backslash \{0\}$ such that

$$\frac{\|(A - \lambda B)^{-1} y\|}{\|y\|} > \frac{1}{\varepsilon}. \tag{3.7}$$

Since $\|E\| \subseteq |\mathbb{K}|$, then there exists $c \in \mathbb{K} \backslash \{0\}$ such that $|c| = \|y\|$. Setting $z = c^{-1}y$, then $\|z\| = 1$. Hence, we obtain

$$\begin{aligned} \|(A - \lambda B)^{-1} z\| &= \|(A - \lambda B)^{-1} c^{-1} y\| \\ &= \frac{\|(A - \lambda B)^{-1} y\|}{|c|} \\ &= \frac{\|(A - \lambda B)^{-1} y\|}{\|y\|}. \end{aligned}$$

From (3.7),

$$\|(A - \lambda B)^{-1} z\| > \frac{1}{\varepsilon}. \tag{3.8}$$

By the same reasoning above, we infer that there exists $c_0 \in \mathbb{K} \backslash \{0\}$ such that $|c_0| = \|(A - \lambda B)^{-1} z\|$. Then, setting $z_0 = c_0^{-1}(A - \lambda B)^{-1} z$, which yields $z_0 \in E$ and $\|z_0\| = 1$. Consequently, we have

$$\begin{aligned} \|(A - \lambda B) z_0\| &= \|(A - \lambda B)(A - \lambda B)^{-1} c_0^{-1} z\| \\ &= \frac{\|z\|}{|c_0|}. \end{aligned}$$

Using the fact that $\|z\| = 1$, we deduce from (3.8) that

$$\begin{aligned}\|(A-\lambda B)z_0\| &= \|(A-\lambda B)^{-1}z\|^{-1}\\ &< \varepsilon.\end{aligned}$$

By Theorem 2.3, there exists $\phi \in E^*$ such that $\phi(z_0) = 1$ and $\|\phi\| = \|z_0\|^{-1} = 1$. We consider the following linear operator given by

$$\text{for all } y \in E,\ Cy = -\phi(y)(A-\lambda B)z_0.$$

Clearly, C is a bounded linear operator on E, since for all $y \in E$,

$$\begin{aligned}\|Cy\| &= \|\phi(y)\|\|(A-\lambda B)z_0\|\\ &< \varepsilon\|y\|.\end{aligned}$$

Then $\|C\| < \varepsilon$. Moreover, we have $(A-\lambda B+C)z_0 = 0$. So, $A-\lambda B+C$ is not invertible. This enables us to conclude that

$$\lambda \in \bigcup_{C\in\mathcal{L}(E):\|C\|<\varepsilon} \sigma(A+C,B).$$

□

Now, we characterize the essential pseudospectrum of a bounded linear operator pencil (A,B) in ultrametric case.

Definition 3.5 *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $A,B \in \mathcal{L}(E)$ and $\varepsilon > 0$. The essential pseudospectrum $\sigma_{e,\varepsilon}(A,B)$ of a bounded linear operator pencil (A,B) is defined by*

$$\sigma_{e,\varepsilon}(A,B) = \mathbb{K}\backslash\{\lambda \in \mathbb{K} : A+C-\lambda B \in \Phi_0(E) \quad \text{for all} \quad C \in \mathcal{L}(E), \|C\| < \varepsilon\},$$

where $\Phi_0(E)$ designates the set of all Fredholm operators on E of index 0.

Theorem 3.3 *Let E be an ultrametric Banach space over a spherically complete field $\mathbb{K}$, let $A,B \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A,B) = \bigcup_{C\in\mathcal{L}(E):\|C\|<\varepsilon} \sigma_e(A+C,B).$$

Proof: Let $A,B \in \mathcal{L}(E)$ and $\varepsilon > 0$, let $\lambda \notin \sigma_{e,\varepsilon}(A,B)$, then for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$,

$$A+C-\lambda B \in \Phi(E) \text{ and } ind(A+C-\lambda B) = 0.$$

Hence $\lambda \notin \sigma_e(A+C,B)$ for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$. This is equivalent to say that

$$\lambda \notin \bigcup_{C\in\mathcal{L}(E):\|C\|<\varepsilon} \sigma_e(A+C,B).$$

Consequently,

$$\bigcup_{C\in\mathcal{L}(E):\|C\|<\varepsilon} \sigma_e(A+C,B) \subseteq \sigma_{e,\varepsilon}(A,B).$$

Conversely, let $\lambda \notin \bigcup_{C\in\mathcal{L}(E):\|C\|<\varepsilon} \sigma_e(A+C,B)$. Then, for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$, we have $\lambda \notin \sigma_e(A+C,B)$. Hence $A+C-\lambda B \in \Phi(E)$ and $ind(A+C-\lambda B) = 0$ for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$. Then, $\lambda \notin \sigma_{e,\varepsilon}(A,B)$. □

Theorem 3.4 *Let E be an ultrametric Banach space over $\mathbb{Q}_p$, let $A,B \in \mathcal{L}(E)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e,\varepsilon}(A,B) = \sigma_{e,\varepsilon}(A+K,B) \text{ for all } K \in \mathcal{K}(E).$$

Proof: Let $A, B \in \mathcal{L}(E)$ and $\varepsilon > 0$, let $\lambda \notin \sigma_{e,\varepsilon}(A, B)$, then for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$,

$$A + C - \lambda B \in \Phi(E) \text{ and } ind(A + C - \lambda B) = 0.$$

Consequently, by Theorem 2.2, for all $K \in \mathcal{K}(E)$ and $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$, we have

$$A + C + K - \lambda B \in \Phi(E) \text{ and } ind(A + C + K - \lambda B) = 0.$$

Thus

$$\lambda \notin \sigma_{e,\varepsilon}(A + K, B).$$

Then

$$\sigma_{e,\varepsilon}(A + K, B) \subseteq \sigma_{e,\varepsilon}(A, B).$$

The opposite inclusion follows from symmetry. □

Remark 3.2 From Theorem 3.4, it follows that the essential pseudospectrum of bounded linear operator pencils is invariant under perturbation of compact operators on ultrametric Banach spaces over $\mathbb{Q}_p$.

The following result gives a characterization of the essential pseudospectrum of bounded linear operator pencils by means of the spectra of all perturbed compact operators on ultrametric Banach spaces over $\mathbb{Q}_p$.

Theorem 3.5 *Let E be an ultrametric Banach space over $\mathbb{Q}_p$ and let $A, B \in \mathcal{L}(E)$ such that $\|E\| \subseteq |\mathbb{Q}_p|$ and $\varepsilon > 0$. Then*

$$\sigma_{e,\varepsilon}(A, B) = \bigcap_{K \in \mathcal{K}(E)} \sigma_\varepsilon(A + K, B).$$

Proof: If $\lambda \notin \bigcap_{K \in \mathcal{K}(E)} \sigma_\varepsilon(A + K, B)$, then there exists $K \in \mathcal{K}(E)$ such that $\lambda \notin \sigma_\varepsilon(A + K,\ B)$. By Theorem 3.2, we have $\lambda \in \rho(A + K + C, B)$ for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$. We have

$$A + C + K - \lambda B \in \Phi(E) \text{ and } ind(A + C + K - \lambda B) = 0.$$

Moreover, by Theorem 2.2, we have

$$A + C - \lambda B \in \Phi(E) \textit{ and } ind(A + C - \lambda B) = 0. \tag{3.9}$$

We conclude that

$$\lambda \notin \sigma_{e,\varepsilon}(A, B).$$

Then,

$$\sigma_{e,\varepsilon}(A, B) \subseteq \bigcap_{K \in \mathcal{K}(E)} \sigma_\varepsilon(A + K, B). \tag{3.10}$$

Conversely, let $\lambda \notin \sigma_{e,\varepsilon}(A, B)$. Using Theorem 3.3, we have for all for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$, $\lambda \notin \sigma_e(A + C, B)$, then from Theorem 2.4, there exists $K \in \mathcal{K}(E)$ such that $\lambda \notin \sigma(A + K + C, B)$, hence for all $C \in \mathcal{L}(E)$ such that $\|C\| < \varepsilon$, $\lambda \in \rho(A + K + C, B)$. Thus

$$\lambda \in \bigcap_{C \in \mathcal{L}(E): \|C\| < \varepsilon} \rho(A + K + C, B). \tag{3.11}$$

It follows from Theorem 3.2, $\lambda \notin \sigma_\varepsilon(A + K, B)$. Consequently,

$$\lambda \notin \bigcap_{K \in \mathcal{K}(E)} \sigma_\varepsilon(A + K, B).$$

Thus

$$\sigma_{e,\varepsilon}(A, B) = \bigcap_{K \in \mathcal{K}(E)} \sigma_\varepsilon(A + K, B).$$

□

We finish with the following example.

Example 3.2 *Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A, B \in \mathcal{L}(\mathbb{E}_\omega)$ be two diagonal operators such that for all $i \in \mathbb{N}$, $Ae_i = a_i e_i$ and $Be_i = b_i e_i$, where $a_i, b_i \in \mathbb{Q}_p$ such that $\sup\limits_{i\in\mathbb{N}} |a_i|$ and $\sup\limits_{i\in\mathbb{N}} |b_i|$ are finite, then by Proposition 3.5, we have*

$$\sigma(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda b_i| = 0\},$$

and for all $\lambda \in \rho(A, B)$, we have

$$\begin{aligned} \|(A - \lambda B)^{-1}\| &= \sup_{i\in\mathbb{N}} \frac{\|(A - \lambda B)^{-1} e_i\|}{\|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \left| \frac{1}{a_i - \lambda b_i} \right| \\ &= \frac{1}{\inf\limits_{i\in\mathbb{N}} |a_i - \lambda b_i|}. \end{aligned}$$

Hence,

$$\left\{\lambda \in \mathbb{Q}_p : \|(A - \lambda B)^{-1}\| > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda b_i| < \varepsilon\right\}.$$

Consequently,

$$\sigma_\varepsilon(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda b_i| = 0\} \cup \left\{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda b_i| < \varepsilon\right\}.$$

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