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Functions with a maximal number of finite invariant or internally-1-quasi-invariant sets or supersets *

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ABSTRACT: A relaxation of the notion of invariant set, known as k-quasi-invariant set, has appeared several times in the literature in relation to group dynamics. The results obtained in this context depend on the fact that the dynamic is generated by a group. In our work, we consider the notions of invariant and 1-internally-quasi-invariant sets as applied to an action of a function f on a set I. We answer several questions of the following type, where $k \in \{0, 1\}$: what are the functions f for which every finite subset of I is internally-k-quasi-invariant? More restrictively, if $I = \mathbb{N}$, what are the functions f for which every finite interval of I is internally-k-quasi-invariant? Last, what are the functions f for which every finite subset of I admits a finite internally-k-quasi-invariant superset? This parallels a similar investigation undertaken by C. E. Praeger in the context of group actions.

Key Words: orbit, discrete dynamics, invariant set, quasi-invariant set, almost-invariant set, bounded movement, totally ordered set.

Contents

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1. Introduction

Invariant sets play an essential role in the qualitative study of dynamical systems [7,5]. In group dynamics, a relaxation of this notion, known as *k-quasi-invariant sets*, or *k-almost-invariant sets* (where $k \in \mathbb{N}$), appears in the works [3,10,4,8,9,2,1] and the references therein. Most of the results contained in these articles crucially depend on the fact that G is a group.
In our work, we depart from the group setting by extending the notion of a k-quasi-invariant set under a group action to the context of an action of a set A on another set I (this simply consists of a function $\rho : A \times I \to I$ with no additional requirement). The lack of bijectivity of the functions $\{\rho(a,\cdot)\}_{a\in A}$ forces us to make the following definitions. If we denote by $|.|$ the cardinality function and $\Lambda^a := \{\rho(a,x) : x \in \Lambda\}$ for $a \in A$ and $\Lambda \subseteq I$, then Λ is *externally-k-quasi-invariant* under the action ρ if

$$\forall a \in A : |\Lambda^a \setminus \Lambda| \leq k,$$

or equivalently

$$\forall a \in A : \exists P \subseteq I : |P| \leq k \text{ and } \Lambda^a \subseteq \Lambda \cup P,$$

and is *internally-k-quasi-invariant* under the action ρ if

$$\forall a \in A : \exists P \subseteq I : |P| \leq k \text{ and } (\Lambda \setminus P)^a \subseteq \Lambda.$$

We focus in this article on internally-k-quasi-invariant sets when A is a singleton and $k \in \{0,1\}$. In this case, the action $\rho : A \times I \to I$ becomes just a function $f : I \to I$, and therefore a set Λ is internally-0-quasi-invariant under the action of f if it is a (forward)-invariant set of f, i.e.

$$f(\Lambda) \subseteq \Lambda \tag{1.1}$$

and is internally-1-quasi-invariant under the action of f if

$$\exists a \in \Lambda : f(\Lambda \setminus \{a\}) \subseteq \Lambda. \tag{1.2}$$

Note that the notions of external or internal k-quasi-invariance under the action of f that we have just defined, differ from external or internal k-quasi-invariance under the action of the monoid $\mathbb{N}$ (by iterates of a function f), which we will not be concerned about in this article.

Most of the findings in [3,10,4,8,9,2,1] about k-quasi-invariant sets under a group action tend to be wrong for internally-k-quasi-invariant sets under the action of $f : I \to I$. In [10], the author shows that if every finite subset of I is k-quasi-invariant under a group action with no fixed points, then I must be finite. The analogous statement for internally-1-quasi-invariant sets under the action of a single function f is wrong, as can be attested by the function $succ : \mathbb{N} \to \mathbb{N}$. However, the problem considered by the author in [10] is interesting, and can be adapted to our setting: what are the functions $f : I \to I$ for which every finite subset of I is internally-k-quasi-invariant? More restrictively, if $I = \mathbb{N}$, what are the functions $f : I \to I$ for which every finite interval of I is internally-k-quasi-invariant? Last, what are the functions $f : I \to I$ for which every finite subset of I admits a finite internally-k-quasi-invariant superset? In this article, we investigate these types of questions in the $k \in \{0,1\}$ cases. Besides, we inform the reader that the idea of this article emerged when we were looking for an example to a proposition related to linear independence in ([6], Proposition 6.1 and Example 6.2).

2. Notations

In the sequel, $\mathbb{N}$ denotes the set $\{0,1,2,\cdots\}$ of natural numbers including 0. $\mathbb{N}^*$ denotes $\mathbb{N} \setminus \{0\}$.
If A is a set, we denote by $|A|$ the cardinality of A, $\mathcal{P}(A)$ the powerset of A, $\mathcal{P}_\omega(A)$ the set $\{B \subseteq A : |B| < \infty\}$, $\mathcal{P}_{\omega,*}(A)$ the set $\{B \subseteq A : 0 < |B| < \infty\}$, and $\mathcal{P}_{\omega,n^+}(A)$ the set $\{B \subseteq A : n \leq |B| < \infty\}$.
If $(A, \leq)$ is a non-empty totally ordered set, we denote by $Int_{\omega,*}(A)$ the set $\{[a,b] : a \leq b, |[a,b]| < \infty\}$, and by $Int_{\omega,n^+}(A)$ the set $\{[a,b] : a \leq b, n \leq |[a,b]| < \infty\}$.
If A is a set, $\phi : A \to A$ a self map and $n \in \mathbb{N}^*$, we denote by ϕ^n the composition of ϕ with itself n times : $\phi \circ \cdots \circ \phi : A \to A$. In addition, we define ϕ^0 to be the identity function on A. Moreover, if $a \in A$, we denote by $\mathcal{O}^+_\phi(a)$ the forward orbit of a under the iterates of ϕ : $\{\phi^n(a) : n \in \mathbb{N}\}$.

3. Preliminaries on discrete time forward orbits

3.1. Elementary lemmas

The following well-known lemmas are easy to prove and are only reminded for convenience.

Lemma 3.1. *Let I be an infinite set, $a \in I$ and $\phi : I \to I$. Then*

$$\mathcal{O}_\phi^+(a) \text{ is infinite } \Leftrightarrow a, \phi(a), \phi(\phi(a)), \cdots \text{ are distinct}$$

or equivalently

$$\mathcal{O}_\phi^+(a) \text{ is finite } \Leftrightarrow \text{ the sequence } a, \phi(a), \phi(\phi(a)), \cdots \text{ is eventually periodic.}$$

Proof. ($\Rightarrow$) Suppose that $\mathcal{O}_\phi^+(a)$ is infinite. Suppose that $\phi^n(a) = \phi^m(a)$ for some $n < m$. By induction, we have $\phi^{n+j}(a) = \phi^{m+j}(a)$ for all $j \in \mathbb{N}$. Let $e \geq n$. Let $e - n = q(m-n) + r$ be the division with remainder of $e-n \in \mathbb{N}$ by $m-n \in \mathbb{N}^*$. If $q \geq 1$, we have $\phi^e(a) = \phi^{n+q(m-n)+r}(a) = \phi^{m+(q-1)(m-n)+r}(a) = \phi^{n+(q-1)(m-n)+r}(a)$. By immediate induction, we have that $\phi^e(a) = \phi^{n+r}(a)$, where $0 \leq r < m-n$. So $\mathcal{O}_\phi^+(a) = \{a, \phi(a), \cdots, \phi^{m-1}(a)\}$ is finite, contradiction.
($\Leftarrow$) This is clear. $\square$

Therefore, for any map $\phi : I \to I$ and $a \in I$, the following simple conjugation result for $\phi_{|\mathcal{O}_\phi^+(a)}$ follows.

Lemma 3.2. *Let I be an infinite set, $a \in I$ and $\phi : I \to I$.*

1. *Suppose that $\mathcal{O}_\phi^+(a)$ is infinite. Then $\phi_{|\mathcal{O}_\phi^+(a)}$ is conjugate to* $succ : \begin{cases} \mathbb{N} & \to \mathbb{N} \\ n & \mapsto n+1 \end{cases}$.
2. *Suppose that $\mathcal{O}_\phi^+(a)$ is finite. Then $\exists (u,v) \in \mathbb{N} \times \mathbb{N}^*$ and a bijection $\alpha : [0, u+v-1] \to \mathcal{O}_\phi^+(a)$ such that*
$$\mathcal{O}_\phi^+(a) = \{a, \phi(a), \cdots, \phi^{u-1}(a)\} \sqcup \{\phi^u(a), \cdots, \phi^{u+v-1}(a)\},$$
$$\phi_{|\{a,\phi(a),\cdots,\phi^{u-1}(a)\}} \text{ is conjugated to } succ_{[u]} : \begin{cases} [0, u-1] & \to [1, u] \\ n & \mapsto n+1 \end{cases} \text{ by } \alpha_{|[0,u-1]},$$
and
$$\phi_{|\{\phi^u(a),\cdots,\phi^{u+v-1}(a)\}} \text{ is conjugated to}$$
$$cycle_{[v]} : \begin{cases} [u, u+v-1] & \to [u, u+v-1] \\ n & \mapsto n+1 \text{ if } n \in [u, u+v-2] \\ u+v-1 & \mapsto u \end{cases} \text{ by } \alpha_{|[u,u+v-1]}.$$

Proof. The map α that realizes the conjugation is, in both cases, the one that sends n to $\phi^n(a)$. $\square$

Thus, we can deduce

Corollary 3.3. *Let I be an infinite set, $a \in I$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a)$ is infinite.
Then $\phi_{|\mathcal{O}_\phi^+(a)} : \mathcal{O}_\phi^+(a) \to \mathcal{O}_\phi^+(a) \setminus \{a\}$ is well-defined and bijective.

Lemma 3.4. *Let I be an infinite set, $(a,b) \in I^2$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a)$ is infinite and $\mathcal{O}_\phi^+(b)$ is finite.
Then $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ don't have an intersection point.

Proof. Assume by way of contradiction that $(\exists u, v \in \mathbb{N}) : \phi^u(a) = \phi^v(b)$. Then

$$\mathcal{O}_\phi^+(a) = \{a, \phi(a), \cdots, \phi^{u-1}(a)\} \cup \mathcal{O}_\phi^+(\phi^v(b)) \subseteq \{a, \phi(a), \cdots, \phi^{u-1}(a)\} \cup \mathcal{O}_\phi^+(b),$$

which is finite, a contradiction. □

Lemma 3.5. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite and $\mathcal{O}_\phi^+(b)$ is infinite.
Then $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ have at least one intersection point.

Proof. Assume by way of contradiction that $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ don't intersect. Then $\mathcal{O}_\phi^+(b) \subseteq I \setminus \mathcal{O}_\phi^+(a)$. But this is impossible since $\mathcal{O}_\phi^+(b)$ is infinite and $I \setminus \mathcal{O}_\phi^+(a)$ is finite. Hence the result. □

Corollary 3.6. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite.
Then $\mathcal{O}_\phi^+(a)$ intersects $\mathcal{O}_\phi^+(b)$ if and only if $\mathcal{O}_\phi^+(b)$ is infinite.

Proof. Combine lemmas 3.4 and 3.5. □

Lemma 3.7. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite and $\mathcal{O}_\phi^+(b)$ is infinite. Then $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b)$, and consequently $\mathcal{O}_\phi^+(b)$, are cofinite.

Proof. First, notice that by lemma 3.1, $a, \phi(a), \phi^2(a), \cdots$ are distinct. From the hypothesis, there exist $u, v \in \mathbb{N}$ such that $\phi^u(a) = \phi^v(b)$. So

$$\begin{aligned} I \setminus (\mathcal{O}_\phi^+(b) \cap \mathcal{O}_\phi^+(b)) &= \Big(I \setminus \mathcal{O}_\phi^+(a)\Big) \cup \Big[\Big(I \setminus \mathcal{O}_\phi^+(\phi^v(b))\Big) \setminus \{b, \cdots, \phi^{v-1}(b)\}\Big] \\ &= \Big(I \setminus \mathcal{O}_\phi^+(a)\Big) \cup \Big[\Big(I \setminus \mathcal{O}_\phi^+(\phi^u(a))\Big) \setminus \{b, \cdots, \phi^{v-1}(b)\}\Big] \\ &= \Big(I \setminus \mathcal{O}_\phi^+(a)\Big) \\ &\quad \cup \Big[\Big((I \setminus \mathcal{O}_\phi^+(a)) \cup \{a, \cdots, \phi^{u-1}(a)\}\Big) \setminus \{b, \cdots, \phi^{v-1}(b)\}\Big] \\ &\subseteq (I \setminus \mathcal{O}_\phi^+(a)) \cup \{a, \cdots, \phi^{u-1}(a)\} \end{aligned}$$

is finite. The fact that $\mathcal{O}_\phi^+(b)$ is also cofinite is due to the inclusion $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b) \subseteq \mathcal{O}_\phi^+(b)$. □

Corollary 3.8. *Let I be an infinite set, $a \in I$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}_\phi^+(a) = I$. Then $\forall b \in I : \mathcal{O}_\phi^+(b)$ is cofinite

Proof. This can be seen directly or as a consequence of corollary 3.6 and lemma 3.7 (since $\forall b \in I : b \in \mathcal{O}_\phi^+(a)$). □

Remark 3.9. *The converse statement is false. Indeed, let* $\phi : \begin{cases} \mathbb{N} \cup \{\bullet\} & \to \mathbb{N} \cup \{\bullet\} \\ n & \mapsto n+1 \\ \bullet & \mapsto 1 \end{cases}$. *Then* $\forall a \in \mathbb{N} \cup \{\bullet\} : (\mathcal{O}_\phi^+(a)$ *is cofinite and* $\mathcal{O}_\phi^+(a) \neq \mathbb{N} \cup \{\bullet\})$.

3.2. Nearest element to a finite set lying in the intersection of orbits of elements of that set

We now associate to each self-map $\phi : I \to I$ on an infinite set I, a particular *noncanonical* map $\xi_\phi : D_\phi \to I$, which gives for each $I^* \in D_\phi$ one of the nearest elements to I^* that lies in the intersection of orbits of elements of I^*. The value of $\xi_\phi(I^*)$, for $I^* \in D_\phi$, is uniquely determined if all the orbits of elements in I^* are infinite (use lemmas 3.12 and 3.1), which arises if and only if $\cap_{a\in I^*}\mathcal{O}^+_\phi(a)$ is infinite by corollary 3.14. However, if $\cap_{a\in I^*}\mathcal{O}^+_\phi(a)$ is finite, which arises if and only if all the orbits of elements in I^* are finite by corollary 3.15, then $\xi_\phi(I^*)$ may be non-uniquely determined as in the case where I^* is precisely a finite cycle of a map $\phi : I \to I$. This definition will be used in the rest of this article.

Definition 3.10. *Let I be an infinite set and $\phi : I \to I$.*
Let $D_\phi = \{I^* \in \mathcal{P}_{\omega,*}(I) : \bigcap_{a\in I^*} \mathcal{O}^+_\phi(a) \neq \emptyset\} \subseteq \mathcal{P}_{\omega,*}(I)$.
For all $I^* \in D_\phi$, $z \in \bigcap_{a\in I^*} \mathcal{O}^+_\phi(a)$ and $a \in I^*$, we define m^z_a as $\min\{m \in \mathbb{N} : \phi^m(a) = z\}$.
We define a noncanonical map $\xi_\phi : D_\phi \to I$ by selecting for all $I^* \in D_\phi$, $\xi_\phi(I^*) \in \bigcap_{a\in I^*} \mathcal{O}^+_\phi(a)$ with $\sum_{a\in I^*} m^{\xi_\phi(I^*)}_a$ minimal.

Example 3.11. *If I is infinite and $I^* \in \mathcal{P}_{\omega,*}(I)$ is such that $(\forall a \in I^*) : \mathcal{O}^+_\phi(a)$ is cofinite, then $I^* \in D_\phi$, because a finite intersection of cofinite sets is cofinite and hence non-empty.*

Lemma 3.12. *Let I be an infinite set, $\phi : I \to I$ and $I^* \in \mathcal{P}_{\omega,*}(I)$ such that $\bigcap_{a\in I^*} \mathcal{O}^+_\phi(a) \neq \emptyset$. Then $\bigcap_{a\in I^*} \mathcal{O}^+_\phi(a) = \mathcal{O}^+_\phi(\xi_\phi(I^*))$.*

Proof. Suppose that $z \in \bigcap_{a\in I^*} \mathcal{O}^+_\phi(a)$. So $(\forall a \in I^*) : z = \phi^{m^z_a}(a)$. Also, we have $(\forall a \in I^*) : \xi_\phi(I^*) = \phi^{m^{\xi_\phi(I^*)}_a}(a)$ where the $m^{\xi_\phi(I^*)}_a$'s satisfy a minimality property.
Hence $\sum_{a\in I^*} m^z_a \geq \sum_{a\in I^*} m^{\xi_\phi(I^*)}_a$, and so $(\exists a \in I^*) : m^z_a \geq m^{\xi_\phi(I^*)}_a$. This implies that

$$z = \phi^{m^z_a}(a) = \phi^{m^z_a - m^{\xi_\phi(I^*)}_a}(\phi^{m^{\xi_\phi(I^*)}_a}(a)) = \phi^{m^z_a - m^{\xi_\phi(I^*)}_a}(\xi_\phi(I^*)) \in \mathcal{O}^+_\phi(\xi_\phi(I^*)).$$

Conversely, suppose that $z \in \mathcal{O}^+_\phi(\xi_\phi(I^*))$. So $(\exists n \in \mathbb{N}) : z = \phi^n(\xi_\phi(I^*))$. Also, we have $(\forall a \in I^*) : \xi_\phi(I^*) = \phi^{m^{\xi_\phi(I^*)}_a}(a)$. Therefore $(\forall a \in I^*) : z = \phi^{n+m^{I^*}_a}(a) \in \mathcal{O}^+_\phi(a)$.
Hence the two sets are equal. □

Lemma 3.13. *Let I be an infinite set, $(a, b, c) \in I^3$ and $\phi : I \to I$.*
Suppose that $\mathcal{O}^+_\phi(a)$ and $\mathcal{O}^+_\phi(b)$ are infinite and have at least one intersection point, and that $c \in \mathcal{O}^+_\phi(b)$ Then $\mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(c)$ is infinite.

Proof. Since $\mathcal{O}^+_\phi(a)$ and $\mathcal{O}^+_\phi(b)$ intersect, there exist $(m, n) \in \mathbb{N}^2$ such that $\phi^n(a) = \phi^m(b)$. Also, since $c \in \mathcal{O}^+_\phi(b)$, there exists $p \in \mathbb{N}$ such that $c = \phi^p(b)$. Therefore $\phi^{n+p}(a) = \phi^{m+p}(b) = \phi^m(c)$, and so this common element belongs to the orbits of a and c. Moreover, since $\mathcal{O}^+_\phi(a)$ is infinite, and this common element and its iterates by ϕ belong to it, we have by lemma 3.1 that the orbit of this element is infinite. This orbit is included in the orbits of a and c, and so in their intersection, which proves that $\mathcal{O}^+_\phi(a)\cap\mathcal{O}^+_\phi(c)$ is infinite. □

Corollary 3.14. *Let I be an infinite set and $\phi : I \to I$.*
Let $I^* = \{a_1, \cdots, a_n\} \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| = +\infty\})$ such that the orbits $\mathcal{O}^+_\phi(a_i)$ intersect jointly or in pairs. Then $\cap_{a\in I^*}\mathcal{O}^+_\phi(a)$ is infinite.

Proof. If the orbits intersect jointly, then they intersect in pairs. Therefore it suffices to consider the latter case. Now, we have $\forall i \in [\![1, n]\!] : |\mathcal{O}^+_\phi(a_i)| = +\infty$, so using lemma 3.13 (in conjunction with lemma 3.12) repeatedly, we see that $|\bigcap_{i\in[\![1,n]\!]} \mathcal{O}^+_\phi(a_i)| = \infty$. □

The following lemma asserts that we cannot find a subset $I^* \in D_\phi$ containing both an element of infinite orbit and another element of finite orbit.

Lemma 3.15. *Let I be an infinite set and $\phi : I \to I$. Then we have* $D_\phi \subseteq \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| = +\infty\}) \cup \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| < \infty\})$

Proof. Let $I^* \in D_\phi$. Suppose by way of contradiction that $\{a, b\} \subseteq I^*$ with $\mathcal{O}^+_\phi(a) = +\infty$ and $\mathcal{O}^+_\phi(b) < +\infty$. Let $c \in \mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b)$. If $\mathcal{O}^+_\phi(c)$ were finite, we would have $\phi^n(c) = \phi^m(c)$ for some $m \neq n$ by lemma 3.1, and so $\phi^{n+u}(a) = \phi^{m+u}(a)$ for some $u \in \mathbb{N}$ since $c \in \mathcal{O}^+_\phi(a)$, which contradicts lemma 3.1 since $\mathcal{O}^+_\phi(a)$ is infinite. So $\mathcal{O}^+_\phi(c)$ must be infinite. But $c \in \mathcal{O}^+_\phi(b)$ implies $\mathcal{O}^+_\phi(c) \subseteq \mathcal{O}^+_\phi(b)$ which is finite, a contradiction. So either $I^* \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| = +\infty\})$ or $I^* \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| < \infty\})$. □

3.3. Maps with pairwise intersecting infinite orbits

The next definition will be helpful in the next section.

Definition 3.16. *Let I be a set and $\phi : I \to I$. We let $\widetilde{P}(\phi)$ be the proposition*

$$\forall a, b \in I : \Big[(|\mathcal{O}^+_\phi(a)| = |\mathcal{O}^+_\phi(b)| = \infty) \Rightarrow |\mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b)| = \infty\Big].$$

Remark 3.17. *Using lemmas 3.13 and 3.4, $\widetilde{P}(\phi)$ is easily seen to be equivalent to*

$$\forall a, b \in I : \Big[(|\mathcal{O}^+_\phi(a)| = |\mathcal{O}^+_\phi(b)| = \infty) \Rightarrow \mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b) \neq \emptyset\Big].$$

Example 3.18.
1. *$\widetilde{P}(\phi)$ is vacuously true when I is finite.*
2. *$\widetilde{P}(\phi)$ is true when $(\exists a \in I) : \mathcal{O}^+_\phi(a)$ is cofinite, see lemma 3.20.*

Lemma 3.19. *Let I be an infinite set and $\phi : I \to I$. Then we have* $\widetilde{P}(\phi) \Leftrightarrow D_\phi \supseteq \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| = +\infty\})$.

Proof. ($\Rightarrow$) Suppose that $\widetilde{P}(\phi)$ is true. Let $I^* = \{a_1, \cdots, a_n\} \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\phi(a)| = +\infty\})$. Then it follows from corollary 3.14 that $I^* \in D_\phi$.
($\Leftarrow$) Suppose that $D_\phi \supseteq \mathcal{P}_{\omega,*}(\{x \in I : |\mathcal{O}^+_\phi(x)| = +\infty\})$. Let $a, b \in I$ such that $|\mathcal{O}^+_\phi(a)| = |\mathcal{O}^+_\phi(b)| = +\infty$. So $\{a, b\} \in \mathcal{P}_{\omega,*}(\{x \in I : |\mathcal{O}^+_\phi(x)| = +\infty\}) \subseteq D_\phi$. Therefore $\mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b) \neq \emptyset$ by definition of D_ϕ. Hence $\widetilde{P}(\phi)$ is true by remark 3.17. □

Lemma 3.20. *Let I be an infinite set, $\phi : I \to I$, and $a \in I$ such that $\mathcal{O}^+_\phi(a)$ is cofinite. Then $\widetilde{P}(\phi)$ holds.*

Proof. Let $b_1, b_2 \in I$ such that $|\mathcal{O}^+_\phi(b_1)| = |\mathcal{O}^+_\phi(b_2)| = \infty$. Then by lemma 3.7, $\mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b_1)$ is non-empty and cofinite. By lemma 3.12, $\mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b_1) = \mathcal{O}^+_\phi(\xi_\phi(\{a, b_1\}))$. Again, by lemma 3.7, $\mathcal{O}^+_\phi(\xi_\phi(\{a, b_1\})) \cap \mathcal{O}^+_\phi(b_2) = \mathcal{O}^+_\phi(a) \cap \mathcal{O}^+_\phi(b_1) \cap \mathcal{O}^+_\phi(b_2)$ is non-empty and cofinite. Since this set is included in $\mathcal{O}^+_\phi(b_1) \cap \mathcal{O}^+_\phi(b_2)$, we have that $\mathcal{O}^+_\phi(b_1) \cap \mathcal{O}^+_\phi(b_2)$ is also non-empty. and cofinite. Therefore $\widetilde{P}(\phi)$ holds.
□

4. Functions with a maximal number of finite invariant sets

The results of this section are obvious and only proved for completeness. They express the fact that, as long as k is strictly less than the cardinality of I, the identity function is the only function with a maximal number of finite invariant sets in $\mathcal{P}_{\omega,k^+}(I)$ or $Int_{\omega,k^+}(\mathbb{N})$.

4.1. Preservation of all finite sets

Proposition 4.1. *Let I be a set, $\alpha : I \to I$, and k a natural number such that $1 \leq k \leq |I|$. Suppose that*

$$\forall I^* \in \mathcal{P}_{\omega,k^+}(I) : \alpha(I^*) \subseteq I^*.$$

Then we have
(a) if $1 \leq k < |I|$, then $\alpha = Id$.
(b) if $k = |I|$, then α can be arbitrary.

Proof. Suppose $1 \leq k < |I|$. If $\alpha(a) \neq a$ for some $a \in I$, one can choose $I^* \in \mathcal{P}_{\omega,k^+}(I)$ such that $a \in I^*$ but $\alpha(a) \notin I^*$ to obtain a contradiction. Conversely the identity function satisfies the requirement.
Suppose now that $k = |I|$. Then I is a finite set and $\mathcal{P}_{\omega,k^+}(I) = \{I\}$, and every function $\alpha : I \to I$ obviously preserves I. □

4.2. Preservation of all finite intervals in $\mathbb{N}$

Proposition 4.2. *Let $\alpha : \mathbb{N} \to \mathbb{N}$, and k a natural number such that $k \geq 1$. Suppose that*

$$\forall [\![a, b]\!] \in Int_{\omega,k^+}(\mathbb{N}) : \alpha([\![a, b]\!]) \subseteq [\![a, b]\!].$$

Then $\alpha = Id$.

Proof. Let $a \in \mathbb{N}$. If $\alpha(a) > a$, one can choose $[\![a - k + 1, a]\!]$ to obtain a contradiction. If $\alpha(a) < a$, one can choose $[\![a, a + k - 1]\!]$ to obtain a contradiction. Conversely the identity function satisfies the requirement. □

5. Functions with a maximal number of finite internally-1-quasi-invariant sets

5.1. Preservation of all finite sets up to one element

If I is a set such that $|I| \geq 3$, the next proposition shows that there are two 3-parameter families of functions $\beta : I \to I$ such that there exists a function $w : \mathcal{P}_{\omega,*}(I) \to I$ such that

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : w(I^*) \in I^* \text{ and } \beta(I^* \setminus \{w(I^*)\}) \subseteq I^*.$$

Note that we cannot require that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \beta(w(I^*)) \notin I^*$ since these functions have many fixed points (except if $|I| = 3$).

Proposition 5.1. *Let I be a set such that $|I| \geq 3$, $\beta : I \to I$ and $w : \mathcal{P}_{\omega,*}(I) \to I$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : w(I^*) \in I^* \text{ and } \beta(I^* \setminus \{w(I^*)\}) \subseteq I^*.$$

Then either
Case ①.

$$\exists (a, b, c) \in I^3 : (\beta(a) = b \text{ and } \beta(b) = c) \text{ and } (\forall x \in I \setminus \{a, b\} : \beta(x) = x)$$

and

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \to I \\ I^* & \mapsto b \text{ if } \{a, b\} \subseteq I^* \\ I^* & \mapsto a \text{ if } a \in I^* \text{ and } b \notin I^* \\ I^* & \mapsto b \text{ if } a \notin I^* \text{ and } b \in I^* \\ I^* & \mapsto \text{any element of } I^* \text{ otherwise} \end{cases}$$

or
Case ②.

$$\exists (a, b, c) \in I^3 \text{ with } a \neq b, b \neq c, c \neq a, \beta(a) = b, \beta(b) = c, \beta(c) = a$$

and

$$\forall x \in I \setminus \{a,b,c\} : \beta(x) = x$$

and

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \to I \\ I^* & \mapsto a \text{ if } a \in I^* \text{ and } b,c \notin I^* \\ I^* & \mapsto b \text{ if } b \in I^* \text{ and } a,c \notin I^* \\ I^* & \mapsto c \text{ if } c \in I^* \text{ and } a,b \notin I^* \\ I^* & \mapsto b \text{ if } \{a,b\} \subseteq I^* \text{ and } c \notin I^* \\ I^* & \mapsto a \text{ if } \{a,c\} \subseteq I^* \text{ and } b \notin I^* \\ I^* & \mapsto c \text{ if } \{b,c\} \subseteq I^* \text{ and } a \notin I^* \\ I^* & \mapsto \text{any element of } I^* \text{ otherwise} \end{cases}.$$

Proof. If β is the identity function, then β is clearly of type (1).
Otherwise, there exists $p \in I : \beta(p) \neq p$.
If β is the identity function on $I \setminus \{p\}$, then β is clearly of type (1).
Otherwise, let $q \in I \setminus \{p\}$ such that $\beta(q) \neq q$.
If for all $(m,n) \in I \setminus \{p\}$ with $m \neq n$ we had $w(\{p,m\}) = m$ and $w(\{p,n\}) = n$, then $\beta(p) \in \{p,m\} \cap \{p,n\} = \{p\}$, a contradiction. So for all $(m,n) \in I \setminus \{p\}$ with $m \neq n$, we have $w(\{p,m\}) = p$ or $w(\{p,n\} = p$ which implies $\beta(m) \in \{p,m\}$ or $\beta(n) \in \{p,n\}$. Similarly, for all $(m,n) \in I \setminus \{q\}$ with $m \neq n$, $\beta(m) \in \{q,m\}$ or $\beta(n) \in \{q,n\}$.
Let $n \in I \setminus \{p,q,\beta(q)\}$. If $w(\{q,n\}) = n$, then $\beta(q) \in \{q,n\}$ so $\beta(q) = n$, a contradiction. So $w(\{q,n\}) = q$ and therefore $\beta(n) \in \{q,n\}$.
If $\exists r \in I \setminus \{p,q,\beta(q)\}$ such that $\beta(r) \neq r$, then $\beta(r) = q \notin \{p,r\}$, which implies $\forall m \in I \setminus \{p,r\} : \beta(m) \in \{p,m\}$. In particular $\beta(q) \in \{p,q\}$ and so $\beta(q) = p$. Since $\beta(r) \neq r$, we can prove in the same way as before that $\forall m,n \in I \setminus \{r\}$ with $m \neq n$, $\beta(m) \in \{r,m\}$ or $\beta(n) \in \{r,n\}$. So either β is the identity function on $I \setminus \{p,r\}$ or there exists $s \in I \setminus \{p,r\}$ such that $\forall n \in I \setminus \{p,r,s\} : \beta(n) \in \{q,n\} \cap \{r,n\} = \{n\}$ and so β is the identity function on $I \setminus \{p,r,s\}$. The first case is included in the second, and we have necessarily $s = q$ since $q \notin \{p,r\}$ and $\beta(q) = p \neq q$. Since $p \neq q$ and $p,q \in I \setminus \{r\}$, we have $\beta(p) \in \{r,p\}$ or $\beta(q) \in \{r,q\}$. The second case is impossible so $\beta(p) \in \{r,p\}$ which implies $\beta(p) = r$. So β is of type (2) $(a = p, b = r, c = q)$.
Otherwise, we have $\forall n \in I \setminus \{p,q,\beta(q)\} : \beta(n) = n$.
If $\beta(q) = p$, then β is clearly of type (1).
Suppose then that $\beta(q) \neq p$.
If $\beta(p) \neq q$, then $\beta(\{p,q\} \setminus \{w(\{p,q\})\}) \subseteq \{p,q\}$ leads to a contradiction independently of $w(\{p,q\} \in \{p,q\}$. So $\beta(p) = q$.
Since $q \neq \beta(q)$ and $q, \beta(q) \in I \setminus \{p\}$, we have $\beta(q) \in \{p,q\}$ or $\beta(\beta(q)) \in \{p, \beta(q)\}$. The first case is impossible. So $\beta(\beta(q)) \in \{p,\beta(q)\}$.
If $\beta(\beta(q)) = p$ then β is of type (2) $(a = p, b = q, c = \beta(q))$.
Otherwise $\beta(\beta(q)) = \beta(q)$ and so β is of type (1) $(a = p, b = q, c = \beta(q))$.
The statements about w are easy to prove. ∎

5.2. Preservation of all finite intervals up to one element in $\mathbb{N}$

Apart from the successor function $\beta = succ$ with $w : \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b \end{cases}$, there exist other functions $\beta : \mathbb{N} \to \mathbb{N}$ such that there exists a function $w : Int_{\omega,*}(\mathbb{N}) \to \mathbb{N}$ such that

$$\forall [\![a,b]\!] \in Int_{\omega,*}(\mathbb{N}) : w([\![a,b]\!]) \in [\![a,b]\!] \text{ and } \beta([\![a,b]\!] \setminus \{w([\![a,b]\!])\}) \subseteq [\![a,b]\!],$$

as shown by the next proposition.

Proposition 5.2. *Let $\beta : \mathbb{N} \to \mathbb{N}$ such that there exists a function $w : Int_{\omega,*}(\mathbb{N}) \to \mathbb{N}$ such that*

$$\forall [\![a,b]\!] \in Int_{\omega,*}(\mathbb{N}) : w([\![a,b]\!]) \in [\![a,b]\!] \text{ and } \beta([\![a,b]\!] \setminus \{w([\![a,b]\!])\}) \subseteq [\![a,b]\!].$$

Let $\{b_n\}_{n\in[\![0,l]\!]}$ the sequence of non-fixed points of β in strictly increasing order, where $l \in \mathbb{N} \cup \{-1, \infty\}$. We extend $\{b_n\}_{n\in[\![0,l]\!]}$ and β by setting $b_{-1} = -1$ and $\beta(-1) = 0$, and if $l < \infty$, we set $b_{l+1} = \infty$. Then we have either

***Case** (1).* $\forall n \in [\![0,l]\!] : b_n < \beta(b_n) \leq b_{n+1}$ *and*

$$w = \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b_n \text{ if } \exists! n \in [\![0,l]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) > b \\ [\![a,b]\!] & \mapsto \text{any element of } [\![a,b]\!] \text{ otherwise} \end{cases}.$$

***Case** (2). There exists $n^* \in [\![-1, l-1]\!]$ such that $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) < b_{n^*+1}$, $\forall m \in [\![n^*+2, l]\!] : b_{m-1} \leq \beta(b_m) < b_m$, and*

$$w = \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b_n \text{ if } \exists! n \in [\![0,n^*]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) > b \\ [\![a,b]\!] & \mapsto b_n \text{ if } \exists! n \in [\![n^*+1,l]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) < a \\ [\![a,b]\!] & \mapsto \text{any element of } [\![a,b]\!] \text{ otherwise} \end{cases}.$$

***Case** (3). There exists $n^* \in [\![-1, l-1]\!]$ such that $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) > b_{n^*+2}$, $\forall m \in [\![n^*+2, l]\!] : b_{m-1} \leq \beta(b_m) < b_m$, and*

$$w = \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b_n \text{ if } \exists! n \in [\![0,n^*]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) > b \\ [\![a,b]\!] & \mapsto b_n \text{ if } \exists! n \in [\![n^*+1,l]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) < a \\ [\![a,b]\!] & \mapsto \text{any element of } [\![a,b]\!] \text{ otherwise} \end{cases}.$$

Proof. Let's show that

$$\boxed{\forall n \in [\![0,l]\!] : (\beta(b_n) < b_n \Rightarrow \forall m \in [\![n+1,l]\!] : b_{m-1} \leq \beta(b_m) < b_m)} \tag{5.1}$$

and

$$\boxed{\forall n \in [\![0,l]\!] : (\beta(b_n) > b_{n+1} \Rightarrow \forall m \in [\![n+1,l]\!] : b_{m-1} \leq \beta(b_m) < b_m)}. \tag{5.2}$$

Suppose that $\beta(b_n) < b_n$ for a certain $n \in \mathbb{N}$ and $n+1 \leq l$. We have

$$\beta([\![b_n, b_{n+1}]\!] \setminus \{w([\![b_n, b_{n+1}]\!])\}) \subseteq [\![b_n, b_{n+1}]\!].$$

If $w([\![b_n, b_{n+1}]\!]) = b_{n+1}$, then $b_n \leq \beta(b_n)$ which is a contradiction. Therefore $w([\![b_n, b_{n+1}]\!]) \neq b_{n+1}$ and so $b_n \leq \beta(b_{n+1}) < b_{n+1}$ since b_{n+1} is not a fixed point. By immediate induction, we have $\forall m \in [\![n+1,l]\!] : b_{m-1} \leq \beta(b_m) < b_m$.
Therefore assertion 5.1 holds.
Suppose that $\beta(b_n) > b_{n+1}$ for a certain $n \in \mathbb{N}$ and $n+1 \leq l$. We have

$$\beta([\![b_n, b_{n+1}]\!] \setminus \{w([\![b_n, b_{n+1}]\!])\}) \subseteq [\![b_n, b_{n+1}]\!].$$

If $w([\![b_n, b_{n+1}]\!]) = b_{n+1}$, then $\beta(b_n) \leq b_{n+1}$ which is a contradiction. Therefore $w([\![b_n, b_{n+1}]\!]) \neq b_{n+1}$ and so $b_n \leq \beta(b_{n+1}) < b_{n+1}$ since b_{n+1} is not a fixed point. By immediate induction, we have

$$\forall m \in [\![n+1,l]\!] : b_{m-1} \leq \beta(b_m) < b_m.$$

Therefore assertion 5.2 holds.

With these assertions in hand, we can now start the proof. We treat the $l = \infty$ and $l < \infty$ cases simultaneously.
Case (1). Suppose $\forall n \in [\![0, l]\!] : b_n < \beta(b_n) \leq b_{n+1}$.
Let $a \leq b \in \mathbb{N}$.

Subcase (1.1). Suppose $\exists n \in [\![0, l]\!] : b_n \in [\![a, b]\!]$ and $\beta(b_n) > b$. Let's show that there is a unique such n. Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 \geq n_1 + 1$, and since $(b_n)_{n\in\mathbb{N}}$ is strictly increasing, we have

$$b \geq b_{n_2} \geq b_{n_1+1} \geq \beta(b_{n_1}) > b,$$

which is a contradiction. Therefore there exists a unique $n(a, b) \in [\![0, l]\!]$ such that $b_{n(a,b)} \in [\![a, b]\!]$ and $\beta(b_{n(a,b)}) > b$. If we had $w([\![a, b]\!]) \neq b_{n(a,b)}$, then since $\beta([\![a, b]\!] \setminus \{w([\![a, b]\!])\}) \subseteq [\![a, b]\!]$, we would have $\beta(b_{n(a,b)}) \in [\![a, b]\!]$, which contradicts $\beta(b_{n(a,b)}) > b$. Therefore $w([\![a, b]\!]) = b_{n(a,b)}$.

Subcase (1.2). Suppose $\forall n \in [\![0, l]\!] : (b_n \in [\![a, b]\!] \Rightarrow \beta(b_n) \leq b)$. There is no additional constraint in this subcase, we can choose $w([\![a, b]\!])$ arbitrarily from $[\![a, b]\!]$.
Conversely. If $l \in \mathbb{N} \cup \{-1, \infty\}$, $\{b_n\}_{n\in[\![0,l]\!]}$ is a strictly increasing sequence, $\beta : \mathbb{N} \to \mathbb{N}$ is a function such that $\mathbb{N} \setminus \{b_n\}_{n\in[\![0,l]\!]}$ are the fixed points of β, $\forall n \in [\![0, l]\!] : b_n < \beta(b_n) \leq b_{n+1}$, and

$$w = \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a, b]\!] & \mapsto b_{n(a,b)} \text{ if } \exists n \in [\![0, l]\!] : b_n \in [\![a, b]\!] \text{ and } \beta(b_n) > b \\ [\![a, b]\!] & \mapsto \text{any element of } [\![a, b]\!] \text{ otherwise} \end{cases},$$

then $\forall a \leq b \in [\![a, b]\!] : \beta([\![a, b]\!] \setminus \{w([\![a, b]\!])\}) \subseteq [\![a, b]\!]$. Indeed, let $a \leq b \in \mathbb{N}$. Because of the uniqueness property of $b_{n(a,b)}$ (when it exists), any $x \in [\![a, b]\!] \setminus \{w([\![a, b]\!])\}$ is either a fixed point (in which case $\beta(x) = x \in [\![a, b]\!]$), or satisfies $x = b_n$ with $\beta(b_n) \leq b$, in which case, since $b_n < \beta(b_n)$, we have $a \leq x = b_n < \beta(b_n) = \beta(x) \leq b$.

Suppose now $\neg\,(\forall n \in [\![0, l]\!] : b_n < \beta(b_n) \leq b_{n+1})$. Necessarily, $l \geq 0$. Let $n^* = \max\{n \in [\![-1, l]\!] : \forall m \leq n : b_m < \beta(b_m) \leq b_{m+1}\}$. Necessarily, $n^* \in [\![-1, l-1]\!]$. Notice that if $\beta(b_{n^*+1}) \leq b_{n^*+2}$, then by maximality of n^*, we have $\beta(b_{n^*+1}) < b_{n^*+1}$ since b_{n^*+1} is not a fixed point of β. Therefore we have either $\beta(b_{n^*+1}) < b_{n^*+1}$ or $\beta(b_{n^*+1}) > b_{n^*+2}$.

Case (2). Suppose $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$ and $\beta(b_{n^*+1}) < b_{n^*+1}$. By assertion 5.1, we have $\forall m \in [\![n^* + 2, l]\!] : b_{m-1} \leq \beta(b_m) < b_m$.
Let $a \leq b \in \mathbb{N}$.

Subcase (2.1). Suppose $\exists n \in [\![0, n^*]\!] : b_n \in [\![a, b]\!]$ and $\beta(b_n) > b$. Let's show that there is a unique such n. Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 \geq n_1 + 1$, and since $(b_n)_{n\in\mathbb{N}}$ is strictly increasing, we have

$$b \geq b_{n_2} \geq b_{n_1+1} \geq \beta(b_{n_1}) > b,$$

which is a contradiction. Therefore there exists a unique $n(a, b) \in [\![0, n^*]\!]$ such that $b_{n(a,b)} \in [\![a, b]\!]$ and $\beta(b_{n(a,b)}) > b$. If we had $w([\![a, b]\!]) \neq b_{n(a,b)}$, then since $\beta([\![a, b]\!] \setminus \{w([\![a, b]\!])\}) \subseteq [\![a, b]\!]$, we would have $\beta(b_{n(a,b)}) \in [\![a, b]\!]$, which contradicts $\beta(b_{n(a,b)}) > b$. Therefore $w([\![a, b]\!]) = b_{n(a,b)}$.

Subcase (2.2). Suppose $\exists n \in [\![n^* + 1, l]\!] : b_n \in [\![a, b]\!]$ and $\beta(b_n) < a$. Let's show that there is a unique such n. Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 - 1 \geq n_1$ and $n_2 \geq n^* + 2$, and since $(b_n)_{n\in\mathbb{N}}$ is strictly increasing, we have

$$a > \beta(b_{n_2}) \geq b_{n_2-1} \geq b_{n_1} \geq a,$$

which is a contradiction. Therefore there exists a unique $n'(a, b) \in [\![n^* + 1, l]\!]$ such that $b_{n'(a,b)} \in [\![a, b]\!]$ and $\beta(b_{n'(a,b)}) > b$. If we had $w([\![a, b]\!]) \neq b_{n'(a,b)}$, then since $\beta([\![a, b]\!] \setminus \{w([\![a, b]\!])\}) \subseteq [\![a, b]\!]$, we would have $\beta(b_{n'(a,b)}) \in [\![a,b]\!]$, which contradicts $\beta(b_{n'(a,b)}) > b$. Therefore $w([\![a,b]\!]) = b_{n'(a,b)}$. Let's show that subcases 2.1 and 2.1 are disjoint. Suppose the contrary. Then

$$a > \beta(b_{n'(a,b)}) \geq b_{n'(a,b)-1} > \beta(b_{n'(a,b)-1}) \geq b_{n'(a,b)-2} > \cdots$$
$$> \beta(b_{n^*+2}) \geq b_{n^*+1} \geq \beta(b_{n^*}) > b_{n^*} \geq \beta(b_{n^*-1}) > \cdots > \beta(b_{n(a,b)}) > b,$$

which is a contradiction.

Subcase (2.3). Suppose $\forall n \in [\![0,n^*]\!] : (b_n \in [\![a,b]\!] \Rightarrow \beta(b_n) \leq b)$ and $\forall n \in [\![n^*+1,l]\!] : (b_n \in [\![a,b]\!] \Rightarrow \beta(b_n) \geq a)$. There is no additional constraint in this subcase, we can choose $w([\![a,b]\!])$ arbitrarily from $[\![a,b]\!]$.

Conversely. If $l \geq 0$, $\{b_n\}_{n\in[\![0,l]\!]}$ is a strictly increasing sequence, $\beta : \mathbb{N} \to \mathbb{N}$ is a function such that $\mathbb{N}\backslash\{b_n\}_{n\in[\![0,l]\!]}$ are the fixed points of β, $n^* \in [\![-1,l-1]\!]$, $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) < b_{n^*+1}$, $\forall m \in [\![n^*+2,l]\!] : b_{m-1} \leq \beta(b_m) < b_m$, and

$$w = \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b_{n(a,b)} \text{ if } \exists n \in [\![0,n^*]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) > b \\ [\![a,b]\!] & \mapsto b_{n'(a,b)} \text{ if } \exists n \in [\![n^*+1,l]\!] : b_n \in [\![a,b]\!] \text{ and } \beta(b_n) < a \\ [\![a,b]\!] & \mapsto \text{any element of } [\![a,b]\!] \text{ otherwise} \end{cases},$$

then $\forall a \leq b \in [\![a,b]\!] : \beta([\![a,b]\!] \setminus \{w([\![a,b]\!])\}) \subseteq [\![a,b]\!]$. Indeed, let $a \leq b \in \mathbb{N}$. Because of the uniqueness properties of $b_{n(a,b)}$ and $b_{n'(a,b)}$ (when they exist) and the disjointness of subcases 2.1 and 2.2, any $x \in [\![a,b]\!] \setminus \{w([\![a,b]\!])\}$ is either a fixed point (in which case $\beta(x) = x \in [\![a,b]\!]$), or satisfies $x = b_n$ with $n \in [\![0,n^*]\!]$ and $\beta(b_n) \leq b$, in which case, since $b_n < \beta(b_n)$, we have $a \leq x = b_n < \beta(b_n) = \beta(x) \leq b$, or satisfies $x = b_n$ with $n \geq n^*+1$ and $\beta(b_n) \geq a$, in which case, since $\beta(b_n) < b_n$, we have $a \leq \beta(b_n) = \beta(x) < b_n = x \leq b$.

Case (3). Suppose $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$ and $\beta(b_{n^*+1}) > b_{n^*+2}$. By assertion 5.2, we have $\forall m \geq n^*+2 : b_{m-1} \leq \beta(b_m) < b_m$. Then the treatment from now on is analogous to case 2 (since we haven't used the conditions $\beta(b_{n^*+1}) < b_{n^*+1}$ or $\beta(b_{n^*+1}) > b_{n^*+2}$) with the same necessary and sufficient form of the function w.
□

Requiring that for such functions, $\beta(w([\![a,b]\!])) \notin [\![a,b]\!]$ for all $a \leq b \in \mathbb{N}$, leads to

Corollary 5.3. *Let $\beta : \mathbb{N} \to \mathbb{N}$ and $w : Int_{\omega,*}(\mathbb{N}) \to \mathbb{N}$ be two functions such that*

$$\forall [\![a,b]\!] \in Int_{\omega,*}(\mathbb{N}) : \quad w([\![a,b]\!]) \in [\![a,b]\!]$$
$$\text{and } \beta([\![a,b]\!] \setminus \{w([\![a,b]\!])\}) \subseteq [\![a,b]\!] \text{ and } \beta(w([\![a,b]\!])) \notin [\![a,b]\!].$$

Then either

Case (1). $\beta : \begin{cases} \mathbb{N} & \to \mathbb{N} \\ n & \mapsto n+1 \end{cases}$ *and* $w : \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b \end{cases}$

Case (2). *There exists $n^* \neq u \in \mathbb{N}$ such that* $\beta : \begin{cases} \mathbb{N} & \to \mathbb{N} \\ n & \mapsto n+1 \text{ for } n < n^* \\ n^* & \mapsto u \\ n & \mapsto n-1 \text{ for } n > n^* \end{cases}$ *and*

$$w : \begin{cases} Int_{\omega,*}(\mathbb{N}) & \to \mathbb{N} \\ [\![a,b]\!] & \mapsto b \text{ if } b \in [\![0,n^*]\!] \\ [\![a,b]\!] & \mapsto a \text{ if } a \geq n^*+1 \end{cases}.$$

Proof. Notice that for all $a \in \mathbb{N}$, $\beta(w([a,a])) \notin [a,a]$ implies that $\beta(a) \neq a$, and so the sequence of non-fixed points of β in strictly increasing order is $\{b_n\}_{n\in\mathbb{N}} = \{n\}_{n\in\mathbb{N}}$. We can then apply proposition 5.2.
□

6. Functions with a maximal number of finite invariant supersets

6.1. Solution to the general problem

Proposition 6.1. *Let $\alpha : I \to I$ and $G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I)$. Then the following conditions are equivalent*

1. *$\forall I^* \in \mathcal{P}_{\omega,*}(I) : I^* \subseteq G(I^*)$ and $\alpha(G(I^*)) \subseteq G(I^*)$.*
2. *The orbits of α are finite and there exists $H : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega}(I)$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_\alpha^+(a).$$

Proof. (1) $\Rightarrow$ (2). Let $a \in I$. We have $\alpha(a) \in G(\{a\})$ since $a \in G(\{a\})$ and $\alpha(G(\{a\})) \subseteq G(\{a\})$. By immediate induction, we have $\mathcal{O}_\alpha^+(a) \subseteq G(\{a\})$. Since $G(\{a\})$ is finite, this implies that $\mathcal{O}_\alpha^+(a)$ is finite. Therefore the orbits of α are finite. Furthermore, let $H(I^*) = G(I^*)$ for all $I^* \in \mathcal{P}_{\omega,*}(I)$. Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have $I^* \cup H(I^*) = I^* \cup G(I^*) = G(I^*)$. Let $a \in G(I^*)$. We have $\alpha(a) \in G(I^*)$ since $a \in G(I^*)$ and $\alpha(G(I^*)) \subseteq G(I^*)$. By immediate induction, we have $\mathcal{O}_\alpha^+(a) \subseteq G(I^*)$. Therefore $\bigcup_{a \in G(I^*)} \mathcal{O}_\alpha^+(a) \subseteq G(I^*)$. The reverse inclusion is obvious since $a \in \mathcal{O}_\alpha^+(a)$ for all $a \in G(I^*)$.
(2) $\Rightarrow$ (1). Let $I^* \in \mathcal{P}_{\omega,*}(I)$. To show that $I^* \subseteq G(I^*)$, let $a \in I^*$. We have $a \in \mathcal{O}_\alpha^+(a) \subseteq G(I^*)$. Therefore $I^* \subseteq G(I^*)$. To show that $\alpha(G(I^*)) \subseteq G(I^*)$, let $a \in G(I^*)$. We have

$$\alpha(a) \in \alpha\left(\bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_\alpha^+(a)\right) = \bigcup_{a \in I^* \cup H(I^*)} \alpha\left(\mathcal{O}_\alpha^+(a)\right) \subseteq \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_\alpha^+(a).$$

Therefore $\alpha(G(I^*)) \subseteq G(I^*)$. □

6.2. Solution to a more constrained problem in $\mathbb{N}$

The following proposition gives a characterization of the functions $\alpha : \mathbb{N} \to \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega,*}(\mathbb{N})$ that satisfy the conditions of proposition 6.1, plus the additional hypothesis that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$.

Proposition 6.2. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega,*}(\mathbb{N})$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*),\ \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*).$$

Then α admits an infinite number of fixed points, and letting this sequence be $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, there exists a function $j : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\forall a \in \mathbb{N} : j(a) = n \text{ if } \exists n \in \mathbb{N} : a = b_n$$
$$j(a) \geq r(a) \text{ if } a \notin \{b_n\}_{n \in \mathbb{N}}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \{a, \alpha(a)\}$ and $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$. Conversely, any such α and G are valid solutions.

Proof. **Direct implication.** First, let's show that

$$\boxed{\forall n \in \mathbb{N} : \alpha(\alpha(n)) = \alpha(n) \geq n} \tag{6.1}$$

We have $\forall n \in \mathbb{N} : \max(G(\{n\})) \in \alpha(\{n\})$, so $\alpha(n) = \max(G(\{n\})) \in G(\{n\})$. Let $n \in \mathbb{N}$. We have $\alpha(\alpha(n)) \in G(\{n\}$ since $\alpha(n) \in G(\{n\})$ (we use the hypothesis $\alpha(G(I^*)) \subseteq G(I^*)$ for $I^* = \{n\}$), and so $\alpha(\alpha(n)) \leq \alpha(n)$ since $\alpha(n) = \max(G(\{n\}))$. Moreover, we have $n \in \{n\} \subseteq G(\{n\})$, and so $n \leq \max(G(\{n\})) = \alpha(n)$. Replacing n by $\alpha(n)$, we have $\forall n \in \mathbb{N} : \alpha(\alpha(n)) \geq \alpha(n)$.
Therefore $\forall n \in \mathbb{N} : \alpha(\alpha(n)) = \alpha(n) \geq n$, and so assertion 6.1 holds.
Notice that if α is injective, then by assertion 6.1 we have $\forall n \in \mathbb{N} : \alpha(n) = n$, and α is the identity function. Otherwise, α admits a sequence of fixed points $\{\alpha(n)\}_{n\in\mathbb{N}}$ tending to infinity since $\forall n \in \mathbb{N} : \alpha(n) \geq n$. Let $\{b_n\}_{n\in\mathbb{N}}$ be the sequence of all fixed points of α in strictly increasing order. Let $b_{-1} = -1$ and extend α by setting $\alpha(-1) = -1$. Then $\forall n \in \mathbb{N} \cup \{-1\} : \forall m \in [b_n + 1, b_{n+1} - 1] : \exists k \geq n + 1 : \alpha(m) = b_k$ since $\alpha(m)$ is a fixed point and $\alpha(m) \geq m > b_n$. Moreover, we have $r(m) \leq n + 1$ since $b_{n+1} > m$, and so $r(m) \leq k$. Therefore we can set $j(m) = n$ if $m = b_n$ and $j(m) = k$ if $m \in [b_n + 1, b_{n+1} - 1]$ for some $n \in \mathbb{N} \cup \{-1\}$.
Moreover, from proposition 6.1, there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a\in I^*\cup H(I^*)} \mathcal{O}^+_\alpha(a) = \bigcup_{a\in I^*\cup H(I^*)}\{a, \alpha(a)\}$. It's easy to show that the condition $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$ means that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$.

Conversely. Given a function $\alpha : \mathbb{N} \to \mathbb{N}$ which admits an infinite number of fixed points $\{b_n\}_{n\in\mathbb{N}}$ in strictly increasing order, such that there exists a function $j : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n &\text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) \geq r(a) &\text{ if } a \notin \{b_n\}_{n\in\mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and such that there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a\in I^*\cup H(I^*)}\{a, \alpha(a)\}$ and $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$, then it's easy to check that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \alpha(G(I^*)) \subseteq G(I^*)$ and $\max(G(I^*)) \in \alpha(I^*)$. ☐

Remark 6.3. *If α is assumed to be bijective, then we can also use proposition 7.15 to deduce $\alpha = Id$. Indeed, suppose that $\alpha : \mathbb{N} \to \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathcal{P}_{\omega,*}(\mathbb{N})$ are such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*),\ \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*),$$

and let $\beta : \begin{cases} \mathbb{N} & \to \mathbb{N} \\ n & \mapsto n+1 \end{cases}$, $\phi = \beta \circ \alpha$, *and for all* $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, $u(I^*) \in I^*$ *such that* $\alpha(u(I^*)) = \max(G(I^*))$.
Then since $P_2(\phi, G, u)$ is true and $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in G(I^*) : |\mathcal{O}^+_\beta(a)| = +\infty$, we have $\alpha = Id$.

In the following proposition, we require furthermore that $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \to Int_{\omega,*}(\mathbb{N})$.

Proposition 6.4. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \to Int_{\omega,*}(\mathbb{N})$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*),\ \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*).$$

Then α admits an infinite number of fixed points, and if this sequence is $\{b_n\}_{n\in\mathbb{N}}$ in strictly increasing order, there exists a function $j : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)},$$

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n &\text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) = r(a) &\text{ if } a > b_0 \text{ and } a \notin \{b_n\}_{n\in\mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$*, and*

$$j(0) \geq j(1) \cdots \geq j(b_0 - 1),$$

and there exists a function $u : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathbb{N}$ *such that* $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = [\![u(I^*), \max\{\alpha(a)\}_{a \in I^*}]\!]$*, and*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\} \leq u(I^*) \leq \min(I^*),$$

Conversely, any such α *and* G *are valid solutions.*

Proof. **Direct implication.** α and G satisfy of course all the previous properties of proposition 6.2.
For all $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, we let $[\![u(I^*), v(I^*)]\!] := G(I^*)$. Assume by way of contradiction that $v(I^*) \neq \max\{\alpha(a)\}_{a \in I^*}$. We have $\forall a \in I^* \subseteq G(I^*) : \alpha(a) \leq v(I^*)$, so $\max\{\alpha(a)\}_{a \in I^*} \leq v(I^*)$. Since $v(I^*) \neq \max\{\alpha(a)\}_{a \in I^*}$, we have $\max\{\alpha(a)\}_{a \in I^*} < v(I^*)$ which is a contradiction since $v(I^*) = \max(G(I^*)) \in \alpha(I^*)$. So $v(I^*) = \max\{\alpha(a)\}_{a \in I^*}$.
Applying this to $I^* = \{b_n, b_{n+1}\}$ for $n \in \mathbb{N}$ gives $G(\{b_n, b_{n+1}\}) = [u(\{b_n, b_{n+1}\}), b_{n+1}]$ where $u(\{b_n, b_{n+1}\}) \leq b_n$, and so $\forall x \in [\![b_n, b_{n+1}]\!] : \alpha(x) \leq b_{n+1}$ since $G(I^*)$ is preserved by α. Hence $\forall x \in]\!]b_n, b_{n+1}]\!] : \alpha(x) = b_{n+1}$ (since $j(x) \geq n+1$ by proposition 6.2).
Applying it for $I^* = \{x\}$ for $x \in [0, b_0[$ gives $G(I^*) = [\![u(\{x\}), \alpha(x)]\!]$ where $u(\{x\}) \leq x$. Since $\alpha(x)$ is a fixed point, we have $\alpha(x) \geq b_0$. Therefore, $\forall y \in [a, b_0[$: $\alpha(y) \leq \alpha(x)$ since $G(I^*)$ is preserved by α. So $\alpha(0) \geq \alpha(1) \cdots \geq \alpha(b_0 - 1)$, which is equivalent to $j(0) \geq j(1) \geq \cdots \geq j(b_0 - 1)$.
Moreover, for all $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, we have $u(I^*) \leq \min(I^*)$ since $I^* \subseteq G(I^*)$.
Furthermore, for all $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, the set $\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\}$ is non-empty (it contains b_0), and letting $u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\}$, we have necessarily $u(I^*) \geq u^*(I^*)$. Indeed, if $u(I^*) > b_0$, the inequality is obvious. Otherwise, it is true since $\alpha(u(I^*)) \in G(I^*)$ (since $\alpha(G(I^*)) \subseteq G(I^*)$ and $u(I^*) \in G(I^*)$), and therefore $\alpha(u(I^*)) \leq \max(G(I^*)) = \max\{\alpha(a)\}_{a \in I^*}$.

Conversely. Given a function $\alpha : \mathbb{N} \to \mathbb{N}$ which admits an infinite number of fixed points $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, such that there exists a function $j : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\forall a \in \mathbb{N} : j(a) = n \text{ if } \exists n \in \mathbb{N} : a = b_n$$
$$j(a) = r(a) \text{ if } a > b_0 \text{ and } a \notin \{b_n\}_{n \in \mathbb{N}}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and

$$j(0) \geq j(1) \cdots \geq j(b_0 - 1),$$

and such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = [\![u(I^*), \max\{\alpha(a)\}_{a \in I^*}]\!]$, where $u : \mathcal{P}_{\omega,*}(\mathbb{N}) \to \mathbb{N}$ is a function such that

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\} \leq u(I^*) \leq \min(I^*),$$

then it's clear that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$. To check that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \alpha(G(I^*)) \subseteq G(I^*)$, we take $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$ and $x \in G(I^*)$, and distinguish between the cases $x \geq b_0$ and $x < b_0$. In the first case, since $x \leq \max\{\alpha(a)\}_{a \in I^*}$, $\max\{\alpha(a)\}_{a \in I^*}$ is a fixed point, and $x \geq b_0$, then it follows that $u(I^*) \leq x \leq \alpha(x) \leq \max\{\alpha(a)\}_{a \in I^*}$. In the second case, we have $u^*(I^*) \leq u(I^*) \leq x < b_0$, therefore $\alpha(x) \leq \alpha(u^*(I^*)) \leq \max\{\alpha(a)\}_{a \in I^*}$ since α is non-increasing on $[\![0, b_0 - 1]\!]$ and by the definition of $u^*(I^*)$. Moreover, since $\alpha(x)$ is a fixed point, it is obvious that $\alpha(x) \geq b_0 > x \geq u(I^*)$. Therefore we are done in both cases and it follows that $\alpha(G(I^*)) \subseteq G(I^*)$. □

7. Functions with a maximal number of finite internally-1-quasi-invariant supersets

Below, we define two similar predicates (in the sense of propositions depending on mathematical objects) that model the property of direct image preservation by $\phi : I \to I$ of finite supersets of finite subsets of I, up to one element, denoted by $P_1(\phi, G, u)$ and $P_2(\phi, G, u)$. We will be concerned with these predicates in the following three subsections.

Definition 7.1. *We say that a 3-tuple* $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ *satisfies property* $P_1(\phi, G, u)$ *if*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in G(I^*), I^* \subseteq G(I^*) \text{ and } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

Definition 7.2. *We say that a 3-tuple* $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ *satisfies property* $P_2(\phi, G, u)$ *if*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in I^*, I^* \subseteq G(I^*) \text{ and } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

The following simple lemma is often used (without mention) in the paper.

Lemma 7.3. *Let* I *be an infinite set and* $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ *a 3-tuple satisfying* $P_1(\phi, G, u)$*. Then we have*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(|\mathcal{O}^+_\phi(u(I^*))| = \infty \Rightarrow \phi(u(I^*)) \notin G(I^*) \text{ and } \phi(u(I^*)) \notin I^*\right).$$

Proof. Let $I^* \in \mathcal{P}_{\omega,*}(I)$ such that $|\mathcal{O}^+_\phi(u(I^*))| = \infty$. From lemma 3.1, $u(I^*), \phi(u(I^*), \phi^2(u(I^*)), \cdots$ must be distinct. In particular, $\forall n \geq 1, \phi^n(u(I^*)) \neq u(I^*)$. Thus we have $\phi(u(I^*)) \notin G(I^*)$ (otherwise we would have $\mathcal{O}^+_\phi(u(I^*)) \subseteq G(I^*)$ which is impossible since $G(I^*)$ is finite and $|\mathcal{O}^+_\phi(u(I^*))| = \infty$). Since $I^* \subseteq G(I^*)$, this implies that $\phi(u(I^*)) \notin I^*$. □

We also have

Lemma 7.4. *Let* I *be an infinite set and* $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ *a 3-tuple satisfying* $P_2(\phi, G, u)$*. Then we have*

$$\left(\forall I^* \in \mathcal{P}_{\omega,*}(I) : |\mathcal{O}^+_\phi(u(I^*))| = \infty\right) \Rightarrow (\forall b \in I : \forall m \in \mathbb{N} : u(\{b, \cdots, \phi^m(b)\} = \phi^m(b)).$$

Proof. Suppose that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : |\mathcal{O}^+_\phi(u(I^*))| = \infty$. Let $b \in I$ and $m \in \mathbb{N}$. Since we should have $u(\{b, \cdots, \phi^m(b)\}) \in \{b, \cdots, \phi^m(b)\}$ and $\phi(u(\{b, \cdots, \phi^m(b)\})) \notin \{b, \cdots, \phi^m(b)\}$ by lemma 7.3, we have necessarily $u(\{b, \cdots, \phi^m(b)\}) = \phi^m(b)$. □

7.1. Solution to problem $P_1(\phi, G, u)$

We first prove the following structural proposition.

Proposition 7.5. *Let* I *be an infinite set and* $\beta : I \to I$*.*
Let $G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I)$ *and* $v : \mathcal{P}_{\omega,*}(I) \to I$*. Then*
$P_1(\beta, G, v) \Leftrightarrow$ *there exist three functions* $H, \overline{H}, \widetilde{H} : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_\omega(I)$ *such that for all* $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}^+_\beta(a)| = +\infty\}$, $\overline{H}(I^*), \widetilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}^+_\beta(a)| < +\infty\}$, *and*
(a) $\exists z \in G(I^*) : |\mathcal{O}^+_\beta(z)| = +\infty$ *and*

$$\begin{cases} H(I^*) & \neq \emptyset \\ \overline{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}^+_\beta(a)\right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \cdots, \xi_\beta(H(I^*))\}\right) \\ v(I^*) & = \xi_\beta(H(I^*)) \end{cases}$$

or
(b) $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$ *and*

$$\begin{cases} H(I^*) & = \emptyset \\ \overline{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}_\beta^+(a)\right) \cup \left(\bigcup_{a\in\overline{H}(I^*)}\{a, \beta(a), \beta^2(a), \cdots, v(I^*)\}\right) \end{cases}.$$

Proof. $(\Rightarrow)$: Suppose $P_1(\beta, G, v)$.
Let

$$H : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty\} \end{cases},$$

$$\overline{H} : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty \text{ and } v(I^*) \in \mathcal{O}_\beta^+(a)\} \end{cases}$$

and

$$\widetilde{H} : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty \text{ and } v(I^*) \notin \mathcal{O}_\beta^+(a)\} \end{cases}.$$

Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have $G(I^*) = H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$. Therefore $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$.
(a) Suppose that $\exists z \in G(I^*) : |\mathcal{O}_\beta^+(z)| = +\infty$. So $H(I^*) \neq \emptyset$.
Let's show that $v(I^*) = \xi_\beta(H(I^*))$. Assume by way of contradiction that $\exists b \in H(I^*) : v(I^*) \notin \mathcal{O}_\beta^+(b)$. Then we must have $\mathcal{O}_\beta^+(b) \subseteq G(I^*)$, which is impossible since $|\mathcal{O}_\beta^+(b)| = +\infty$ and $G(I^*)$ is finite. So $v(I^*) \in \bigcap_{b\in H(I^*)} \mathcal{O}_\beta^+(b) = \mathcal{O}_\beta^+(\xi_\beta(H(I^*)))$. In particular, $|\mathcal{O}_\beta^+(v(I^*))| = +\infty$ and so $v(I^*) \in H(I^*)$. So there exist $m, n \in \mathbb{N}$ such that $v(I^*) = \beta^m(\xi_\beta(H(I^*)))$ and $\xi_\beta(H(I^*)) = \beta^n(v(I^*))$. This can only work if $m = n = 0$ due to lemma 3.1 and therefore $v(I^*) = \xi_\beta(H(I^*))$.
Now we must have $\overline{H}(I^*) = \emptyset$ (otherwise there exists $a' \in G(I^*)$ such that $|\mathcal{O}_\beta^+(a')| < +\infty$ and $v(I^*) = \xi_\beta(H(I^*)) \in \mathcal{O}_\beta^+(z) \cap \mathcal{O}_\beta^+(a') = \emptyset$ by lemma 3.15, a contradiction).
Let's show that $G(I^*) = \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}_\beta^+(a)\right) \cup \left(\bigcup_{a\in H(I^*)}\{a, \beta(a), \beta^2(a), \cdots, \xi_\beta(H(I^*))\}\right)$. $P_1(\beta, G, v)$ shows that the inclusion $\supseteq$ is true. Conversely, let $a \in G(I^*)$. We have either $a \in H(I^*)$ or $a \in \widetilde{H}(I^*)$. Therefore the inclusion $\subseteq$ is also true.
(b) Suppose that $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$. So $H(I^*) = \emptyset$ and $\overline{H}(I^*) \neq \emptyset$.
Let's show that $G(I^*) = \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}_\beta^+(a)\right) \cup \left(\bigcup_{a\in\overline{H}(I^*)}\{a, \beta(a), \beta^2(a), \cdots, v(I^*)\}\right)$.
$P_1(\beta, G, v)$ shows that the inclusion $\supseteq$ is true. Conversely, let $a \in G(I^*)$. We have $a \in \widetilde{H}(I^*)$ or $a \in \overline{H}(I^*)$ and so the inclusion $\subseteq$ is true.
$(\Leftarrow)$: Suppose that $H, \overline{H}$ and $\widetilde{H}$ exist with the desired properties. Let $I^* \in \mathcal{P}_{\omega,*}(I)$.
(a) If $\exists z \in G(I^*) : |\mathcal{O}_\beta^+(z)| = +\infty$, then $\overline{H}(I^*) = \emptyset$ and so $G(I^*) \supseteq \widetilde{H}(I^*) \cup H(I^*) \supseteq I^*$ using the hypotheses. Moreover, we have clearly $v(I^*) \in G(I^*)$ and $\beta(G(I^*) \setminus \{v(I^*)\}) \subseteq G(I^*)$.
(b) If $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$, we have $H(I^*) = \emptyset$ and we can check similarly that $P_1(\beta, G, v)$ is true. □

The following proposition characterizes the existential problem $\exists G \exists v : P_1(\beta, G, v)$ to $\widetilde{P}(\beta)$ in terms of $\widetilde{P}(\beta)$.

Proposition 7.6. *Let I be an infinite set and $\beta : I \to I$. Then*

$$(\exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : \exists v : \mathcal{P}_{\omega,*}(I) \to I : P_1(\beta, G, v)) \Leftrightarrow \widetilde{P}(\beta).$$

Proof. $(\Rightarrow)$: Let $a, b \in I$ such that $|\mathcal{O}_\beta^+(a)| = |\mathcal{O}_\beta^+(b)| = +\infty$. Assume by way of contradiction that $\mathcal{O}_\beta^+(a) \cap \mathcal{O}_\beta^+(b) = \emptyset$. So $a \neq b$. Let $I^* = \{a, b\}$. Let $G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I)$ and $v : \mathcal{P}_{\omega,*}(I) \to I$

such that $P_1(\beta, G, v)$. Suppose by way of contradiction that $v(I^*) \in \mathcal{O}^+_\beta(a)$. Then $v(I^*) \notin \mathcal{O}^+_\beta(b)$ since $\mathcal{O}^+_\beta(a) \cap \mathcal{O}^+_\beta(b) = \emptyset$. Using the hypothesis, we obtain $\mathcal{O}^+_\beta(b) \subseteq G(I^*)$ which is impossible since $G(I^*)$ is finite and $\mathcal{O}^+_\beta(b)$ is infinite. Therefore $v(I^*) \notin \mathcal{O}^+_\beta(a)$. But this is also impossible for the same reason. So we must have $\mathcal{O}^+_\beta(a) \cap \mathcal{O}^+_\beta(b) \neq \emptyset$.
$(\Leftarrow)$: Suppose that $\widetilde{P}(\beta)$ is true. Let

$$H : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in I^* : |\mathcal{O}^+_\beta(a)| = +\infty\} \end{cases},$$

$$\widetilde{H} : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in I^* : |\mathcal{O}^+_\beta(a)| < +\infty\} \end{cases},$$

$$G : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_{\omega,*}(I) \\ I^* & \mapsto \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}^+_\beta(a)\right) \cup \left(\bigcup_{a\in H(I^*)}\{a, \beta(a), \beta^2(a), \cdots, \xi_\beta(H(I^*))\}\right) \end{cases},$$

and

$$v : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to I \\ I^* & \mapsto \xi_\beta(H(I^*)) \text{ if } H(I^*) \neq \emptyset \\ I^* & \mapsto \text{an arbitrary element of } I^* \text{ otherwise} \end{cases},$$

where we used the fact $\mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}^+_\beta(a)| = +\infty\}) \subseteq D_\beta$ (see lemma 3.19). Then $P_1(\beta, G, v)$ holds. □

7.2. Solution to problem $P_2(\phi, G, u)$

Proposition 7.5 implies, trivially, the following characterization of $P_2(\phi, G, u)$. It is not known to the authors if $P_2(\phi, G, u)$ admits a simpler description.

Proposition 7.7. *Let I be an infinite set and $\beta : I \to I$.
Let $G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I)$ and $v : \mathcal{P}_{\omega,*}(I) \to I$. Then
$P_2(\beta, G, v) \Leftrightarrow$ there exist three functions $H, \overline{H}, \widetilde{H} : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_\omega(I)$ such that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}^+_\beta(a)| = +\infty\}$, $\overline{H}(I^*), \widetilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}^+_\beta(a)| < +\infty\}$, and
(a) $\exists z \in G(I^*) : |\mathcal{O}^+_\beta(z)| = +\infty$ and*

$$\begin{cases} H(I^*) & \neq \emptyset \\ \overline{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}^+_\beta(a)\right) \cup \left(\bigcup_{a\in H(I^*)}\{a, \beta(a), \beta^2(a), \cdots, \xi_\beta(H(I^*))\}\right) \\ v(I^*) & = \xi_\beta(H(I^*)) \in I^* \end{cases}$$

*or
(b) $\forall a \in G(I^*) : |\mathcal{O}^+_\beta(a)| < +\infty$ and*

$$\begin{cases} H(I^*) & = \emptyset \\ \overline{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a\in\widetilde{H}(I^*)} \mathcal{O}^+_\beta(a)\right) \cup \left(\bigcup_{a\in\overline{H}(I^*)}\{a, \beta(a), \beta^2(a), \cdots, v(I^*)\}\right) \\ v(I^*) & \in I^* \end{cases}.$$

In what follows, we direct our attention to the existential problem $\exists G \exists u : P_2(\phi, G, u)$, which we try to characterize in terms of a condition on ϕ. To achieve that, let's first prove

Proposition 7.8. *Let I be an infinite set and $\beta : I \to I$. Then, if $D_\beta = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I)$: $\xi_\beta(I^*) \in I^*$, then $(\exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : \exists v : \mathcal{P}_{\omega,*}(I) \to I : P_2(\beta, G, v))$.*

Proof. Suppose that $D_\beta = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\beta(I^*) \in I^*$. Then, by lemma 3.15, and using the previous notations of proposition 7.6, we have $\forall I^* \in \mathcal{P}_{\omega,*}(I) : H(I^*) \in \{I^*, \emptyset\}$ and so $v(I^*) \in I^*$. Therefore, there exist G and v with the desired properties. □

Definition 7.9. *Let I be a set and $\phi : I \to I$.*
We define the partial order $\preceq_\phi$ on I by : $a \preceq_\phi b \Leftrightarrow b \in \mathcal{O}_\phi^+(a)$.

The following proposition characterizes, under the condition that all the orbits of ϕ are infinite, the existential problem $\exists G \exists u : P_2(\phi, G, u)$ in terms of the orbital structure of ϕ.

Proposition 7.10. *Let I be an infinite set, $\phi : I \to I$ such that $(\forall a \in I) : \mathcal{O}_\phi^+(a)$ is infinite.*
Then the following conditions are equivalent :

1. $\exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \to I : P_2(\phi, G, u)$.
2. $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^* \neq \emptyset$,
3. $D_\phi = \mathcal{P}_{\omega,*}(I)$ *and* $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^* = \{\xi_\phi(I^*)\}$.
4. $D_\phi = \mathcal{P}_{\omega,*}(I)$ *and* $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\phi(I^*) \in I^*$,
5. $\forall n \in \mathbb{N}^* : \forall (a_1, \cdots, a_n) \in I^n$ *which are distinct* : $\exists d \in [\![1, n]\!] : \exists (m_1, \cdots, m_n) \in \mathbb{N}^n$ *with* $m_d = 0$: $\forall i \in [\![1, n]\!] : a_d = \phi^{m_i}(a_i)$.
6. $\forall (a, b) \in I^2$ *with* $a \neq b$, $(\exists n \in \mathbb{N} : a = \phi^n(b))$ *or* $(\exists n \in \mathbb{N} : b = \phi^n(a))$.
7. $\preceq_\phi$ *is a total order on I.*

Proof. (1) ⇒ (2) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have from proposition 7.5 that

$$u(I^*) = \xi_\phi(H(I^*)) \in \cap_{a\in H(I^*)} \mathcal{O}_\phi^+(a) = \cap_{a\in G(I^*)} \mathcal{O}_\phi^+(a) \subseteq \cap_{a\in I^*} \mathcal{O}_\phi^+(a),$$

where we used the fact that here we have necessarily $H(I^*) = G(I^*) \supseteq I^*$. Moreover, we have $u(I^*) \in I^*$. So $\left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^* \neq \emptyset$.
(2) ⇒ (3) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Since $\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a) \neq \emptyset$, then $I^* \in D_\phi$. Let $x \in \left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^*$. Since $x \in \bigcap_{a\in I^*} \mathcal{O}_\phi^+(a) = \mathcal{O}_\phi^+(\xi_\phi(I^*))$ by lemma 3.12 and $\xi_\phi(I^*) = \phi^{m_x^{\xi_\phi(I^*)}}(x)$ because $x \in I^*$ (see definition 3.10), then $\exists n \geq m_x^{\xi_\phi(I^*)} : x = \phi^n(x)$. Since $\{x, \phi(x), \phi^2(x), \cdots\}$ are distinct (see lemma 3.1), we have necessarily $n = 0$ and so $m_x^{\xi_\phi(I^*)} = 0$. This implies $\xi_\phi(I^*) = \phi^{m_x^{\xi_\phi(I^*)}}(x) = x \in I^*$. Since this is true for all $x \in \left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^*$, we have $\left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^* = \{\xi_\phi(I^*)\}$.
(3) ⇒ (4) : This is clear.
(4) ⇒ (1) : This is proposition 7.8.
(4) ⇒ (5) : Let $n \in \mathbb{N}^*$ and let $a_1, \cdots, a_n$ be distinct elements of I. Applying (4) to $I^* = \{a_1, \cdots, a_n\}$ gives $\xi_\phi(I^*) \in I^*$ which is the content of (5).
(5) ⇒ (2) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Write I^* as $\{a_1, \cdots, a_n\}$ where n is the cardinality of I^*. From (5), we have $a_d \in \left(\bigcap_{a\in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^*$, and so this last set is not empty.
(5) ⇒ (6) : This is clear.
(6) ⇒ (5) : Let $n \in \mathbb{N}^*$ and let $a_1, \cdots, a_n$ be distinct elements of I. Assume by way of contradiction that $\forall d \in [\![1, n]\!] : \exists l(d) \in [\![1, n]\!] \setminus \{d\} : a_d \notin \mathcal{O}_\phi^+(a_{l(d)})$. By (6), this implies that $\forall d \in [\![1, n]\!] : a_{l(d)} \in \mathcal{O}_\phi^+(a_d)$. Besides, we have necessarily $\exists d \in [\![1, n]\!] : \exists e \in \mathbb{N}^* : l^e(d) = d$. Otherwise, we would have $\forall d \in [\![1, n]\!] : \forall e \in \mathbb{N}^* : l^e(d) \neq d$, which implies that $\forall \widetilde{d} \in [\![1, n]\!] : \{\widetilde{d}, l(\widetilde{d}), l^2(\widetilde{d}), \cdots\}$ are distinct. This is impossible since this infinite set is included in $[\![1, n]\!]$. Therefore, $\exists (n_1, \cdots, n_e) \in \mathbb{N}^e : a_d = a_{l^e(d)} = \phi^{n_e}(a_{l^{e-1}(d)}) = \phi^{n_e}(\phi^{n_{e-1}}(a_{l^{e-2}(d)})) = \cdots = \phi^{\sum_{i=1}^e n_i}(a_d)$. Since $\mathcal{O}_\phi^+(a_d)$ is infinite, we have by lemma

3.1 that $\forall i \in [1,e] : n_i = 0$, which implies in particular that $a_{l(d)} = a_d$, a contradiction. Therefore we have the result.
(6) $\Leftrightarrow$ (7) : This is just the definition of a total order. $\square$

From this proposition, the following corollary characterizing the existential problem $\exists G \exists u : P_2(\phi, G, u)$ with no additionnal assumption can be easily derived.

Corollary 7.11. *Let I be an infinite set and $\phi : I \to I$. Denote by I_ϕ^{inf} (resp. I_ϕ^{fin}) the subsets of I consisting of the elements which have an infinite (resp. finite) orbit under ϕ. Notice that I_ϕ^{inf} and I_ϕ^{fin} are both invariant under ϕ and that $\{I_\phi^{inf}, I_\phi^{fin}\}$ is a partition of I. Then the following conditions are equivalent :*

1. $\exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \to I : P_2(\phi, G, u)$.
2. $(\preceq_\phi)_{|I_\phi^{inf}}$ *is a total order on* I_ϕ^{inf}.

Lemma 7.12. *Let I be a set and $\phi : I \to I$.*
Suppose that $\exists a \in I : \mathcal{O}_\phi^+(a) = I$.
Then $\preceq_\phi$ is a total order on I.

Proof. Let $x, y \in I$. We can write $x = \phi^n(a)$ and $y = \phi^m(a)$ for $m, n \in \mathbb{N}$. If $m \leq n$, we have $y \preceq_\phi x$. Otherwise we have $x \preceq_\phi y$. Hence $\preceq_\phi$ is a total order. $\square$

The following proposition is a supplement to proposition 7.10 where we also suppose the existence of a cofinite orbit.

Proposition 7.13. *Let I be an infinite set and $\phi : I \to I$ such that $\forall a \in I : \mathcal{O}_\phi^+(a)$ is infinite. Then we have*

$$\begin{aligned}
&(\exists \widetilde{a} \in I : \mathcal{O}_\phi^+(\widetilde{a}) \text{ is cofinite}) \\
&\text{ and } (\exists u : \mathcal{P}_{\omega,*}(I) \to I : \exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : P_2(\phi, G, u)) \\
&\Leftrightarrow \left(\exists \widetilde{a} \in I : \mathcal{O}_\phi^+(\widetilde{a}) \text{ is cofinite}\right) \text{ and } (\preceq_\phi \text{ is a total order on } I) \\
&\Leftrightarrow \exists \alpha : (\mathbb{N}, \leq) \to (I, \preceq_\phi) \text{ such that } \alpha \text{ is an increasing bijection} \\
&\Leftrightarrow \exists \widetilde{a} \in I : \mathcal{O}_\phi^+(\widetilde{a}) = I.
\end{aligned}$$

Proof. Suppose that $\exists \widetilde{a} \in I : \mathcal{O}_\phi^+(\widetilde{a}) = I$. Let $\alpha = \begin{cases} \mathbb{N} & \to I \\ n & \mapsto \phi^n(\widetilde{a}) \end{cases}$. Clearly, α is increasing. It is injective by lemma 3.1. It is surjective by the hypothesis.
Conversely, suppose that there is such an increasing bijection α from $\mathbb{N}$ to I. For all $n \in \mathbb{N}$, we have $\alpha(n+1) = \phi(\alpha(n))$. Indeed, since α is increasing, we have $\exists l(n) \in \mathbb{N} : \alpha(n+1) = \phi^{l(n)}(\alpha(n))$. Suppose by way of contradiction that $l(n) \neq 1$. If $l(n) = 0$, we would have $\alpha(n+1) = \alpha(n)$, contradicting the injectivity of α. If $l(n) \geq 2$, let $m \in \mathbb{N}$ such that $\alpha(m) = \phi(\alpha(n))$. If $m \geq n+1$, then $\alpha(n+1) \preceq_\phi \alpha(m)$, and so $\phi(\alpha(n)) = \alpha(m) \in \mathcal{O}_\phi^+(\alpha(n+1)) = \mathcal{O}_\phi^+(\phi^{l(n)}(\alpha(n)))$. This implies $\exists u \in \mathbb{N} : \phi(\alpha(n)) = \phi^{l(n)-1+u}(\phi(\alpha(n)))$ where $l(n) - 1 + u \geq 1$, contradicting lemma 3.1. If $m \leq n$, then $\alpha(m) \preceq_\phi \alpha(n)$, and so $\alpha(n) \in \mathcal{O}_\phi^+(\alpha(m)) = \mathcal{O}_\phi^+(\phi(\alpha(n)))$. This implies $\exists u \in \mathbb{N} : \alpha(n) = \phi^{u+1}(\alpha(n))$ where $u + 1 \geq 1$, contradicting again lemma 3.1. Hence $l(n) = 1$ and the relation $\alpha(n+1) = \phi(\alpha(n))$ is proved. Now consider $\widetilde{a} = \alpha(0)$. Then clearly $\mathcal{O}_\phi^+(\widetilde{a}) = \alpha(\mathbb{N}) = I$. This establishes the equivalence of the last two statements.
Using lemma 7.12, the last statement clearly implies the second. Let's prove that the second one implies the last one. Using lemma 3.7, we have that all the orbits of ϕ are cofinite. Let $b \in I$ such that $|I \setminus \mathcal{O}_\phi^+(b)|$ is minimal. Suppose by way of contradiction that $\mathcal{O}_\phi^+(b) \neq I$ and let $c \in I \setminus \mathcal{O}_\phi^+(b)$. Since $\preceq_\phi$ is a total

order, we have necessarily $b \in \mathcal{O}_\phi^+(c)$. Specifically, we have $\exists n \geq 1 : b = \phi^n(c)$. Since $c, \phi(c), \phi^2(c), \cdots$ are distinct by lemma 3.1, we have, thus, $|I \setminus \mathcal{O}_\phi^+(c)| < |I \setminus \mathcal{O}_\phi^+(b)|$ which is a contradiction. Hence $\mathcal{O}_\phi^+(b) = I$. The first and second statements are equivalent by corollary 3.8 and proposition 7.10. ☐

Note that we can also prove explicitly that

$$\exists a \in I : \mathcal{O}_\phi^+(a) = I \Rightarrow (\exists u : \mathcal{P}_{\omega,*}(I) \to I : \exists G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I) : P_2(\phi, G, u)).$$

Indeed, suppose $\exists a \in I : \mathcal{O}_\phi^+(a) = I$ and define

$$n : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathbb{N} \\ I^* & \mapsto \max\{n \in \mathbb{N} : \phi^n(a) \in I^*\} \end{cases}$$

which is well-defined because $\{a, \phi(a), \phi^2(a), \cdots\}$ are distinct (lemma 3.1) and let

$$u : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to I \\ I^* & \mapsto \phi^{n(I^*)}(a) \end{cases}$$

and

$$G : \begin{cases} \mathcal{P}_{\omega,*}(I) & \to \mathcal{P}_{\omega,*}(I) \\ I^* & \mapsto \{a, \phi(a), \cdots, \phi^{n(I^*)}(a)\} \end{cases}.$$

Then u and G clearly satisfy the requirements.

Remark 7.14. *There are many maps $\phi : \mathbb{N} \to \mathbb{N}$ such that $\exists a \in \mathbb{N} : \mathcal{O}_\phi^+(a) = \mathbb{N}$ (not just the successor function). They are conjugated to the successor function and can be found by choosing a bijection $\alpha : \mathbb{N} \to \mathbb{N}$ and setting $\phi = \alpha \circ succ \circ \alpha^{-1}$. For example, there is the function*

$$\begin{cases} \mathbb{N} & \to \mathbb{N}^* \\ 0 & \mapsto 2 \\ 2i+1 & \mapsto 2i+4 \text{ for all } i \in \mathbb{N} \\ 2i+2 & \mapsto 2i+1 \text{ for all } i \in \mathbb{N} \end{cases}$$

7.3. An indivisibility property for 3-tuples (ϕ, G, u) satisfying $P_2(\phi, G, u)$

3-tuples $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ satisfying $P_2(\phi, G, u)$ enjoy the following indivisibility property with respect to composition of maps $\beta : I \to I$ satisfying $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$ with bijective maps $\alpha : I \to I$ satisfying $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$.

Proposition 7.15. *Let I be a set and $(\phi : I \to I, G : \mathcal{P}_{\omega,*}(I) \to \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \to I)$ a 3-tuple satisfying $P_2(\phi, G, u)$.*
Then if $\phi = \beta \circ \alpha$ where $\beta : I \to I$ is a map satisfying $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$ and $\alpha : I \to I$ is a bijective map such that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$, then $\alpha = id$ and $\beta = \phi$.

Proof. First, notice that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, α bijective and $\alpha(G(I^*)) \subseteq G(I^*)$ implies that α induces a bijection on $G(I^*)$.
Let $v = \begin{cases} \mathcal{P}_{\omega,*}(I) & \to I \\ I^* & \mapsto \alpha(u(I^*)) \in G(I^*) \end{cases}$. Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Since α induces a bijection on $G(I^*)$, it is easily seen that $\beta(G(I^*) \setminus \{v(I^*)\}) \subseteq G(I^*)$. Since $\exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$, it follows from proposition 7.5 that $v(I^*) = \xi_\beta(H(I^*))$, where we have used the notation of that proposition. Specifying this relation for $I^* = \{b\}$, one has $\alpha(b) = \alpha(u(\{b\})) = v(\{b\}) = \xi_\beta(H(\{b\})$ for all $b \in I$.
Since $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$, we have $\forall b \in I : \alpha(b) \in \alpha(G(\{b\})) \subseteq G(\{b\})$, and by induction $\mathcal{O}_\alpha^+(b) \subseteq G(\{b\})$. Since $G(\{b\})$ is finite and α is bijective, we have from lemma 3.1 $\exists n_b \geq 1 : \alpha^{n_b}(b) = b$.

This implies in particular that $\mathcal{O}_\beta^+(b) = \mathcal{O}_\beta^+(\alpha^{n_b}(b)) = \mathcal{O}_\beta^+(\xi_\beta(H(\{\alpha^{n_b-1}(b)\}))) = \bigcap_{c\in H(\{\alpha^{n_b-1}(b)\})} \mathcal{O}_\beta^+(c)$ is an orbit arising as a finite intersection of infinite orbits. Thus it is infinite, which is true for all $b \in I$. Now let $b \in I$. We have $|\mathcal{O}_\beta^+(b)| = |\mathcal{O}_\beta^+(\alpha(b))| = +\infty$. Let $I^* = H(\{\alpha(b)\})$. Since $\alpha(b) \in I^*$, we have $\xi_\beta(I^*) \in \mathcal{O}_\beta^+(\alpha(b))$. Hence $\alpha(\alpha(b)) = \xi_\beta(I^*) \in \mathcal{O}_\beta^+(\alpha(b))$. So $\forall b \in I : \exists k_b \in \mathbb{N} : \alpha(\alpha(b)) = \beta^{k_b}(\alpha(b))$. We have $\forall b \in I : \alpha(\alpha(\alpha(b))) = \beta^{k_{\alpha(b)}}(\alpha(\alpha(b))) = \beta^{k_{\alpha(b)}+k_b}(\alpha(a))$. In general, $\forall b \in I : \forall n \geq 1 : \alpha^n(b) \in \mathcal{O}_\beta^+(\alpha(b))$. Let $b \in I$ and choose $n = n_{\alpha(b)} \geq 1$. Then $\alpha(b) = \alpha^{n_{\alpha(b)}}(\alpha(b)) \in \mathcal{O}_\beta^+(\alpha(\alpha(b)))$, and so $\exists m \in \mathbb{N} : \alpha(b) = \beta^m(\alpha(\alpha(b)))$. We saw previously that $\alpha(\alpha(b)) \in \mathcal{O}_\beta^+(\alpha(b))$, so $\exists l \in \mathbb{N} : \alpha(\alpha(b)) = \beta^l(\alpha(b))$. Combining the two results, we have $\alpha(b) = \beta^{m+l}(\alpha(b))$ and so $m = l = 0$ since $|\mathcal{O}_\beta^+(\alpha(b))| = +\infty$. In turn, this implies that $\alpha(\alpha(b)) = \alpha(b)$ and so $\alpha(b) = b$ since α is a bijection.
Hence $\alpha = id$ and $\beta = \phi$. □

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