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$\mathcal{H}$-scatteredness in Minimal Spaces with Hereditary Classes

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ABSTRACT: Quite recently, a new minimal structure $m_H^\star$ has been introduced in [14] by using a minimal structure m and a hereditary class $\mathcal{H}$. In this paper, we introduce and investigate the notion of $\mathcal{H}$-scattedness in a hereditary minimal space $(X, m, \mathcal{H})$.

Key Words: Minimal structure, hereditary class, $\mathcal{H}$-isolated, $\mathcal{H}$-accumulation, $\mathcal{H}$-scattered.

Contents

1. Introduction

The notion of ideals in topological spaces was introduced by Kuratowski [10]. Janković and Hamlett [8] defined the local function on an ideal topological space $(X, \tau, \mathcal{I})$. By using it they obtained a new topology $\tau^\star$ for X and investigated relations between τ and $\tau^\star$. In [14], Noiri and Popa introduced the minimal local function on a minimal space (X, m) with a hereditary class $\mathcal{H}$ and constructed a minimal structure $m_H^\star$ which contains m. They showed that many properties related to τ and $\tau^\star$ remain similarly valid on m and $m_H^\star$.

In this paper, we introduce the notions of $\mathcal{H}$-isolated points and $\mathcal{H}$-accumulation points of a subset in a hereditary minimal space $(X, m, \mathcal{H})$. Moreover, we introduce the notion of $\mathcal{H}$-scattedness in $(X, m, \mathcal{H})$ and obtain the characterizations and several properties of $\mathcal{H}$-scattered spaces. Also papers [2,3,4,5] have introduced some property related to minimal spaces with hereditary classes.

2. Minimal Structures

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [15] on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Definition 2.2. Let (X, m) be an m-space and A a subset of X. The *m-closure* $\mathrm{mCl}(A)$ of A [11] is defined by $\mathrm{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$.

Lemma 2.3. (Maki et al. [11]).*Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mCl}(A) = A$ *if A is m-closed,*
(2) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$,
(3) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$,
(4) $\mathrm{mCl}(A) \cup \mathrm{mCl}(B) \subset \mathrm{mCl}(A \cup B)$,
(5) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$.

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Definition 2.4. A minimal structure m on a set X is said to have
(1) *property* $\mathcal{B}$ [11] if the union of any collection of elements of m is an element of m,
(2) *property* $\mathcal{F}$ if m is closed under finite intersections.

Lemma 2.5. (Popa and Noiri [15]). *Let (X, m) be an m-space and A a subset of X.*
(1) $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$.
(2) Let m have property $\mathcal{B}$. Then the following properties hold:
(i) A is m-closed if and only if $\mathrm{mCl}(A) = A$,
(ii) $\mathrm{mCl}(A)$ is m-closed.

Definition 2.6. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [7] if it satisfies the following property: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$.

Definition 2.7. *[14] Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the* minimal local function *$A^{\star}_{mH}(\mathcal{H}, m)$ of A is defined as follows:*

$$A^{\star}_{mH}(\mathcal{H},\ m) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in m(x)\}.$$

Hereafter, $A^{\star}_{mH}(\mathcal{H}, m)$ is simply denoted by $A^{\star}_{mH}$. Also $mCl^{\star}_{H}(A) = A \cup A^{\star}_{mH}$.

Remark 2.8. *[14] Let $(X, m, \mathcal{H})$ be a hereditary m-space and A a subset of X. If $\mathcal{H} = \{\emptyset\}$ (resp. $\mathcal{P}(X)$), then $A^{\star}_{mH} = \mathrm{mCl}(A)$ (resp. $A^{\star}_{mH} = \emptyset$).*

Lemma 2.9. *[14] Let $(X, m, \mathcal{H})$ be a hereditary m-space. For subsets A and B of X, the following properties hold:*

1. *If $A \subset B$, then $A^{\star}_{mH} \subset B^{\star}_{mH}$,*
2. *$A^{\star}_{mH} = \mathrm{mCl}(A^{\star}_{mH}) \subset \mathrm{mCl}(A)$,*
3. *$A^{\star}_{mH} \cup B^{\star}_{mH} \subset (A \cup B)^{\star}_{mH}$,*
4. *$(A^{\star}_{mH})^{\star}_{mH} \subset (A \cup A^{\star}_{mH})^{\star}_{mH} = A^{\star}_{mH}$,*
5. *If $A \in \mathcal{H}$, then $A^{\star}_{mH} = \emptyset$.*

Similar study may also be considered through grill as well as generalized topological spaces [1,13].

Lemma 2.10. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If $U \in m$ and $U \cap A \in \mathcal{H}$, then $U \cap A^{\star}_{mH} = \emptyset$.*

Definition 2.11. *A subset A in a hereditary m-space $(X, m, \mathcal{H})$ is said to be $\mathcal{H}$-dense [12] (resp. m-dense, $m^{\star}_{H}$-dense) if $A^{*}_{mH} = X$ (resp. $mCl(A) = X$, $mCl^{\star}_{H}(A) = X$).*

The collection of all $\mathcal{H}$-dense (resp. m-dense, $m^{\star}_{H}$-dense) is denoted by $D_{\mathcal{H}}(X, m)$ (resp. $D(X, m)$, $D^{\star}_{\mathcal{H}}(X, m)$).

Example 2.12. *Let $X = \{a, b, c, d\}$, $m = \{X, \emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. If $A = \{a, c\}$ then $A^{\star}_{mH} = X$. So that $mCl^{\star}_{H}(A) = X$ and A is $m^{\star}_{H}$-dense.*

Example 2.13. *Let $X = \{a, b, c, d\}$, $m = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. If $A = \{a, c\}$ then $A^{\star}_{mH} = \{a, c, d\}$. So that $mCl^{\star}_{H}(A) \neq X$ and A is not $m^{\star}_{H}$-dense.*

Theorem 2.14. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then the following properties hold:*
(1) $D_{\mathcal{H}}(X, m) \subset D^{\star}_{\mathcal{H}}(X, m) \subset D(X, m)$,
(2) If for some $U \in m$, $U \cap D \in \mathcal{H}$ implies $U \cap (X - D) \notin \mathcal{H}$, then $D_{\mathcal{H}}(X, m) = D^{\star}_{\mathcal{H}}(X, m)$.

Proof. (1) Let $D \in D_{\mathcal{H}}(X,m)$. Then $mCl^*_H(D) = D \cup D^*_{mH} = X$, i.e. $D \in D^{\star}_{\mathcal{H}}(X,m)$. Therefore, $D_{\mathcal{H}}(X,m) \subseteq D^{\star}_{\mathcal{H}}(X,m)$. Since $m \subset m^{\star}_H$, $mCl^{\star}_H(A) \subset mCl(A)$ for any subset A of X. Hence $D^{\star}_{\mathcal{H}}(X,m) \subset D(X,m)$.

(2) Let $D \in D^{\star}_{\mathcal{H}}(X,m)$. Then $mCl^*_H(D) = D \cup D^*_{mH} = X$. We prove that $D^*_{mH} = X$. Let $x \in X$ such that $x \notin D^*_{mH}$. Then, there exists $\phi \neq U \in m(x)$ such that $U \cap D \in \mathcal{H}$. Hence, $U \cap (X-D) \notin \mathcal{H}$ and hence $U \cap (X-D) \neq \phi$. Let $x_0 \in U \cap (X-D)$. Then $x_0 \notin D$ and also $x_0 \notin D^*_{mH}$. Because $x_0 \in D^*_{mH}$ implies that $U \cap D \notin \mathcal{H}$ which is contrary to $U \cap D \in \mathcal{H}$. Thus $x_0 \notin D \cup D^*_{mH} = mCl^*_H(D) = X$. This is a contradiction. Therefore, we obtain $D \in D_{\mathcal{H}}(X,m)$ and, $D^{\star}_{\mathcal{H}}(X,m) \subseteq D_{\mathcal{H}}(X,m)$. Hence $D_{\mathcal{H}}(X,m) = D^{\star}_{\mathcal{H}}(X,m)$. □

Corollary 2.15. *Let $(X,m,\mathcal{H})$ be a hereditary m-space. Then for $x \in X$, $X - \{x\}$ is $\mathcal{H}$-dense if and only if $\Gamma^*_{mH}(\{x\}) = \emptyset$, where $\Gamma^*_{mH}(\{A\}) = X - (X-A)^*_{mH}$ for any subset A of X.*

Proof. The proof follows from the definition of $\mathcal{H}$-dense sets, since $\Gamma^*_{mH}(\{x\}) = X - (X - \{x\})^*_{mH} = \emptyset$ if and only if $X = (X - \{x\})^*_{mH}$. □

3. $\mathcal{H}$-isolated Points and $\mathcal{H}$-derived Sets

Let (X,m) be an m-space and let $x \in X$ and $A \subseteq X$. Then x is called an m-accumulation point of A in X if $U \cap (A - \{x\}) \neq \emptyset$ for every $U \in m(x)$. The m-derived set of A in X, denoted by $d_m(A)$, is the set of all m-accumulation points of A in X and x is called an m-isolated point of A in X if there exists $U \in m(x)$ such that $U \cap A = \{x\}$. We denote the set of all m-isolated points of A in X by $I_m(A)$. It is well known that $I_m(A) = A - d_m(A)$ and $mCl(A) = d_m(A) \cup A$.

Now, we introduce the concepts of $\mathcal{H}$-isolated points and $\mathcal{H}$-derived sets in a hereditary m-space $(X,m,\mathcal{H})$.

Definition 3.1. *Let $(X,m,\mathcal{H})$ be a hereditary m-space and let $x \in X$ and $A \subseteq X$.*

1. *x is called an $\mathcal{H}$-isolated point of A in X if there exists $U \in m^*_H(x)$ such that $U \cap A = \{x\}$. We denote the set of all $\mathcal{H}$-isolated points of A in X by $I_{\mathcal{H}}(A)$.*
2. *x is called an $\mathcal{H}$-accumulation point of A in X if $U \cap (A - \{x\}) \neq \emptyset$ for every $U \in m^*_H(x)$. The $\mathcal{H}$-derived set of A in X, denoted by $d_{\mathcal{H}}(A)$, is the set of all $\mathcal{H}$-accumulation point of A in X.*

Example 3.2. *Let $X = \{a,b,c\}$, $m = \{X, \emptyset, \{a\}, \{b\}, \{b,c\}\}$.*

1. *If $A = \{a,b\}$ then m-derived of A is $d_m(A) = \{c\}$. So that m-isolated of A is $I_m(A) = A - d_m(A) = \{a,b\}$.*
2. *If $B = \{a,c\}$ then m-derived of B is $d_m(B) = \emptyset$. So that m-isolated of B is $I_m(B) = B - d_m(B) = \{a,c\}$.*
3. *If $C = \{b,c\}$ then m-derived of C is $d_m(A) = \{c\}$. So that m-isolated of C is $I_m(C) = C - d_m(C) = \{b\}$.*

Example 3.3. *Let $X = \{a,b,c,d\}$, $m = \{X, \emptyset, \{a\}\}$ with $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $m^* = \{X, \emptyset, \{a\}, \{b,c,d\}\}$.*

1. *If $A = \{a,b\}$ then $\mathcal{H}$-derived of A is $d_{\mathcal{H}}(A) = \{c,d\}$. So that $\mathcal{H}$-isolated of A is $I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A) = \{a,b\}$.*
2. *If $B = \{b,c\}$ then $\mathcal{H}$-derived of B is $d_{\mathcal{H}}(B) = \{b,c,d\}$. So that $\mathcal{H}$-isolated of B is $I_{\mathcal{H}}(B) = B - d_{\mathcal{H}}(B) = \emptyset$.*

Proposition 3.4. *Let $(X,m,\mathcal{H})$ be a hereditary m-space and m have property $\mathcal{F}$. Then for A, $B \subseteq X$, the following properties hold:*

1. *$I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A)$.*
2. *$I_m(A) \subseteq I_{\mathcal{H}}(A) \subseteq A$.*

3. (a) $A = I_{\mathcal{H}}(A) \cup [d_{\mathcal{H}}(A) \cap A]$;
   (b) $d_{\mathcal{H}}(A) \cap A = A - I_{\mathcal{H}}(A)$.
4. *If* $A \in m_H^* - \{\emptyset\}$ *and* $A \subseteq B$, *then* $I_{\mathcal{H}}(A) \subseteq I_{\mathcal{H}}(B)$.
5. (a) $I_{\mathcal{H}}(A) \cap I_{\mathcal{H}}(B) \subseteq I_{\mathcal{H}}(A \cap B)$;
   (b) $I_{\mathcal{H}}(A \cup B) \subseteq I_{\mathcal{H}}(A) \cup I_{\mathcal{H}}(B)$.

Proof. (1) Let $x \in I_{\mathcal{H}}(A)$. Then $U \cap A = \{x\}$ for some $U \in m_H^*(x)$. This implies $U \cap (A - \{x\}) = \emptyset$. Then $x \notin d_{\mathcal{H}}(A)$. Thus $x \in A - d_{\mathcal{H}}(A)$ and so $I_{\mathcal{H}}(A) \subseteq A - d_{\mathcal{H}}(A)$. Conversely, let $x \in A - d_{\mathcal{H}}(A)$. Since $x \notin d_{\mathcal{H}}(A)$, we have $U \cap (A - \{x\}) = \emptyset$ for some $U \in m_H^*(x)$. Note that $U \cap A = \{x\}$. Then $x \in I_{\mathcal{H}}(A)$ and so $A - d_{\mathcal{H}}(A) \subseteq I_{\mathcal{H}}(A)$. Hence $I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A)$.
(2) This is obvious.
(3) (a) For any $x \in A$ and $U \in m_H^*(x)$, $U \cap A = \{x\}$ or $U \cap (A - \{x\}) \neq \emptyset$, then $x \in I_{\mathcal{H}}(A) \cup d_{\mathcal{H}}(A)$ and $A \subseteq I_{\mathcal{H}}(A) \cup d_{\mathcal{H}}(A)$. Thus $A \subseteq (I_{\mathcal{H}}(A) \cup d_{\mathcal{H}}(A)) \cap A = I_{\mathcal{H}}(A) \cup [d_{\mathcal{H}}(A) \cap A]$. And $A \supseteq (I_{\mathcal{H}}(A) \cup d_{\mathcal{H}}(A)) \cap A$. Hence $A = I_{\mathcal{H}}(A) \cup [d_{\mathcal{H}}(A) \cap A]$.
(b) This holds by (1).
(4) Let $x \in I_{\mathcal{H}}(A)$. Then $U \cap A = \{x\}$ for some $U \in m_H^*(x)$. Since $A \in m_H^* - \{\emptyset\}$, $U \cap A \in m_H^* - \{\emptyset\}$. Note that $(U \cap A) \cap B = \{x\}$. Then $x \in I_{\mathcal{H}}(B)$. Thus $I_{\mathcal{H}}(A) \subseteq I_{\mathcal{H}}(B)$.
(5) This is obvious. □

Proposition 3.5. *Let* $(X, m, \mathcal{H})$ *and* $(X, m, \mathcal{J})$ *be two hereditary spaces with* $\mathcal{J} \subseteq \mathcal{H}$. *Then for* $A \subseteq X$, $I_{\mathcal{J}}(A) \subseteq I_{\mathcal{H}}(A)$.

Proof. Let $x \in I_{\mathcal{J}}(A)$. Then $U \cap A = \{x\}$ for some $U \in m_J^*(x)$. It is clear that $\mathcal{J} \subseteq \mathcal{H}$ implies that $m_J^* \subseteq m_H^*$. So $U \in m_H^*$ and thus $x \in I_{\mathcal{H}}(A)$. Hence $I_{\mathcal{J}}(A) \subseteq I_{\mathcal{H}}(A)$. □

Proposition 3.6. *Let* $(X, m, \mathcal{H})$ *and* $(X, n, \mathcal{H})$ *be two hereditary spaces with* $n \subseteq m$. *Then for* $A \subseteq X$, $I_{n\mathcal{H}}(A) \subseteq I_{m\mathcal{H}}(A)$.

Proof. Let $x \in I_{n\mathcal{H}}(A)$. Then $U \cap A = \{x\}$ for some $U \in n_H^*(x)$. It is clear that $n \subseteq m$ implies $n_H^* \subseteq m_H^*$. So $U \in m_H^*$ and thus $x \in I_{m\mathcal{H}}(A)$. Hence $I_{n\mathcal{J}}(A) \subseteq I_{m\mathcal{H}}(A)$. □

Definition 3.7. *A hereditary m-space* $(X, m, \mathcal{H})$ *is said to be* $\mathcal{H}$*-scattered if* $I_{\mathcal{H}}(A) \neq \emptyset$ *for any nonempty* $A \in \mathcal{P}(X)$.
Example 3.8. *Let* $X = \{a, b, c\}$, $m = \{X, \emptyset, \{a\}, \{b\}\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}\}$. *Then*

$$m^* = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

It is clear that $m^*(a) = \{X, \{a\}, \{a, b\}\}$, $m^*(b) = \{X, \{b\}, \{a, b\}\}$ *and* $m^*(c) = \{X\}$.
Then a hereditary m-space $(X, m, \mathcal{H})$ *is* $\mathcal{H}$*-scattered because* $I_{\mathcal{H}}(A) \neq \emptyset$ *for any nonempty* $A \in \mathcal{P}(X)$ *as the following table.*

	$d_{\mathcal{H}}(A)$	$I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A)$
$A = \{a\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{a\}$
$A = \{b\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{b\}$
$A = \{c\}$	$d_{\mathcal{H}}(A) = \emptyset$	$I_{\mathcal{H}}(A) = \{c\}$
$A = \{a, b\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{a, b\}$
$A = \{a, c\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{a\}$
$A = \{b, c\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{b\}$
$A = \{a, b, c\}$	$d_{\mathcal{H}}(A) = \{c\}$	$I_{\mathcal{H}}(A) = \{a, b\}$

Let $(X, m, \mathcal{H})$ be a hereditary m-space. The family of all $m_H^\star$-dense of X is denoted by $D_{\mathcal{H}}^* = D_{\mathcal{H}}^\star(X, m)$. For the subspace $(Y, m_Y, \mathcal{H}_Y)$, the family of all $m_H^\star$-dense subsets of Y is denoted by $D_{\mathcal{H}}^\star(Y) = \{A \subseteq Y : mCl_{H_Y}^*(A) = Y\}$.

Lemma 3.9. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then $A \subseteq X$ is $m^\star_H$-dense in X if and only if $U \cap A \neq \emptyset$ for every nonempty set $U \in m^*_H$.*

Proof. Let A be $m^\star_H$-dense in X and let U be nonempty $U \in m^*_H$. Pick $x \in U$. Then $x \in X = mCl^*_H(A) = A \cup A^*_{mH}$. Then if $x \in A$, then $x \in A \cap U$ and $A \cap U \neq \emptyset$. Suppose that $x \in A^*_{mH}$ and $A \cap U = \emptyset$. Since $X - U$ is m^*_H-closed in X, $(X - U)^*_H \subseteq X - U$. Then $U \subseteq X - (X - U)^*_H$. By $x \in U$, $x \notin (X - U)^*_H$. It follows that $V \cap (X - U) \in \mathcal{H}$ for some $V \in m(x)$. By $A \cap U = \emptyset$, $A \subseteq X - U$. $V \cap A \subseteq V \cap (X - U) \in \mathcal{H}$. Then $V \cap A \in \mathcal{H}$. Hence $x \notin A^*_{mH}$. This is a contradiction. Thus, $U \cap A \neq \emptyset$. Conversely, suppose $mCl^*_H(A) \neq X$. Put $U = X - mCl^*_H(A)$, then U is a nonempty set in m^*_H. But $U \cap A = [X - mCl^*_H(A)] \cap A = \emptyset$. This is a contradiction.

□

Theorem 3.10. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and m have property $\mathcal{F}$. The following are equivalent.*

1. *X is $\mathcal{H}$-scattered;*
2. *$I_{\mathcal{H}}(A) \in D^\star_{\mathcal{H}}(A)$, for any nonempty set $A \in \mathcal{P}(X)$;*
3. *$D \in D^\star_{\mathcal{H}}(A)$ if and only if $I_{\mathcal{H}}(A) \subseteq D$, for any nonempty set $A \in \mathcal{P}(X)$;*
4. *$d_{\mathcal{H}}(A) = d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$ for any nonempty set $A \in \mathcal{P}(X)$;*
5. *If A is nonempty m^*_H-closed set, then $I_{\mathcal{H}}(A) \neq \emptyset$.*

Proof. (1) ⇒ (2): Let $\emptyset \neq V \in m^*_{H|_A}$. Then $V = W \cap A$ for some $W \in m^*_H$. Since X is $\mathcal{H}$-scattered, $I_{\mathcal{H}}(V) \neq \emptyset$. Pick $x \in I_{\mathcal{H}}(V)$. Then $U \cap V = \{x\}$ for some $U \in m^*_H(x)$. So $(U \cap W) \cap A = U \cap (W \cap A) = U \cap V = \{x\}$. Note that $U \cap W \in m^*_H(x)$. This implies $x \in I_{\mathcal{H}}(A)$. Then $x \in V \cap I_{\mathcal{H}}(A)$ and so $V \cap I_{\mathcal{H}}(A) \neq \emptyset$. By Lemma 3.9 $mCl^*_{H_A}[I_{\mathcal{H}}(A)] = A$. Thus $I_{\mathcal{H}}(A) \in D^\star_{\mathcal{H}}(A)$.
(2) ⇒ (3): Let $I_{\mathcal{H}}(A) \subseteq D$. By (2), $A = mCl^*_{H_A}[I_{\mathcal{H}}(A)] \subseteq mCl^*_{H_A}[D]$. Thus $D \in D^\star_{\mathcal{H}}(A)$. Conversely, suppose $I_{\mathcal{H}}(A) \nsubseteq D$ for some $D \in D^\star_{\mathcal{H}}(A)$. Then $I_{\mathcal{H}}(A) - D \neq \emptyset$. Pick $x \in I_{\mathcal{H}}(A) - D$. Then $U \cap A = \{x\}$ for some $U \in m^*_H(x)$. Note that $U \cap A \in m^*_{H|_A}(x)$ and $D \in D^\star_{\mathcal{H}}(A)$. By Lemma 3.9, $D \cap (U \cap A) \neq \emptyset$. But $D \cap (U \cap A) = D \cap \{x\} = \emptyset$. This is a contradiction.
(3) ⇒ (4): Since $I_{\mathcal{H}}(A) \subseteq A$, $d_{\mathcal{H}}[I_{\mathcal{H}}(A)] \subseteq d_{\mathcal{H}}(A)$. Suppose $d_{\mathcal{H}}(A) \nsubseteq d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$. Then $d_{\mathcal{H}}(A) - d_{\mathcal{H}}[I_{\mathcal{H}}(A)] \neq \emptyset$. Pick up $x \in d_{\mathcal{H}}(A) - d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$. By Proposition 3.4(1), $I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A)$. Then $x \notin I_{\mathcal{H}}(A)$ and $x \notin d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$ implies $U \cap [I_{\mathcal{H}}(A) - \{x\}] = \emptyset$ for some $U \in m^*_H(x)$. Note that $x \notin I_{\mathcal{H}}(A)$. Then $(U \cap A) \cap I_{\mathcal{H}}(A) \subseteq U \cap I_{\mathcal{H}}(A) = \emptyset$ with $U \cap A \in m^*_{H|_A}$. By (3) $I_{\mathcal{H}}(A) \in D^\star_{\mathcal{H}}(A)$. Then $V \cap I_{\mathcal{H}}(A) \neq \emptyset$ for every $V \in m^*_{H|_A}$. This is a contradiction. Hence $d_{\mathcal{H}}(A) \subseteq d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$ and hence $d_{\mathcal{H}}(A) = d_{\mathcal{H}}[I_{\mathcal{H}}(A)]$.
(4) ⇒ (1): Suppose $I_{\mathcal{H}}(A) = \emptyset$ for some nonempty set $A \in \mathcal{P}(X)$. By (4), $d_{\mathcal{H}}(A) = d_{\mathcal{H}}[I_{\mathcal{H}}(A)] = d_{\mathcal{H}}(\emptyset) = \emptyset$. By Proposition 3.4(3), $A = I_{\mathcal{H}}(A) \cup [d_{\mathcal{H}}(A) \cap A] = \emptyset$, a contradiction. Hence X is $\mathcal{H}$-scattered.
(1) ⇒ (5): This is obvious.
(5) ⇒ (1): Let $\emptyset \neq A \in \mathcal{P}(X)$. Since $mCl^*_H(A)$ is m^*_H-closed, by (5), $I_{\mathcal{H}}[mCl^*_H(A)] \neq \emptyset$. Pick $x \in I_{\mathcal{H}}[mCl^*_H(A)]$. Then $U \cap [mCl^*_H(A)] = \{x\}$ for some $U \in m^*_H(x)$. Suppose $U \cap A = \emptyset$. We have $A \subseteq X - U$. Then $mCl^*_H(A) \subseteq X - U$. So $U \cap mCl^*_H(A) = \emptyset$. This is a contradiction. Thus $U \cap A \neq \emptyset$. Since $U \cap A \subseteq U \cap [mCl^*_H(A)] = \{x\}$, we have $U \cap A = \{x\}$. So $x \in I_{\mathcal{H}}(A)$. This implies $I_{\mathcal{H}}(A) \neq \emptyset$. Hence X is $\mathcal{H}$-scattered. □

Definition 3.11. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Put $X^0 = X$ and $X^1 = \{x \in X : x$ is not an $\mathcal{H}$-isolated point in $X\}$. Let α be any order number. If X^β is already defined for all order $\beta < \alpha$, then we put*

$$X^\alpha = \begin{cases} (X^\beta)^1, & \text{if } \alpha = \beta + 1 \text{ and } \beta \text{ is an ordinal number;} \\ \bigcap\limits_{\beta < \alpha} X^\beta, & \text{if } \alpha \text{ is a limit ordinal number.} \end{cases}$$

Remark 3.12.

1. $X^1 = X - I_{\mathcal{H}}(X) = X \cap d_{\mathcal{H}}(X)$.

2. $X^\beta \subseteq X^\alpha$ *whenever* $\alpha \leq \beta$.

3. $X^\alpha = X^{\alpha-1} - I_{\mathcal{H}}(X^{\alpha-1}) = X^{\alpha-1} \cap d_{\mathcal{H}}(X^{\alpha-1})$ *for any successor ordinal number* α.

4. *If* α *is a successor ordinal number and* $X^\alpha = \emptyset$, *then* $X = \underset{\beta \leq \alpha-1}{\cup} I_{\mathcal{H}}(X^\beta)$.

Lemma 3.13. *Let* $(X, m, \mathcal{H})$ *be a hereditary m-space. If m has property* $\mathcal{F}$, *the following properties hold.*

1. X^α *is* m^*_H*-closed for any ordinal number* α.

2. $Y \subseteq X$, *then* $Y^\alpha \subseteq X^\alpha$ *for any ordinal number* α.

Proof. 1. We use induction on α.
1) $\alpha = 1$. Let $x \in I_{\mathcal{H}}(X)$. Then $U_x \cap X = \{x\}$ for some $U_x \in m^*_H(x)$. This implies $\{x\} = U_x \in m^*_H$. Thus $I_{\mathcal{H}}(X) = \underset{x \in I_{\mathcal{H}}(X)}{\cup} \{x\} \in m^*_H$. Thus $X^1 = X - I_{\mathcal{H}}(X)$ is m^*_H-closed.
2) Suppose X^β is m^*_H-closed for any $\beta < \alpha$. We will prove X^α is m^*_H-closed in the following cases.
(a) α is a successor ordinal number. Let $x \in I_{\mathcal{H}}(X^{\alpha-1})$. $U_x \cap X^{\alpha-1} = \{x\}$ for some $U_x \in m^*_H(x)$. By Remark 3.12, $X^\alpha = X^{\alpha-1} - I_{\mathcal{H}}(X^{\alpha-1})$. So

$$X^\alpha = X^{\alpha-1} - \underset{x \in I_{\mathcal{H}}(X^{\alpha-1})}{\cup} \{x\} = [X - \underset{x \in I_{\mathcal{H}}(X^{\alpha-1})}{\cup} U_x] \cap X^{\alpha-1}.$$

By induction hypothesis, $X^{\alpha-1}$ is m^*_H-closed. Thus X^α is m^*_H-closed.
(b) α is a limit ordinal number. By induction hypothesis, X^β is m^*_H-closed for any $\beta < \alpha$. Thus $X^\alpha = \underset{\beta<\alpha}{\cap} X^\beta$ is m^*_H-closed.
2. Let $Y \subseteq X$. We will prove $Y^\alpha \subseteq X^\alpha$ for any ordinal number α.
1) $Y^1 = Y \cap d_{\mathcal{H}}(Y) \subseteq X \cap d_{\mathcal{H}}(X) = X^1$. This show $Y^\alpha \subseteq X^\alpha$ when $\alpha = 1$.
2) Suppose $Y^\beta \subseteq X^\beta$ for any $\beta < \alpha$. We consider the following cases
(a) α is a successor ordinal number. By induction hypothesis, $Y^{\alpha-1} \subseteq X^{\alpha-1}$. By Remark 3.12, $Y^\alpha = Y^{\alpha-1} \cap d_{\mathcal{H}}(Y^{\alpha-1}) \subseteq X^{\alpha-1} \cap d_{\mathcal{H}}(X^{\alpha-1}) = X^\alpha$.
(b) α is a limit ordinal number. By induction hypothesis, $Y^\beta \subseteq X^\beta$ for any $\beta < \alpha$. Thus $Y^\alpha = \underset{\beta<\alpha}{\cap} Y^\beta \subseteq \underset{\beta<\alpha}{\cap} X^\beta = X^\alpha$. By 1) and 2) we have $Y^\alpha \subseteq X^\alpha$ for any ordinal number α.

□

Definition 3.14. *Let* $(X, m, \mathcal{H})$ *be a hereditary m-space.*

1. *An ordinal number* β *is called the derived length of* X *if* $\beta = min\{\alpha : X^\alpha = \emptyset\}$. β *is denoted by* $\delta(X)$.

2. X *is said to have a derived length if there is an ordinal number* α *such that* $X^\alpha = \emptyset$.

Lemma 3.15. $X^\delta = X^{\delta+1}$ *for some ordinal number* δ.

Theorem 3.16. *Let* $(X, m, \mathcal{H})$ *be a hereditary m-space. Then* X *is* $\mathcal{H}$*-scattered if and only if* X *has a derived length.*

Proof. Sufficiency. Suppose that X is not $\mathcal{H}$-scattered. Then $I_{\mathcal{H}}(A) = \emptyset$ for some nonempty set $A \subseteq X$. We claim $A \subseteq X^\alpha$ for any ordinal number α.
(1) Let $x \in A$ and $U \in m^*_H(x)$. Since $I_{\mathcal{H}}(A) = \emptyset$, $U \cap A \neq \{x\}$. Note that $x \in U \cap A$. Then $|U \cap A| \geq 2$ and so $U \cap (A - \{x\}) \neq \emptyset$. Now $U \cap (A - \{x\}) \subseteq U \cap (X - \{x\})$. Then $U \cap (X - \{x\}) \neq \emptyset$. This implies $x \in X \cap d_{\mathcal{H}}(X)$. By Remark 3.12, $x \in X^1$. Thus $A \subseteq X - I_{\mathcal{H}}(X) = X^1$. i.e., $A \subseteq X^\alpha$ when $\alpha = 1$.
(2) Suppose $A \subseteq X^\beta$ for any $\beta < \alpha$. We will prove $A \subseteq X^\alpha$ in the following cases.
a) α is a successor ordinal number. Let $x \in A$ and $U \in m^*_H(x)$. By (1) $U \cap (A - \{x\}) \neq \emptyset$. By induction hypothesis, $A \subseteq X^{\alpha-1}$. Then $U \cap (X^{\alpha-1} - \{x\}) \neq \emptyset$. This implies $x \in X^{\alpha-1} \cap d_{\mathcal{H}}(X^{\alpha-1})$. By Remark 3.12, $x \in X^\alpha$. Hence $A \subseteq X^\alpha$.
b) α is a limit ordinal number. By induction hypothesis, $A \subseteq X^\beta$ for any $\beta < \alpha$. Then $\underset{\beta<\alpha}{\cap} X^\beta = X^\alpha$.
Since X has a derived length, $X^\delta = \emptyset$ for some ordinal number δ. By claim, $A \subseteq X^\beta$. Then $A = \emptyset$, a contradiction.

Necessity. Conversely, suppose that X has no derived length. By Lemma 3.15, $X^{\delta+1} = X^{\delta}$ and Remark 3.12, $X^{\delta+1} = X^{\delta} - I_{\mathcal{H}}(X^{\delta})$. Then $I_{\mathcal{H}}(X^{\delta}) = \emptyset$. Note that X has no derived length. Then $X^{\delta} \neq \emptyset$. It follows that X is not $\mathcal{H}$-scattered. This is a contradiction.

□

4. Characterizations of Scattered Spaces

Corollary 4.1. *(1) Let $(X, m, \mathcal{H})$ and $(X, m, \mathcal{J})$ be two hereditary spaces with $\mathcal{J} \subseteq \mathcal{H}$. If $(X, m, \mathcal{J})$ is $\mathcal{J}$-scattered, then $(X, m, \mathcal{H})$ is $\mathcal{H}$-scattered.*
(2) Let $(X, m, \mathcal{H})$ and $(X, n, \mathcal{H})$ be two hereditary spaces with $n \subseteq m$. If $(X, n, \mathcal{H})$ is $\mathcal{H}$-scattered, then $(X, m, \mathcal{H})$ is $\mathcal{H}$-scattered.

Proof. These hold by Proposition 3.5 and Proposition 3.6. □

An m-space (X, m) is said to be scatted if $I_m(A) \neq \emptyset$ for any nonempty set $A \in \mathcal{P}(X)$.

Theorem 4.2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then the following are equivalent.*

1. *(X, m) is scattered.*
2. *$(X, m, \mathcal{H})$ is $\mathcal{H}$-scattered for any hereditary $\mathcal{H}$ on X.*
3. *$(X, m, \{\emptyset\})$ is $\{\emptyset\}$-scattered*

Proof. (1) $\Rightarrow$ (2): This follows from Proposition 3.4 (2).
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (1): Since $m = m_H^*$ whenever $\mathcal{H} = \{\emptyset\}$, $I_m(A) = I_{\mathcal{H}}(A) \neq \emptyset$. Thus (X, m) is scattered. □

Theorem 4.3. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and Y be nonempty subset of X. If X is $\mathcal{H}$-scattered, then $(Y, m_Y, \mathcal{H}_Y)$ is $\mathcal{H}_Y$-scattered.*

Proof. Let A be nonempty set of Y. Since X is $\mathcal{H}$-scattered, $I_{\mathcal{H}}(A) \neq \emptyset$. Pick $x \in I_{\mathcal{H}}(A)$. Then $U \cap A = \{x\}$ for some $U \in m_H^*(x)$. Note that $U \cap Y \in {m_Y}_H^*(x)$ and $(U \cap Y) \cap A = (U \cap A) \cap Y = \{x\}$. Then $x \in I_{\mathcal{H}_Y}(A)$ and so $I_{\mathcal{H}_Y}(A) \neq \emptyset$. Hence $(Y, m_Y, \mathcal{H}_Y)$ is $\mathcal{H}_Y$-scattered. □

Lemma 4.4. *If every $\mathcal{H}_\alpha$ is a hereditary on X_α ($\alpha \in \Delta$), then $\bigcup_{\alpha \in \Delta} \{H_\alpha : H_\alpha \in \mathcal{H}_\alpha\}$ is a hereditary on $\bigcup_{\alpha \in \Delta} X_\alpha$.*

Definition 4.5. *A hereditary m-space $(X, m, \mathcal{H})$ is called $\mathcal{H}$-resolvable if X has two disjoint $\mathcal{H}$-dense subsets. Otherwise, X is called $\mathcal{H}$-irresolvable.*

Example 4.6. *Let $X = \{a, b, c, d\}$, $m = \{X, \emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$ with $\mathcal{H} = \{\emptyset, \{a\}\}$. Then it is clear that. If $A = \{a, c\}$ then $A^* = X$ and if $B = \{b, d\}$ then $B^* = X$ therefore, A and B is two disjoint $\mathcal{H}$-dense subsets of X. Hence, a hereditary m-space $(X, m, \mathcal{H})$ is $\mathcal{H}$-resolvable.*

Proposition 4.7. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If X is $\mathcal{H}$-scattered, then X is $\mathcal{H}$-irresolvable.*

Proof. Suppose that X is not $\mathcal{H}$-irresolvable. Then X is $\mathcal{H}$-resolvable. For some nonempty sets $A, B \in \mathcal{P}(X)$, we have $A_H^* = B_H^* = X$ and $A \cap B = \emptyset$. Since $A, B \in D_{\mathcal{H}}^*(X)$, by Theorem 3.10, $I_{\mathcal{H}}(X) \subseteq A, B$, and $I_{\mathcal{H}}(X) \subseteq A \cap B$. Since X is $\mathcal{H}$-scattered, $I_{\mathcal{H}}(X) \neq \emptyset$. So $A \cap B \neq \emptyset$. Thus, X is $\mathcal{H}$-irresolvable. □

It is clear that by Proposition 4.7 a hereditary m-space $(X, m, \mathcal{H})$ in Example 3.8 is $\mathcal{H}$-irresolvable.

Definition 4.8. *A mapping $f : (X, m, \mathcal{H}) \to (Y, n, \mathcal{J})$ is said to be $\mathcal{H}$-closed if $f(A)$ is n_J^*-closed in Y for each m_H^*-closed subset A of X.*

Theorem 4.9. *Let $(X, m, \mathcal{H})$ be $\mathcal{H}$-scattered, $(Y, n, \mathcal{J})$ be a hereditary n-space, where m and n has property $\mathcal{F}$, and let $f : (X, m, \mathcal{H}) \to (Y, n, \mathcal{J})$ be $\mathcal{H}$-closed. Suppose that f satisfies the following condition. The set $\{\beta : X^{\beta} \cap f^{-1}(y) \neq \emptyset\}$ contains a largest element for any $y \in Y$. Then the following properties hold:*

1. *$Y^\alpha \subseteq f(X^\alpha)$ for every ordinal number α,*

2. *$\delta(Y) \leq \delta(X)$,*

3. *Y is $\mathcal{J}$-scattered.*

Proof. Since (2) and (3) hold by (1) and Theorem 3.16, we only need to prove (1) i.e. $Y^\alpha \subseteq f(X^\alpha)$ for every ordinal number α.
We use induction on α.

1. Since $Y^0 = Y = f(X) = f(X^0)$, then $Y^\alpha \subseteq f(X^\alpha)$ when $\alpha = 0$.

2. Suppose $Y^\beta \subseteq f(X^\beta)$ when $\beta < \alpha$. It suffices to show $Y^\alpha \subseteq f(X^\alpha)$ in the following two cases,

(a) $\alpha = \beta + 1$ for some ordinal number β.
Suppose $Y^\alpha \nsubseteq f(X^\alpha)$. Then $Y^\alpha - f(X^\alpha) \neq \emptyset$. Pick $y \in Y^\alpha - f(X^\alpha)$. Then $X^\alpha \cap f^{-1}(y) \neq \emptyset$. Put $F = X^\beta - f^{-1}(y)$.
Claim 1. F is m_H^*-closed in X. Put $A = X^\beta \cap f^{-1}(y)$. Then $F = X^\beta - A$. Since $X^\alpha \cap f^{-1}(y) = \emptyset$, $f^{-1}(y) \subseteq X - X^\alpha$. This is implies $A \subseteq X^\beta \cap (X - X^\alpha) = X^\beta - X^\alpha$. By Remark 3.12, $X^\beta - X^\alpha = I_{\mathcal{H}}(X^\beta)$. Thus $A \subseteq I_{\mathcal{H}}(X^\beta)$. For any $x \in A$, $x \in I_{\mathcal{H}}(X^\beta)$. Then $U \cap X^\beta = \{x\}$ for some $U \in m_H^*$. Then $\{x\} \in m_{HX^\beta}^*$ (relative space) and so $A = \underset{x\in A}{\cup}\{x\} \in m_{HX^\beta}^*$. This implies $F = X^\beta - A$ is m_H^*-closed in X^β. By Lemma 3.13 (1) F is m_H^*-closed in X. By induction hypothesis, $Y^\beta \subseteq f(X^\beta)$. Then $Y^\beta - \{y\} \subseteq f(X^\beta) - \{y\}$. Note that $X^\beta \subseteq F \cup f^{-1}(y)$. Then $Y^\beta - \{y\} \subseteq f[F \cup f^{-1}(y)] - \{y\} = f(F)$. Thus $Y^\beta - f(F) \subseteq \{y\}$. Conversely, by $f^{-1}(y) \cap F = \emptyset$, $y \notin f(F)$. Note that $y \in Y^\alpha \subseteq Y^\beta$. Then $\{y\} \subseteq Y^\beta - f(F)$. Hence $\{y\} = Y^\beta - f(F)$. Since f is $\mathcal{H}$-closed, by **Claim 1.**, $f(F)$ is n_J^*-closed. Note that $y \notin f(F)$. Put $U = Y - f(F)$. Then $U \in n_J^*(y)$. By $U \cap Y^\beta = Y^\beta - f(F) = \{y\}$, $y \in I_{\mathcal{J}}(Y^\beta)$. By Remark 3.12, $Y^\beta - Y^\alpha = I_{\mathcal{J}}(Y^\beta)$. This implies $y \notin Y^\alpha$. This is a contradiction. Therefore, $Y^\alpha \subseteq f(X^\alpha)$.
(b) α is a limit ordinal number. Suppose $Y^\alpha \nsubseteq f(X^\alpha)$. Then $Y^\alpha - f(X^\alpha) \neq \emptyset$. Pick $y \in Y^\alpha - f(X^\alpha)$. Put $\pi = max\{\beta : X^\beta \cap f^{-1}(y) \neq \emptyset\}$. By condition of hypothesis, we have $X^\pi \cap f^{-1}(y) \neq \emptyset$. Since $X^\alpha \cap f^{-1}(y) = \emptyset$. We can claim $\pi < \alpha$. Otherwise, we have $\pi \geq \alpha$. Since $X^\pi \cap f^{-1}(y) \neq \emptyset$ and $X^\pi \subseteq X^\alpha$, $X^\alpha \cap f^{-1}(y) \neq \emptyset$. Thus $y \in f(X^\alpha)$. This is a contradiction. But $X^{\pi+1} \cap f^{-1}(y) = \emptyset$. Then $\{y\} \cap f(X^{\pi+1}) = \emptyset$ and so $f^{-1}(y) \cap f^{-1}[f(X^{\pi+1})] = \emptyset$. Put $W = X - f^{-1}[f(X^{\pi+1})]$. Then $f^{-1}(y) \subseteq W$. By Lemma 3.13(1), $X^{\pi+1}$ is m_H^*-closed. By f is $\mathcal{H}$-closed, $f(X^{\pi+1})$ is n_J^*-closed. Put $Z = Y - f(X^{\pi+1})$. Then Z is n_J^*-open and $W = f^{-1}(Z)$. Put $g = f_{|W}$.
Claim 2. $g = f_{|W} : (W, m_W, \mathcal{H}_W) \to (Z, n_Z, \mathcal{J}_Z)$ is $\mathcal{H}_W$-closed. Let K be m_H^*-closed in W. Then $K = F \cap W$ for some m_H^*-closed set F in X. Since f is $\mathcal{H}$-closed, $f(F)$ is n_H^*-closed in Y. Note that $g(K) = f(W \cap F) = f[f^{-1}(Z) \cap F] = Z \cap f(F)$. Then $g(K)$ is n_H^*-closed in Z. Then X is $\mathcal{H}$-scattered, by Theorem 4.3, W is $\mathcal{H}_W$-scattered. By Theorem 3.16, $\delta(W)$ is existence.
Claim 3. $\delta(W) \leq \pi + 1$. $W^{\pi+1} \subseteq W \subseteq X - X^{\pi+1}$. By Lemma 3.13 (2), $W^{\pi+1} \subseteq X^{\pi+1}$. Then $W^{\pi+1} \subseteq X^{\pi+1} \cap [X - X^{\pi+1}] = \emptyset$. Thus $\delta(W) \leq \pi + 1$.
Claim 4. $Y^\alpha \cap Z = Z^\alpha$.

1. $\alpha = 0$. We have $Z^0 = Z = Y \cap Z = Y^0 \cap Z$.

2. Suppose $Y^\beta \cap Z = Z^\beta$ for every $\beta < \alpha$. We will prove $Y^\alpha \cap Z = Z^\alpha$ in the following cases.

(i) α is a successor ordinal number.
By induction hypothesis, $Y^{\alpha-1} \cap Z = Z^{\alpha-1}$. By $Z^\alpha \subseteq Y^\alpha$ and $Z^\alpha \subseteq Z$, we have $Z^\alpha \subseteq Y^\alpha \cap Z$. Let $y \in Y^\alpha \cap Z$. By Remark 3.12, $Y^\alpha = Y^{\alpha-1} \cap d_{\mathcal{H}}(Y^{\alpha-1})$. Then $y \in d_{\mathcal{H}}(Y^{\alpha-1}) \cap Y^{\alpha-1} \cap Z = d_{\mathcal{H}}(Y^{\alpha-1}) \cap Z^{\alpha-1}$. Note that Z is an n_J^*-open set containing y. $y \in d_{\mathcal{H}}(Y^{\alpha-1})$ implies that $[U \cap Z] \cap [Y^{\alpha-1} - \{y\}] \neq \emptyset$ for any n_J^*-open set U containing y. Then $[U \cap Z] \cap [Y^{\alpha-1} - \{y\}] = U \cap Z \cap Y^{\alpha-1} \cap [Y - \{y\}] = U \cap Z^{\alpha-1} \cap [Y - \{y\}] = U \cap [Z^{\alpha-1} - \{y\}] \neq \emptyset$. Thus, $y \in d_{\mathcal{H}}(Z^{\alpha-1})$. By Remark 3.12, $Z^\alpha = Z^{\alpha-1} \cap d_{\mathcal{H}}(Z^{\alpha-1})$. Then $y \in Z^\alpha$. Hence $Y^\alpha \cap Z \subseteq Z^\alpha$. Hence $Y^\alpha \cap Z = Z^\alpha$.
(ii) α is a limit ordinal number.
By induction hypothesis, $Y^\beta \cap Z = Z^\beta$ for any $\beta < \alpha$. Then

$$Y^\alpha \cap Z = \left(\cap_{\beta<\alpha} Y^\beta\right) \cap Z = \cap_{\beta<\alpha}\left(Y^\beta \cap Z\right) = \cap_{\beta<\alpha} Z^\beta = Z^\alpha.$$

By **Claim 2,** $g = f_{|W} : (W, m_W, \mathcal{H}_W) \to (Z, n_Z, \mathcal{J}_Z)$ is $\mathcal{H}_W$-closed. By repeating the proof of (a), we can prove $Z^{\pi+1} \subseteq g(W^{\pi+1})$. By **Claim 3,** $\emptyset = W^{\delta(W)} \supseteq W^{\pi+1}$. This implies $Z^{\pi+1} = \emptyset$. By Remark 3.12(4), $Z = \underset{\beta\leq\pi}{\cup} I_{\mathcal{H}}(Z^{\beta})$. Note that $X^{\pi+1} \cap f^{-1}(y) = \emptyset$. Then $y \notin f(X^{\pi+1})$. So $y \in Z = \underset{\beta\leq\pi}{\cup} I_{\mathcal{H}}(Z^{\beta})$. We obtain $y \in I_{\mathcal{H}}(Z^{\gamma})$ for some $\gamma \leq \pi$. It follows $U \cap Z^{\gamma} = \{y\}$ for some $U \in n^*_{\mathcal{J}}(y)$. By **Claim 4,** $Y^{\gamma} \cap Z = Z^{\gamma}$. Then $(U \cap Z) \cap Y^{\gamma} = U \cap Z^{\gamma} = \{y\}$. Since $U \cap Z \in n^*_{\mathcal{J}}(y)$, we have $y \in I_{\mathcal{H}}(Y^{\gamma}) = Y^{\gamma} - Y^{\gamma+1}$. Since $\pi < \alpha$ and α is a limit ordinal, $\pi + 1 < \alpha$. Then $\gamma + 1 \leq \pi + 1 < \alpha$. By Remark 3.12, $Y^{\gamma+1} \supset Y^{\alpha}$. Then $y \notin Y^{\alpha}$. This is a contradiction. Therefore, $Y^{\alpha} \subseteq f(X^{\alpha})$.

□

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