H-scatteredness in Minimal Spaces with Hereditary Classes

Ahmad Al-Omari and Takashi Noiri

ABSTRACT: Quite recently, a new minimal structure \( m_H^* \) has been introduced in [14] by using a minimal structure \( m \) and a hereditary class \( H \). In this paper, we introduce and investigate the notion of \( H \)-scatteredness in a hereditary minimal space \((X, m, H)\).

Key Words: Minimal structure, hereditary class, \( H \)-isolated, \( H \)-accumulation, \( H \)-scattered.

Contents

1 Introduction
2 Minimal Structures
3 \( H \)-isolated Points and \( H \)-derived Sets
4 Characterizations of Scattered Spaces

1. Introduction

The notion of ideals in topological spaces was introduced by Kuratowski [10]. Janković and Hamlett [8] defined the local function on an ideal topological space \((X, \tau_J)\). By using it they obtained a new topology \( \tau^* \) for \( X \) and investigated relations between \( \tau \) and \( \tau^* \). In [14], Noiri and Popa introduced the minimal local function on a minimal space \((X, m)\) with a hereditary class \( H \) and constructed a minimal structure \( m_H^* \) which contains \( m \). They showed that many properties related to \( \tau \) and \( \tau^* \) remain similarly valid on \( m \) and \( m_H^* \).

In this paper, we introduce the notions of \( H \)-isolated points and \( H \)-accumulation points of a subset in a hereditary minimal space \((X, m, H)\). Moreover, we introduce the notion of \( H \)-scatteredness in \((X, m, H)\) and obtain the characterizations and several properties of \( H \)-scattered spaces. Also papers [2,3,4,5] have introduced some property related to minimal spaces with hereditary classes.

2. Minimal Structures

Definition 2.1. A subfamily \( m \) of the power set \( P(X) \) of a nonempty set \( X \) is called a minimal structure (briefly \( m \)-structure) [15] on \( X \) if \( \emptyset \in m \) and \( X \in m \).

By \((X, m)\), we denote a nonempty set \( X \) with a minimal structure \( m \) on \( X \) and call it an \( m \)-space. Each member of \( m \) is said to be \( m \)-open and the complement of an \( m \)-open set is said to be \( m \)-closed. For a point \( x \in X \), the family \( \{ U : x \in U \text{ and } U \in m \} \) is denoted by \( m(x) \).

Definition 2.2. Let \((X, m)\) be an \( m \)-space and \( A \) a subset of \( X \). The \( m \)-closure \( mCl(A) \) of \( A \) [11] is defined by \( mCl(A) = \cap \{ F \subset X : A \subset F, X \setminus F \in m \} \).

Lemma 2.3. (Maki et al. [11]).Let \( X \) be a nonempty set and \( m \) a minimal structure on \( X \). For subsets \( A \) and \( B \) of \( X \), the following properties hold:

1. \( A \subset mCl(A) \) and \( mCl(A) = A \) if \( A \) is \( m \)-closed,
2. \( mCl(\emptyset) = \emptyset \), \( mCl(X) = X \),
3. If \( A \subset B \), then \( mCl(A) \subset mCl(B) \),
4. \( mCl(A) \cup mCl(B) \subset mCl(A \cup B) \),
5. \( mCl(mCl(A)) = mCl(A) \).

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Definition 2.4. A minimal structure \( m \) on a set \( X \) is said to have
(1) property \( \mathcal{B} \) if the union of any collection of elements of \( m \) is an element of \( m \),
(2) property \( \mathcal{F} \) if \( m \) is closed under finite intersections.

Lemma 2.5. (Popa and Noiri [15]). Let \( (X, m) \) be an \( m \)-space and \( A \) a subset of \( X \).
(1) \( x \in \text{mCl}(A) \) if and only if \( U \cap A \neq \emptyset \) for every \( U \in m(x) \).
(2) Let \( m \) have property \( \mathcal{B} \). Then the following properties hold:
(i) \( A \) is \( m \)-closed if and only if \( \text{mCl}(A) = A \),
(ii) \( \text{mCl}(A) \) is \( m \)-closed.

Definition 2.6. A nonempty subfamily \( \mathcal{H} \) of \( \mathcal{P}(X) \) is called a hereditary class on \( X \) [7] if it satisfies the following property: \( A \in \mathcal{H} \) and \( B \subseteq A \) implies \( B \in \mathcal{H} \). A hereditary class \( \mathcal{H} \) is called an ideal if it satisfies the additional condition: \( A \in \mathcal{H} \) and \( B \subseteq \mathcal{H} \) implies \( A \cup B \in \mathcal{H} \).

A minimal space \((X, m)\) with a hereditary class \( \mathcal{H} \) on \( X \) is called a hereditary minimal space (briefly hereditary \( m \)-space) and is denoted by \((X, m, \mathcal{H})\).

Definition 2.7. [14] Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. For a subset \( A \) of \( X \), the minimal local function \( A^*_{mH}(\mathcal{H}, m) \) of \( A \) is defined as follows:
\[
A^*_{mH}(\mathcal{H}, m) = \{ x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in m(x) \}.
\]
Hereafter, \( A^*_{mH}(\mathcal{H}, m) \) is simply denoted by \( A^*_{mH} \). Also \( \text{mCl}^*_{mH}(A) = A \cup A^*_{mH} \).

Remark 2.8. [14] Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space and \( A \) a subset of \( X \). If \( \mathcal{H} = \{\emptyset\} \) (resp. \( \mathcal{P}(X) \)), then \( A^*_{mH} = \text{mCl}(A) \) (resp. \( A^*_{mH} = \emptyset \)).

Lemma 2.9. [14] Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. For subsets \( A \) and \( B \) of \( X \), the following properties hold:
1. If \( A \subseteq B \), then \( A^*_{mH} \subseteq B^*_{mH} \).
2. \( A^*_{mH} = \text{mCl}(A^*_{mH}) \subseteq \text{mCl}(A) \).
3. \( A^*_{mH} \cup B^*_{mH} \subseteq (A \cup B)^*_{mH} \).
4. \( (A^*_{mH})^*_{mH} \subseteq (A \cup (A^*_{mH}))^*_{mH} = A^*_{mH} \).
5. If \( A \in \mathcal{H} \), then \( A^*_{mH} = \emptyset \).

A similar study may also be considered through grill as well as generalized topological spaces [1,13].

Lemma 2.10. Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. If \( U \in m \) and \( U \cap A \in \mathcal{H} \), then \( U \cap A^*_{mH} = \emptyset \).

Definition 2.11. A subset \( A \) in a hereditary \( m \)-space \((X, m, \mathcal{H})\) is said to be \( \mathcal{H} \)-dense \([12]\) (resp. \( m \)-dense, \( m^*_H \)-dense) if \( A^*_{mH} = X \) (resp. \( \text{mCl}(A) = X \), \( m^*_H(A) = X \)).

The collection of all \( \mathcal{H} \)-dense (resp. \( m \)-dense, \( m^*_H \)-dense) is denoted by \( D_\mathcal{H}(X, m) \) (resp. \( D(X, m) \), \( D^*_H(X, m) \)).

Example 2.12. Let \( X = \{a, b, c, d\} \), \( m = \{X, \emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}\} \) and \( \mathcal{H} = \{\emptyset, \{a\}\} \). If \( A = \{a, c\} \) then \( A^*_{mH} = X \). So that \( \text{mCl}^*_H(A) = X \) and \( A \) is \( m^*_H \)-dense.

Example 2.13. Let \( X = \{a, b, c, d\} \), \( m = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \mathcal{H} = \{\emptyset, \{c\}\} \). If \( A = \{a, c\} \) then \( A^*_{mH} = \{a, c, d\} \). So that \( \text{mCl}^*_H(A) \neq X \) and \( A \) is not \( m^*_H \)-dense.

Theorem 2.14. Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. Then the following properties hold:
(1) \( D_\mathcal{H}(X, m) \subseteq D^*_H(X, m) \subseteq D(X, m) \),
(2) If for some \( U \in m \), \( U \cap D \in \mathcal{H} \) implies \( U \cap (X - D) \notin \mathcal{H} \), then \( D_\mathcal{H}(X, m) = D^*_H(X, m) \).
Corollary 2.15. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. Then for \(x \in X\), \(X - \{x\}\) is \(\mathcal{H}\)-dense if and only if \(\Gamma^*_{mH}(\{x\}) = X - (X - \{x\})^*_{mH}\) for any subset \(A\) of \(X\).

Proof. The proof follows from the definition of \(\mathcal{H}\)-dense sets, since \(\Gamma^*_{mH}(\{x\}) = X - (X - \{x\})^*_{mH}\) if and only if \(X = (X - \{x\})^*_{mH}\).

3. \(\mathcal{H}\)-isolated Points and \(\mathcal{H}\)-derived Sets

Let \((X, m)\) be an \(m\)-space and let \(x \in X\) and \(A \subseteq X\). Then \(x\) is called an \(m\)-accumulation point of \(A\) in \(X\) if \(U \cap (\{A\} - \{x\}) \neq \emptyset\) for every \(U \in m(x)\). The \(m\)-derived set of \(A\) in \(X\), denoted by \(d_m(A)\), is the set of all \(m\)-accumulation points of \(A\) in \(X\) and \(x\) is called an \(m\)-isolated point of \(A\) in \(X\) if there exists \(U \in m(x)\) such that \(U \cap A = \{x\}\). We denote the set of all \(m\)-isolated points of \(A\) in \(X\) by \(I_m(A)\). It is well known that \(I_m(A) = A - d_m(A)\) and \(mCl(A) = d_m(A) \cup A\).

Now, we introduce the concepts of \(\mathcal{H}\)-isolated points and \(\mathcal{H}\)-derived sets in a hereditary \(m\)-space \((X, m, \mathcal{H})\).

Definition 3.1. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and let \(x \in X\) and \(A \subseteq X\).

1. \(x\) is called an \(\mathcal{H}\)-isolated point of \(A\) in \(X\) if there exists \(U \in m^*_H(x)\) such that \(U \cap A = \{x\}\). We denote the set of all \(\mathcal{H}\)-isolated points of \(A\) in \(X\) by \(I_{\mathcal{H}}(A)\).

2. \(x\) is called an \(\mathcal{H}\)-accumulation point of \(A\) in \(X\) if \(U \cap (\{A\} - \{x\}) \neq \emptyset\) for every \(U \in m^*_H(x)\). The \(\mathcal{H}\)-derived set of \(A\) in \(X\), denoted by \(d_{\mathcal{H}}(A)\), is the set of all \(\mathcal{H}\)-accumulation points of \(A\) in \(X\).

Example 3.2. Let \(X = \{a, b, c\}, m = \{X, \emptyset, \{a\}, \{b\}, \{b, c\}\}\).

1. If \(A = \{a, b\}\) then \(m\)-derived of \(A\) is \(d_m(A) = \{c\}\). So that \(m\)-isolated of \(A\) is \(I_m(A) = A - d_m(A) = \{a, b\}\).

2. If \(B = \{a, c\}\) then \(m\)-derived of \(B\) is \(d_m(B) = \emptyset\). So that \(m\)-isolated of \(B\) is \(I_m(B) = B - d_m(B) = \{a, c\}\).

3. If \(C = \{b, c\}\) then \(m\)-derived of \(C\) is \(d_m(A) = \{c\}\). So that \(m\)-isolated of \(C\) is \(I_m(C) = C - d_m(C) = \{b\}\).

Example 3.3. Let \(X = \{a, b, c, d\}, m = \{X, \emptyset, \{a\}\}\) with \(\mathcal{H} = \{\emptyset, \{a\}\}\). Then \(m^* = \{X, \emptyset, \{a\}, \{b, c, d\}\}\).

1. If \(A = \{a, b\}\) then \(\mathcal{H}\)-derived of \(A\) is \(d_{\mathcal{H}}(A) = \{c, d\}\). So that \(\mathcal{H}\)-isolated of \(A\) is \(I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A) = \{a, b\}\).

2. If \(B = \{b, c\}\) then \(\mathcal{H}\)-derived of \(B\) is \(d_{\mathcal{H}}(B) = \{b, c\}\). So that \(\mathcal{H}\)-isolated of \(B\) is \(I_{\mathcal{H}}(B) = B - d_{\mathcal{H}}(B) = \emptyset\).

Proposition 3.4. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and \(m\) have property \(\mathcal{F}\). Then for \(A, B \subseteq X\), the following properties hold:

1. \(I_{\mathcal{H}}(A) = A - d_{\mathcal{H}}(A)\).

2. \(I_m(A) \subseteq I_{\mathcal{H}}(A) \subseteq A\).
3. (a) \( A = I_{3\mathcal{H}}(A) \cup [d_{3\mathcal{H}}(A) \cap A] \);
(b) \( d_{3\mathcal{H}}(A) \cap A = A - I_{3\mathcal{H}}(A) \).

4. If \( A \in m_{3\mathcal{H}}^* - \{\emptyset\} \) and \( A \subseteq B \), then \( I_{3\mathcal{H}}(A) \subseteq I_{3\mathcal{H}}(B) \).

5. (a) \( I_{3\mathcal{H}}(A) \cap I_{3\mathcal{H}}(B) \subseteq I_{3\mathcal{H}}(A \cap B) \);
(b) \( I_{3\mathcal{H}}(A \cup B) \subseteq I_{3\mathcal{H}}(A) \cup I_{3\mathcal{H}}(B) \).

Proof. (1) Let \( x \in I_{3\mathcal{H}}(A) \). Then \( U \cap A = \{x\} \) for some \( U \in m_{3\mathcal{H}}^*(x) \). This implies \( U \cap (A - \{x\}) = \emptyset \). Then \( x \notin d_{3\mathcal{H}}(A) \). Thus \( x \in A - d_{3\mathcal{H}}(A) \) and \( I_{3\mathcal{H}}(A) \subseteq A - d_{3\mathcal{H}}(A) \). Conversely, let \( x \in A - d_{3\mathcal{H}}(A) \). Since \( x \notin d_{3\mathcal{H}}(A) \), we have \( U \cap (A - \{x\}) = \emptyset \) for some \( U \in m_{3\mathcal{H}}^*(x) \). Note that \( U \cap A = \{x\} \). Then \( x \in I_{3\mathcal{H}}(A) \) and \( A - d_{3\mathcal{H}}(A) \subseteq I_{3\mathcal{H}}(A) \). Hence \( I_{3\mathcal{H}}(A) = A - d_{3\mathcal{H}}(A) \).

(2) This is obvious.

(3) (a) For any \( x \in A \) and \( U \in m_{3\mathcal{H}}^+(x) \), \( U \cap A = \{x\} \) or \( U \cap (A - \{x\}) = \emptyset \), then \( x \in I_{3\mathcal{H}}(A) \cup d_{3\mathcal{H}}(A) \) and \( A \subseteq I_{3\mathcal{H}}(A) \cup d_{3\mathcal{H}}(A) \). Thus \( A \subseteq (I_{3\mathcal{H}}(A) \cup d_{3\mathcal{H}}(A)) \cap A = I_{3\mathcal{H}}(A) \cup [d_{3\mathcal{H}}(A) \cap A] \). And \( A \supseteq (I_{3\mathcal{H}}(A) \cup d_{3\mathcal{H}}(A)) \cap A \). Hence \( A = I_{3\mathcal{H}}(A) \cup [d_{3\mathcal{H}}(A) \cap A] \).
(b) This holds by (1).

(4) Let \( x \in I_{3\mathcal{H}}(A) \). Then \( U \cap A = \{x\} \) for some \( U \in m_{3\mathcal{H}}^*(x) \). Since \( A \in m_{3\mathcal{H}}^* - \{\emptyset\} \), \( U \cap A \in m_{3\mathcal{H}}^* - \{\emptyset\} \). Note that \( U \cap A \cap B = \{x\} \). Then \( x \in I_{3\mathcal{H}}(B) \). Thus \( I_{3\mathcal{H}}(A) \subseteq I_{3\mathcal{H}}(B) \).

(5) This is obvious.

\[ \Box \]

Proposition 3.5. Let \((X, m, \mathcal{H})\) and \((X, m, \mathcal{J})\) be two hereditary spaces with \( \mathcal{J} \subseteq \mathcal{H} \). Then for \( A \subseteq X \), \( I_{3\mathcal{J}}(A) \subseteq I_{3\mathcal{H}}(A) \).

Proof. Let \( x \in I_{3\mathcal{J}}(A) \). Then \( U \cap A = \{x\} \) for some \( U \in m_{3\mathcal{J}}^+(x) \). It is clear that \( \mathcal{J} \subseteq \mathcal{H} \) implies that \( m_{3\mathcal{J}}^* \subseteq m_{3\mathcal{H}}^* \). So \( U \in m_{3\mathcal{H}}^+ \) and thus \( x \in I_{3\mathcal{H}}(A) \). Hence \( I_{3\mathcal{J}}(A) \subseteq I_{3\mathcal{H}}(A) \).

Proposition 3.6. Let \((X, m, \mathcal{H})\) and \((X, n, \mathcal{K})\) be two hereditary spaces with \( n \subseteq m \). Then for \( A \subseteq X \), \( I_{n3\mathcal{K}}(A) \subseteq I_{m3\mathcal{K}}(A) \).

Proof. Let \( x \in I_{n3\mathcal{K}}(A) \). Then \( U \cap A = \{x\} \) for some \( U \in n_{3\mathcal{K}}^+(x) \). It is clear that \( n \subseteq m \) implies \( n_{3\mathcal{K}}^* \subseteq m_{3\mathcal{H}}^* \). So \( U \in m_{3\mathcal{H}}^+ \) and thus \( x \in I_{m3\mathcal{K}}(A) \). Hence \( I_{n3\mathcal{K}}(A) \subseteq I_{m3\mathcal{K}}(A) \).

Definition 3.7. A hereditary \( m \)-space \((X, m, \mathcal{H})\) is said to be \( \mathcal{H} \)-scattered if \( I_{3\mathcal{H}}(A) \neq \emptyset \) for any nonempty \( A \in \mathcal{P}(X) \).

Example 3.8. Let \( X = \{a, b, c\} \), \( m = \{X, \emptyset, \{a\}, \{b\}\} \) and \( \mathcal{H} = \{\emptyset, \{c\}\} \). Then
\[ m^* = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}. \]

It is clear that \( m^*(a) = \{X, \{a\}, \{a, b\}\} \), \( m^*(b) = \{X, \{b\}, \{a, b\}\} \) and \( m^*(c) = \{X\} \).

Then a hereditary \( m \)-space \((X, m, \mathcal{H})\) is \( \mathcal{H} \)-scattered because \( I_{3\mathcal{H}}(A) \neq \emptyset \) for any nonempty \( A \in \mathcal{P}(X) \) as the following table.

<table>
<thead>
<tr>
<th>( A \subseteq {a} )</th>
<th>( d_{3\mathcal{H}}(A) = {c} )</th>
<th>( I_{3\mathcal{H}}(A) = \emptyset )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \subseteq {b} )</td>
<td>( d_{3\mathcal{H}}(A) = {c} )</td>
<td>( I_{3\mathcal{H}}(A) = {b} )</td>
</tr>
<tr>
<td>( A \subseteq {c} )</td>
<td>( d_{3\mathcal{H}}(A) = \emptyset )</td>
<td>( I_{3\mathcal{H}}(A) = \emptyset )</td>
</tr>
<tr>
<td>( A \subseteq {a, b} )</td>
<td>( d_{3\mathcal{H}}(A) = {c} )</td>
<td>( I_{3\mathcal{H}}(A) = {a, b} )</td>
</tr>
<tr>
<td>( A \subseteq {a, c} )</td>
<td>( d_{3\mathcal{H}}(A) = {c} )</td>
<td>( I_{3\mathcal{H}}(A) = {a} )</td>
</tr>
<tr>
<td>( A \subseteq {b, c} )</td>
<td>( d_{3\mathcal{H}}(A) = {c} )</td>
<td>( I_{3\mathcal{H}}(A) = {b} )</td>
</tr>
<tr>
<td>( A \subseteq {a, b, c} )</td>
<td>( d_{3\mathcal{H}}(A) = {c} )</td>
<td>( I_{3\mathcal{H}}(A) = {a, b} )</td>
</tr>
</tbody>
</table>

Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. The family of all \( m_{3\mathcal{H}}^* \)-dense of \( X \) is denoted by \( D_{3\mathcal{H}}^* = D_{3\mathcal{H}}^*(X, m) \). For the subspace \((Y, m_Y, \mathcal{H}_Y)\), the family of all \( m_{3\mathcal{H}}^* \)-dense subsets of \( Y \) is denoted by \( D_{3\mathcal{H}}^*(Y) = \{A \subseteq Y : m_{3\mathcal{H}}(A) = Y\} \).
Lemma 3.9. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. Then \(A \subseteq X\) is \(m^*_H\)-dense in \(X\) if and only if \(U \cap A \neq \emptyset\) for every nonempty set \(U \in m^*_H\).

Proof. Let \(A\) be \(m^*_H\)-dense in \(X\) and let \(U\) be nonempty \(U \in m^*_H\). Pick \(x \in U\). Then \(x \in X = m\text{CL}^*_H(A) = A \cup A^*_m\). Then if \(x \in A\), then \(x \in A \cap U\) and \(A \cap U \neq \emptyset\). Suppose that \(x \in A^*_m\) and \(A \cap U = \emptyset\). Since \(X - U\) is \(m^*_H\)-closed in \(X\), \((X - U)^*_H \subseteq X - U\). Then \(U \subseteq X - (X - U)^*_H\). By \(x \in U\), \(x \notin (X - U)^*_H\). It follows that \(V \cap (X - U) \in \mathcal{H}\) for some \(V \in m(x)\). By \(A \cap U = \emptyset\), \(A \subseteq X - U\). \(V \cap A \subseteq V \cap (X - U) \in \mathcal{H}\). Then \(V \cap A \in \mathcal{H}\). Hence \(x \notin A^*_m\). This is a contradiction. Thus, \(U \cap A \neq \emptyset\). Conversely, suppose \(m\text{CL}^*_H(A) \neq X\). Put \(U = X - m\text{CL}^*_H(A)\), then \(U\) is a nonempty set in \(m^*_H\). But \(U \cap A = [X - m\text{CL}^*_H(A)] \cap A = \emptyset\). This is a contradiction. \(\square\)

Theorem 3.10. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and \(m\) have property \(\mathcal{F}\). The following are equivalent.

1. \(X\) is \(\mathcal{H}\)-scattered;
2. \(I_\mathcal{H}(A) \in D^\mathcal{H}_\mathcal{H}(A)\), for any nonempty set \(A \in \mathcal{P}(X)\);
3. \(D \in D^\mathcal{H}_\mathcal{H}(A)\) if and only if \(I_\mathcal{H}(A) \subseteq D\), for any nonempty set \(A \in \mathcal{P}(X)\);
4. \(d_\mathcal{H}(A) = d_\mathcal{H}[I_\mathcal{H}(A)]\) for any nonempty set \(A \in \mathcal{P}(X)\);
5. If \(A\) is nonempty \(m^*_H\)-closed, then \(I_\mathcal{H}(A) \neq \emptyset\).

Proof. (1) \(\Rightarrow\) (2): Let \(\emptyset \neq V \in m^*_H\). Then \(V = W \cap A\) for some \(W \in m^*_H\). Since \(X\) is \(\mathcal{H}\)-scattered, \(I_\mathcal{H}(V) \neq \emptyset\). Pick \(x \in I_\mathcal{H}(V)\). Then \(U \cap V = \{x\}\) for some \(U \in m^*_H(x)\). So \((U \cap W) \cap A = U \cap (W \cap A) = U \cap V = \{x\}\). Note that \(U \cap W \in m^*_H(x)\). This implies \(x \in I_\mathcal{H}(A)\). Then \(x \in V \cap I_\mathcal{H}(A)\) and so \(V \cap I_\mathcal{H}(A) \neq \emptyset\). By Lemma 3.9 \(m\text{CL}^*_H[I_\mathcal{H}(A)] = A\). Thus \(I_\mathcal{H}(A) \in D^\mathcal{H}_\mathcal{H}(A)\).

(2) \(\Rightarrow\) (3): Let \(I_\mathcal{H}(A) \subseteq D\). By (2), \(A = m\text{CL}^*_H[I_\mathcal{H}(A)] \subseteq m\text{CL}^*_H[I_\mathcal{H}(A)]D\). Thus \(D \in D^\mathcal{H}_\mathcal{H}(A)\). Conversely, suppose \(I_\mathcal{H}(A) \not\subseteq D\) for some \(D \in D^\mathcal{H}_\mathcal{H}(A)\). Then \(I_\mathcal{H}(A) - D \neq \emptyset\). Pick \(x \in I_\mathcal{H}(A) - D\). Then \(U \cap A = \{x\}\) for some \(U \in m^*_H(x)\). Note that \(U \cap A \in m^*_H(x)\) and \(D \in D^\mathcal{H}_\mathcal{H}(A)\). By Lemma 3.9, \(D \cap (U \cap A) \neq \emptyset\). But \(D \cap (U \cap A) = D \cap \{x\} = \emptyset\). This is a contradiction.

(3) \(\Rightarrow\) (4): Since \(I_\mathcal{H}(A) \subseteq A\), \(d_\mathcal{H}[I_\mathcal{H}(A)] \subseteq d_\mathcal{H}(A)\). Suppose \(d_\mathcal{H}(A) \not\subseteq d_\mathcal{H}[I_\mathcal{H}(A)]\). Then \(d_\mathcal{H}(A) - d_\mathcal{H}[I_\mathcal{H}(A)] \neq \emptyset\). Pick up \(x \in d_\mathcal{H}(A) - d_\mathcal{H}[I_\mathcal{H}(A)]\). By Proposition 3.4(1), \(I_\mathcal{H}(A) = A - d_\mathcal{H}(A)\). Then \(x \notin I_\mathcal{H}(A)\) and \(x \notin d_\mathcal{H}[I_\mathcal{H}(A)]\) implies \(U \cap I_\mathcal{H}(A) - \{x\}\) for some \(U \in m^*_H(x)\). Note that \(x \notin I_\mathcal{H}(A)\). Then \(U \cap I_\mathcal{H}(A) - \{x\} \subseteq U \cap d_\mathcal{H}(A) - \emptyset\) with \(U \cap A \in m^*_H(A)\). By (3) \(I_\mathcal{H}(A) \in D^\mathcal{H}_\mathcal{H}(A)\). Then \(V \cap I_\mathcal{H}(A) \neq \emptyset\) for every \(V \in m^*_H\). This is a contradiction. Hence \(d_\mathcal{H}(A) \subseteq d_\mathcal{H}[I_\mathcal{H}(A)]\) and hence \(d_\mathcal{H}(A) = d_\mathcal{H}[I_\mathcal{H}(A)]\).

(4) \(\Rightarrow\) (1): Suppose \(I_\mathcal{H}(A) = \emptyset\) for some nonempty set \(A \in \mathcal{P}(X)\). By (4), \(d_\mathcal{H}(A) = d_\mathcal{H}[I_\mathcal{H}(A)] = d_\mathcal{H}(\emptyset) = \emptyset\). By Proposition 3.4(3), \(A = I_\mathcal{H}(A) \cup [d_\mathcal{H}(A) \cap A] = \emptyset\), a contradiction. Hence \(X\) is \(\mathcal{H}\)-scattered.

(5) \(\Rightarrow\) (1): Let \(\emptyset \neq A \in \mathcal{P}(X)\). Since \(m\text{CL}^*_H(A)\) is \(m^*_H\)-closed, by (5), \(I_\mathcal{H}[m\text{CL}^*_H(A)] \neq \emptyset\). Pick \(x \in I_\mathcal{H}[m\text{CL}^*_H(A)]\). Then \(U \cap [m\text{CL}^*_H(A)] = \{x\}\) for some \(U \in m^*_H(x)\). Suppose \(U \cap A = \emptyset\). We have \(A \subseteq X - U\). Then \(m\text{CL}^*_H(A) \subseteq X - U\). So \(U \cap m\text{CL}^*_H(A) = \emptyset\). This is a contradiction. Thus \(U \cap A \neq \emptyset\). Since \(U \cap A \subseteq U \cap m\text{CL}^*_H(A) = \{x\}\), we have \(U \cap A = \{x\}\). So \(x \in I_\mathcal{H}(A)\). This implies \(I_\mathcal{H}(A) \neq \emptyset\). Hence \(X\) is \(\mathcal{H}\)-scattered. \(\square\)

Definition 3.11. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. Put \(X^0 = X\) and \(X^1 = \{x \in X : x\) is not an \(\mathcal{H}\)-isolated point in \(X\}\). Let \(\alpha\) be any order number. If \(X^\beta\) is already defined for all order \(\beta < \alpha\), then we put

\[
X^\alpha = \begin{cases} 
(X^\beta)^1, & \text{if } \alpha = \beta + 1 \text{ and } \beta \text{ is an ordinal number;} \\
\bigcap_{\beta < \alpha} X^\beta, & \text{if } \alpha \text{ is a limit ordinal number.}
\end{cases}
\]

Remark 3.12.

1. \(X^1 = X - I_{\mathcal{H}}(X) = X \cap d_{\mathcal{H}}(X)\).
2. $X^\beta \subseteq X^\alpha$ whenever $\alpha \leq \beta$.

3. $X^\alpha = X^{\alpha-1} - I_{\delta}(X^{\alpha-1}) = X^{\alpha-1} \cap d_{\delta}(X^{\alpha-1})$ for any successor ordinal number $\alpha$.

4. If $\alpha$ is a successor ordinal number and $X^\alpha = \emptyset$, then $X = \bigcup_{\beta \leq \alpha-1} I_{\delta}(X^\beta)$.

**Lemma 3.13.** Let $(X,m,\mathcal{H})$ be a hereditary $m$-space. If $m$ has property $\mathcal{F}$, the following properties hold.

1. $X^\alpha$ is $m^*_H$-closed for any ordinal number $\alpha$.

2. $Y \subseteq X$, then $Y^\alpha \subseteq X^\alpha$ for any ordinal number $\alpha$.

**Proof.** 1. We use induction on $\alpha$.
   1) $\alpha = 1$. Let $x \in I_{\delta}(X)$. Then $U_x \cap X = \{x\}$ for some $U_x \in m^*_H(x)$. This implies $\{x\} = U_x \in m^*_H$. Thus $I_{\delta}(X) = \bigcup_{x \in I_{\delta}(X)} \{x\} \in m^*_H$. Thus $X^1 = X - I_{\delta}(X)$ is $m^*_H$-closed.

   2) Suppose $X^\beta$ is $m^*_H$-closed for any $\beta < \alpha$. We will prove $X^\alpha$ is $m^*_H$-closed in the following cases.
   (a) $\alpha$ is a successor ordinal number. Let $x \in I_{\delta}(X^{\alpha-1})$. $U_x \cap X^{\alpha-1} = \{x\}$ for some $U_x \in m^*_H(x)$. By Remark 3.12, $X^\alpha = X^{\alpha-1} - I_{\delta}(X^{\alpha-1})$. So
   
   $X^\alpha = X^{\alpha-1} - \bigcup_{x \in I_{\delta}(X^{\alpha-1})} \{x\} = [X - x \in I_{\delta}(X^{\alpha-1}) U_x] \cap X^{\alpha-1}$.

   By induction hypothesis, $X^{\alpha-1}$ is $m^*_H$-closed. Thus $X^\alpha$ is $m^*_H$-closed.

   (b) $\alpha$ is a limit ordinal number. By induction hypothesis, $X^\beta$ is $m^*_H$-closed for any $\beta < \alpha$. Thus $X^\alpha = \bigcap_{\beta < \alpha} X^\beta$ is $m^*_H$-closed.

   2. Let $Y \subseteq X$. We will prove $Y^\alpha \subseteq X^\alpha$ for any ordinal number $\alpha$.
   1) $Y^1 = Y \cap d_{\delta}(Y) \subseteq X \cap d_{\delta}(X) = X^1$. This shows $Y^\alpha \subseteq X^\alpha$ when $\alpha = 1$.
   2) Suppose $Y^\beta \subseteq X^\beta$ for any $\beta < \alpha$. We consider the following cases.
   (a) $\alpha$ is a successor ordinal number. By induction hypothesis, $Y^{\alpha-1} \subseteq X^{\alpha-1}$. By Remark 3.12, $Y^\alpha = Y^{\alpha-1} \cap d_{\delta}(Y^{\alpha-1}) \subseteq X^{\alpha-1} \cap d_{\delta}(X^{\alpha-1}) = X^\alpha$.
   (b) $\alpha$ is a limit ordinal number. By induction hypothesis, $Y^\beta \subseteq X^\beta$ for any $\beta < \alpha$. Thus $Y^\alpha = \bigcap_{\beta < \alpha} Y^\beta \subseteq X^\alpha$. By 1) and 2) we have $Y^\alpha \subseteq X^\alpha$ for any ordinal number $\alpha$.

\[\square\]

**Definition 3.14.** Let $(X,m,\mathcal{H})$ be a hereditary $m$-space.

1. An ordinal number $\beta$ is called the derived length of $X$ if $\beta = \min\{\alpha : X^\alpha = \emptyset\}$. $\beta$ is denoted by $\delta(X)$.

2. $X$ is said to have a derived length if there is an ordinal number $\alpha$ such that $X^\alpha = \emptyset$.

**Lemma 3.15.** $X^\delta = X^{\delta+1}$ for some ordinal number $\delta$.

**Theorem 3.16.** Let $(X,m,\mathcal{H})$ be a hereditary $m$-space. Then $X$ is $\mathcal{H}$-scattered if and only if $X$ has a derived length.

**Proof.** Sufficiency. Suppose that $X$ is not $\mathcal{H}$-scattered. Then $I_{\delta}(A) = \emptyset$ for some nonempty set $A \subseteq X$. We claim $A \subseteq X^\alpha$ for any ordinal number $\alpha$.

(1) Let $x \in A$ and $U \in m^*_H(x)$. Since $I_{\delta}(A) = \emptyset$, $U \cap A \neq \{x\}$. Note that $x \in U \cap A$. Then $|U \cap A| \geq 2$ and so $U \cap (A - \{x\}) \neq \emptyset$. Now $U \cap (A - \{x\}) \subseteq U \cap (X - \{x\})$. Then $U \cap (X - \{x\}) \neq \emptyset$. This implies $x \in X \cap d_{\delta}(X)$. By Remark 3.12, $x \in X^1$. Thus $A \subseteq X - I_{\delta}(X) = X^1$. i.e., $A \subseteq X^\alpha$ when $\alpha = 1$.

(2) Suppose $A \subseteq X^\beta$ for any $\beta < \alpha$. We will prove $A \subseteq X^\alpha$ in the following cases.
   a) $\alpha$ is a successor ordinal number. Let $x \in A$ and $U \in m^*_H(x)$. By (1) $U \cap (A - \{x\}) \neq \emptyset$. By induction hypothesis, $A \subseteq X^{\alpha-1}$. Then $U \cap (X^{\alpha-1} - \{x\}) \neq \emptyset$. This implies $x \in X^{\alpha-1} \cap d_{\delta}(X^{\alpha-1})$. By Remark 3.12, $x \in X^\alpha$. Hence $A \subseteq X^\alpha$.

b) $\alpha$ is a limit ordinal number. By induction hypothesis, $A \subseteq X^\beta$ for any $\beta < \alpha$. Then $\bigcap_{\beta < \alpha} X^\beta = X^\alpha$.

Since $X$ has a derived length, $X^\delta = \emptyset$ for some ordinal number $\delta$. By claim, $A \subseteq X^\beta$. Then $A = \emptyset$, a contradiction.
Necessity. Conversely, suppose that \( X \) has no derived length. By Lemma 3.15, \( X^{\delta+1} = X^\delta \) and Remark 3.12, \( X^{\delta+1} = \mathbb{H}(X^\delta) \). Then \( I_\mathbb{H}(X^\delta) = \emptyset \). Note that \( X \) has no derived length. Then \( X^\delta \neq \emptyset \). It follows that \( X \) is not \( \mathcal{H} \)-scattered. This is a contradiction.

\( \square \)

4. Characterizations of Scattered Spaces

**Corollary 4.1.** (1) Let \((X, m, \mathcal{H})\) and \((X, m, \mathcal{J})\) be two hereditary spaces with \( \mathcal{J} \subseteq \mathcal{H} \). If \((X, m, \mathcal{J})\) is \( \mathcal{J} \)-scattered, then \((X, m, \mathcal{H})\) is \( \mathcal{H} \)-scattered.

(2) Let \((X, m, \mathcal{H})\) and \((X, n, \mathcal{H})\) be two hereditary spaces with \( n \leq m \). If \((X, n, \mathcal{H})\) is \( \mathcal{H} \)-scattered, then \((X, m, \mathcal{H})\) is \( \mathcal{H} \)-scattered.

**Proof.** These hold by Proposition 3.5 and Proposition 3.6.

An \( m \)-space \((X, m)\) is said to be scattered if \( I_m(A) \neq \emptyset \) for any nonempty set \( A \in \mathcal{P}(X) \).

**Theorem 4.2.** Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. Then the following are equivalent.

1. \((X, m)\) is scattered.
2. \((X, m, \mathcal{H})\) is \( \mathcal{H} \)-scattered for any hereditary \( \mathcal{H} \) on \( X \).
3. \((X, m, \{\emptyset\})\) is \( \{\emptyset\} \)-scattered

**Proof.** (1) \( \Rightarrow \) (2): This follows from Proposition 3.4 (2).

(2) \( \Rightarrow \) (3): The proof is obvious.

(3) \( \Rightarrow \) (1): Since \( m = m_{\mathcal{H}}^* \) whenever \( \mathcal{H} = \{\emptyset\} \), \( I_m(A) = I_{\mathcal{H}^*}(A) \neq \emptyset \). Thus \((X, m)\) is scattered.

**Theorem 4.3.** Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space and \( Y \) be nonempty subset of \( X \). If \( X \) is \( \mathcal{H} \)-scattered, then \((Y, m_Y, \mathcal{H}_Y)\) is \( \mathcal{H}_Y \)-scattered.

**Proof.** Let \( A \) be nonempty set of \( Y \). Since \( X \) is \( \mathcal{H} \)-scattered, \( I_{\mathcal{H}}(A) \neq \emptyset \). Pick \( x \in I_{\mathcal{H}}(A) \). Then \( U \cap A = \{x\} \) for some \( U \in m_{\mathcal{H}}^*(x) \). Note that \( U \cap Y \in m_Y(\mathcal{H}) \) and \( (U \cap Y) \cap A = (U \cap A) \cap Y = \{x\} \). Then \( x \in I_{\mathcal{H}_Y}(A) \) and so \( I_{\mathcal{H}_Y}(A) \neq \emptyset \). Hence \((Y, m_Y, \mathcal{H}_Y)\) is \( \mathcal{H}_Y \)-scattered.

**Lemma 4.4.** If every \( \mathcal{H}_\alpha \) is a hereditary on \( X_\alpha \) \((\alpha \in \Delta)\), then \( \bigcup_{\alpha \in \Delta} \{H_\alpha : H_\alpha \in \mathcal{H}_\alpha\} \) is a hereditary on \( \bigcup_{\alpha \in \Delta} X_\alpha \).

**Definition 4.5.** A hereditary \( m \)-space \((X, m, \mathcal{H})\) is called \( \mathcal{H} \)-resolvable if \( X \) has two disjoint \( \mathcal{H} \)-dense subsets. Otherwise, \( X \) is called \( \mathcal{H} \)-irresolvable.

**Example 4.6.** Let \( X = \{a, b, c, d\} \), \( m = \{X, \emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}\} \) with \( \mathcal{H} = \{\emptyset, \{a\}\} \). Then it is clear that \( \mathcal{H} \)-dense subsets of \( X \). Hence, a hereditary \( m \)-space \((X, m, \mathcal{H})\) is \( \mathcal{H} \)-resolvable.

**Proposition 4.7.** Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space. If \( X \) is \( \mathcal{H} \)-scattered, then \( X \) is \( \mathcal{H} \)-irresolvable.

**Proof.** Suppose that \( X \) is not \( \mathcal{H} \)-irresolvable. Then \( X \) is \( \mathcal{H} \)-resolvable. For some nonempty sets \( A, B \in \mathcal{P}(X) \), we have \( A^*_H = B^*_H = X \) and \( A \cap B = \emptyset \). Since \( A, B \in D_H^*(X) \), by Theorem 3.10, \( I_{\mathcal{H}}(X) \subseteq A, B \), and \( I_{\mathcal{H}}(X) \subseteq A \cap B \). Since \( X \) is \( \mathcal{H} \)-scattered, \( I_{\mathcal{H}}(X) \neq \emptyset \). So \( A \cap B \neq \emptyset \). Thus, \( X \) is \( \mathcal{H} \)-irresolvable.

It is clear that by Proposition 4.7 a hereditary \( m \)-space \((X, m, \mathcal{H})\) in Example 3.8 is \( \mathcal{H} \)-irresolvable.

**Definition 4.8.** A mapping \( f : (X, m, \mathcal{H}) \to (Y, n, \mathcal{J}) \) is said to be \( \mathcal{H} \)-closed if \( f(A) \) is \( n^*_J \)-closed in \( Y \) for each \( m^*_H \)-closed subset \( A \) of \( X \).

**Theorem 4.9.** Let \((X, m, \mathcal{H})\) be \( \mathcal{H} \)-scattered, \((Y, n, \mathcal{J})\) be a hereditary \( n \)-space, where \( m \) and \( n \) has property \( \mathcal{F} \), and let \( f : (X, m, \mathcal{H}) \to (Y, n, \mathcal{J}) \) be \( \mathcal{H} \)-closed. Suppose that \( f \) satisfies the following condition. The set \( \{\beta : X^\beta \cap f^{-1}(y) \neq \emptyset\} \) contains a largest element for any \( y \in Y \). Then the following properties hold:
1. $Y^\alpha \subseteq f(X^\alpha)$ for every ordinal number $\alpha$.
2. $\delta(Y) \leq \delta(X)$.
3. $Y$ is $\beta$-scattered.

Proof. Since (2) and (3) hold by (1) and Theorem 3.16, we only need to prove (1) i.e. $Y^\alpha \subseteq f(X^\alpha)$ for every ordinal number $\alpha$.

We use induction on $\alpha$.

1. Since $Y^0 = Y = f(X) = f(X^0)$, then $Y^\alpha \subseteq f(X^\alpha)$ when $\alpha = 0$.

2. Suppose $Y^\beta \subseteq f(X^\beta)$ when $\beta < \alpha$. It suffices to show $Y^\alpha \subseteq f(X^\alpha)$ in the following two cases,

(a) $\alpha = \beta + 1$ for some ordinal number $\beta$.
Suppose $Y^\alpha \nsubseteq f(X^\alpha)$. Then $Y^\alpha - f(X^\alpha) \neq \emptyset$. Pick $y \in Y^\alpha - f(X^\alpha)$. Then $X^\alpha \cap f^{-1}(y) \neq \emptyset$. Put $F = X^\beta - f^{-1}(y)$.

Claim 1. $F$ is $m_H^\beta$-closed in $X$. Put $A = X^\beta \cap f^{-1}(y)$. Then $F = X^\beta - A$. Since $X^\beta \cap f^{-1}(y) = \emptyset$, $f^{-1}(y) \subseteq X - X^\alpha$. This implies $A \subseteq X^\beta \cap (X - X^\alpha) = X^\beta - X^\alpha$. By Remark 3.12, $X^\beta - X^\alpha = I_\beta(X^\beta)$. Thus $A \subseteq I_\beta(X^\beta)$. For any $x \in A$, $x \in I_\beta(X^\beta)$. Then $U \cap X^\beta = \{x\}$ for some $U \in m_H^\beta$. Then $\{x\} \in m_H^\beta(\text{relative space})$ and so $A = \bigcup_{x \in A} \{x\} \in m_H^\beta$. This implies $F = X^\beta - A$ is $m_H^\beta$-closed in $X^\beta$. By Lemma 3.13 (1) $F$ is $m_H^\beta$-closed in $X$. By induction hypothesis, $Y^\beta \subseteq f(X^\beta)$. Then $Y^\beta \cap f^{-1}(y) \subseteq f(X^\beta) - \{y\}$. Note that $X^\beta \subseteq F \cup f^{-1}(y)$. Then $Y^\beta \cap \{y\} \subseteq f(F \cup f^{-1}(y)) - \{y\} = f(F)$. Thus $Y^\beta - f(F) \subseteq \{y\}$. Conversely, by $f^{-1}(y) \cap F = \emptyset$, $y \notin f(F)$. Note that $y \in Y^\alpha \subseteq Y^\beta$. Then $\{y\} \subseteq Y^\beta - f(F)$. Hence $\{y\} = Y^\beta - f(F)$. Since $f$ is $\mathcal{H}$-closed, by Claim 1, $f(F)$ is $n_J^\beta$-closed. Note that $y \notin f(F)$. Put $U = Y - f(F)$. Then $U \in n_J^\beta(y)$. By $U \cap Y^\beta = Y^\beta - f(F) = \{y\}$, $y \in I_\beta(Y^\beta)$. By Remark 3.12, $Y^\beta - Y^\alpha = I_\beta(Y^\beta)$. This implies $y \notin Y^\alpha$. This is a contradiction. Therefore, $Y^\alpha \subseteq f(X^\alpha)$.

(b) $\alpha$ is a limit ordinal number. Suppose $Y^\alpha \nsubseteq f(X^\alpha)$. Then $Y^\alpha - f(X^\alpha) \neq \emptyset$. Pick $y \in Y^\alpha - f(X^\alpha)$. Put $\pi = \max\{\beta : X^\beta \cap f^{-1}(y) \neq \emptyset\}$. By condition of hypothesis, we have $X^\pi \cap f^{-1}(y) \neq \emptyset$. Since $X^\alpha \cap f^{-1}(y) = \emptyset$. We can claim $\pi < \alpha$. Otherwise, we have $\pi \geq \alpha$. Since $X^\pi \cap f^{-1}(y) \neq \emptyset$ and $X^\pi \subseteq X^\alpha$, $X^\alpha \cap f^{-1}(y) \neq \emptyset$. Thus $y \in f(X^\alpha)$. This is a contradiction. But $X^{\pi+1} \cap f^{-1}(y) = \emptyset$. Then $\{y\} \cap f(X^{\pi+1}) = \emptyset$ and so $f^{-1}(y) \cap f^{-1}[f(X^{\pi+1})] = \emptyset$. Put $W = X - f^{-1}[f(X^{\pi+1})]$. Then $f^{-1}(y) \subseteq W$. By Lemma 3.13 (1), $X^{\pi+1}$ is $m_H^\beta$-closed. By $f$ is $\mathcal{H}$-closed, $f(X^{\pi+1})$ is $n_J^\beta$-closed. Put $Z = Y - f(X^{\pi+1})$. Then $Z$ is $n_J^\beta$-open and $W = f^{-1}(Z)$. Put $g = f|_W$.

Claim 2. $g = f|_W : (W, m_H^\beta, \mathcal{H}_W) \to (Z, n_J^\beta, \mathcal{H}_Z)$ is $\mathcal{H}_W$-closed. Let $K$ be $m_H^\beta$-closed in $W$. Then $K = F \cap W$ for some $m_H^\beta$-closed set $F$ in $X$. Since $f$ is $\mathcal{H}$-closed, $f(F)$ is $n_J^\beta$-closed in $Y$. Note that $g(K) = f(W \cap F) = f[f^{-1}(Z) \cap F] = Z \cap f(F)$. Then $g(K)$ is $n_H^\beta$-closed in $Z$. Then $X$ is $\mathcal{H}$-scattered, by Theorem 4.3, $W$ is $\mathcal{H}_W$-scattered. By Theorem 3.16, $\delta(W)$ is existence.

Claim 3. $\delta(W) \leq \pi + 1$. $W^{\pi+1} \subseteq W \subseteq X - X^{\pi+1}$. By Lemma 3.13 (2), $W^{\pi+1} \subseteq X^{\pi+1}$. Then $W^{\pi+1} \subseteq X^{\pi+1} \cap [X - X^{\pi+1}] = \emptyset$. Thus $\delta(W) \leq \pi + 1$.

Claim 4. $Y^\alpha \cap Z = Z^\alpha$.

1. $\alpha = 0$. We have $Z^0 = Y = Y \cap Z = Y^0 \cap Z$.

2. Suppose $Y^\beta \cap Z = Z^\beta$ for every $\beta < \alpha$. We will prove $Y^\alpha \cap Z = Z^\alpha$ in the following cases.

(i) $\alpha$ is a successor ordinal number.

By induction hypothesis, $Y^{\alpha-1} \cap Z = Z^{\alpha-1}$. By $Z^\alpha \subseteq Y^\alpha$ and $Z^\alpha \subseteq Z$, we have $Z^\alpha \subseteq Y^\alpha \cap Z$. Let $y \in Y^\alpha \cap Z$. By Remark 3.12, $Y^\alpha = Y^{\alpha-1} \cap d_{\mathcal{Y}}(Y^{\alpha-1})$. Then $y \in d_{\mathcal{Y}}(Y^{\alpha-1}) \cap Y^{\alpha-1} \cap Z = d_{\mathcal{Y}}(Y^{\alpha-1}) \cap Z^{\alpha-1}$. Note that $Z$ is an $n_J^\beta$-open set containing $y$, $y \in d_{\mathcal{Y}}(Y^{\alpha-1})$ implies that $[U \cap Z \cap Y^{\alpha-1} - \{y\}] \neq \emptyset$ for any $n_J^\beta$-open set $U$ containing $y$. Then $[U \cap Z \cap Y^{\alpha-1} - \{y\}] = U \cap Z \cap Y^{\alpha-1} - \{y\} = U \cap Z^{\alpha-1} - \{y\} \neq \emptyset$. Thus, $y \in d_{\mathcal{Y}}(Z^{\alpha-1})$. By Remark 3.12, $Z^{\alpha-1} = Z^{\alpha-1} \cap d_{\mathcal{Y}}(Z^{\alpha-1})$. Then $y \in Z^{\alpha-1}$. Hence $Y^\alpha \cap Z = Z^\alpha$. Hence $Y^\alpha \cap Z = Z^\alpha$.

(ii) $\alpha$ is a limit ordinal number.

By induction hypothesis, $Y^\beta \cap Z = Z^\beta$ for any $\beta < \alpha$. Then $Y^\alpha \cap Z = (\cap_{\beta < \alpha} Y^\beta) \cap Z = \cap_{\beta < \alpha} (Y^\beta \cap Z) = \cap_{\beta < \alpha} Z^\beta = Z^\alpha$. 


By Claim 2, \( g = f_W : (W, m_W, \mathcal{H}(W)) \rightarrow (Z, n_Z, \mathcal{H}(Z)) \) is \( \mathcal{H}(W) \)-closed. By repeating the proof of (a), we can prove \( Z^{\pi+1} \subseteq \gamma \mathcal{H}(W) + 1 \). By Claim 3, \( \emptyset = W^{\delta(W)} \supseteq W^{\pi} + 1 \). This implies \( Z^{\pi+1} \cap f^{-1}(y) = \emptyset \). By Remark 3.12(4), \( Z = \bigcup_{\beta \leq \pi} I_{\mathcal{H}}(Z^\beta) \). Note that \( X^{\pi+1} \cap f^{-1}(y) = \emptyset \). Then \( y \notin f(X^{\pi+1}) \). So \( y \in Z = \bigcup_{\beta \leq \pi} I_{\mathcal{H}}(Z^\beta) \). We obtain \( y \in I_{\mathcal{H}}(Z^\gamma) \) for some \( \gamma \leq \pi \). It follows \( U \cap Z^{\gamma} = \{ y \} \) for some \( U \in n_{\gamma}(y) \). By Claim 4, \( Y^{\gamma} \cap Z = Z^{\gamma} \). Then \( (U \cap Z) \cap Y^{\gamma} = U \cap Z^{\gamma} = \{ y \} \). Since \( U \cap Z \in n_{\gamma}(y) \), we have \( y \in I_{\mathcal{H}}(Y^{\gamma}) = Y^{\gamma} - Y^{\gamma+1} \). Since \( \pi < \alpha \) and \( \alpha \) is a limit ordinal, \( \pi + 1 < \alpha \). Then \( \gamma + 1 \leq \pi + 1 < \alpha \). By Remark 3.12, \( Y^{\gamma+1} \supset Y^{\gamma} \). Then \( y \notin Y^{\gamma} \). This is a contradiction. Therefore, \( Y^{\gamma} \subseteq f(X^{\alpha}) \).

\[ \square \]

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