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Multiplicity of Positive Solutions for Elliptic Problem in Fractional Orlicz-Sobolev Spaces with Discontinuous Nonlinearities

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ABSTRACT: It is established existence and multiplicity of positive solutions for a non-local elliptic problem driven by $(-\Delta)^s_{a(\cdot)}$ operator, with Dirichlet-type boundary conditions. One of these solutions is obtained as a critical point to the energy function associated with the studied elliptic problem by using the well-known mountain pass theorem. The nonlinearities is not satisfied Ambrosetti-Rabinowitz condition, monotonocity or convexity conditions, and can be discontinuous in nature.

Key Words: Fractional Orlicz-Sobolev spaces, elliptic problem, variational approach, discontinuous nonlinearity, mountain pass theorem.

Contents

1. Introduction.

The paper considers the existence and multiplicity of solutions for the following problem

$$\begin{cases} (-\Delta)^{\mathrm{s}}_{\mathrm{a}(\cdot)}\mathrm{u} = \mathrm{H}(\mathrm{u}-\alpha)\mathrm{f}(\mathrm{x},\mathrm{u}) & \text{in } \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \mathbb{R}^{\mathrm{N}} \backslash \mathbb{U}, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $\mathbb{U} \subset \mathbb{R}\,(N \geq 2)$ refers to an open set with Lipschitz boundary $\partial\mathbb{U}$, and H is the Heaviside function. The operator $(-\Delta)^s_{a(\cdot)}$ is named fractional $a(\cdot)$-Laplacian (see [6]), which is given by

$$(-\Delta)^{\mathrm{s}}_{\mathrm{a}(\cdot)}\mathrm{u}(\mathrm{x}) = \mathrm{p.v.} \int_{\mathbb{R}^{\mathrm{N}}} \mathrm{a}\big(|\mathrm{D}^{\mathrm{s}}\mathrm{u}|\big)\mathrm{D}^{\mathrm{s}}\mathrm{u}\frac{\mathrm{dy}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{N}}}, \tag{1.2}$$

where $\mathrm{D}^{\mathrm{s}}\mathrm{u} = \dfrac{\mathrm{u}(\mathrm{x})-\mathrm{u}(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{\mathrm{s}}}$, $\mathrm{d}\phi = \dfrac{\mathrm{dxdy}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{s}}}$ and $\mathrm{a}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous function satisfying:
(a_1) $\tau\mathrm{a}(\tau) \to \infty$ as $\tau \to \infty$ and $\tau\mathrm{a}(\tau) \to 0$ as $\tau \to 0$,
(a_2) $(\tau\mathrm{a}(\tau))' > 0$.
We extend $\tau \mapsto \tau\mathrm{a}(\tau)$ to $\mathbb{R}$ as an odd non-decreasing homeomorphism function φ from $\mathbb{R}$ onto itself with,

$$\varphi(\tau) = \begin{cases} 0 & \text{for for all} \tau = 0, \\ \mathrm{a}(|\tau|)\tau & \text{for } \tau \neq 0. \end{cases} \tag{1.3}$$

The function A is given by

$$\mathrm{A}(\tau) = \int_0^\tau \varphi(\mathrm{r})\,\mathrm{dr}, \quad \tau \geq 0. \tag{1.4}$$

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For more detail on the Orlicz and Orlicz-Sobolev setting, we refer the reader to [19,20]. At the same time, The authors in [6] defined the fractional Orlicz–Sobolev space $\mathbb{F}^s$. They believe that is the natural fractional version of the other fractional space which has been widely discussed since the 50s. Also, they showed that $\mathbb{F}^s$ is a generalization of the fractional Sobolev space $\mathbb{W}^{s,p}$, $p \in (1,\infty)$, which has been the fundamental tool to treat many types of problems as image processing [16], for some physical models [12,22]. However, it is natural to ask what results can be recovered when the fractional diffusion $(-\Delta)^s$ is replaced by $(-\Delta)^s_{\mathrm{a}(\cdot)}$. As far as we know, several properties and results of $\mathbb{W}^{s,p}$ have been extended to $\mathbb{F}^s$. It is impossible to cover every aspect of the subject, so we will only present a few instances for those who are interested [5,6,9,13,14,15,17].
Recently, many results about the existence and multiplicity of boundary value problems with discontinuous nonlinearities have been obtained, e.g., [2,3] and reference therein. More precisely, we refer to Ambrosetti et al. [2] for a comprehensive introduction to the study of problems with discontinuous nonlinearities, they considered the following problem:

$$\begin{cases} -\Delta \mathrm{u} = \mathrm{g}(\mathrm{x}) + \mathrm{f}(\mathrm{u}) & \text{in } \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \partial\mathbb{U}, \end{cases} \tag{1.5}$$

where g is a given function and f is a discontinuous nonlinearity. In the meantime, Ambrosetti et al. in [3] studied the following elliptic problem

$$\begin{cases} -\Delta \mathrm{u} = \mathrm{H}(\mathrm{u}-\alpha)\mathrm{f}(\mathrm{u}) & \text{in } \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \partial\mathbb{U}, \end{cases} \tag{1.6}$$

where H refers to a Heaviside function, $\alpha > 0$, f is increasing in u. The nonlinear term in this case is discontinuous at $\mathrm{u} = \alpha$. which generates an unknown region in $\mathbb{U}$ called "free boundary"to be determined. The authors demonstrated the existence of two positive solutions under appropriate conditions and by applying the dual variational principle. The associated free boundaries are two hypersurfaces in $\mathbb{R}^{\mathrm{N}}$.
The issue regarding nonlocal operators has received a lot of attention recently. A special focus is placed on the fractional Laplacian, which fulfills the same function in the theory of nonlocal operators than the Laplacian does in the theory of local operators. Following Caffarelli and Silvestre's ground-breaking 2007 paper [8], which used the Dirichlet-Neumann map of a suitable degenerate elliptic operator, there has been a great deal of research done on partial differential equations. The authors introduced a new local realization of the fractional Laplacian $(-\Delta)^s$, for all $s \in (0,1)$. After then, a large portion of the papers focuses on the investigation of fractional operators. In his study of a recent one [23], Sabri considers the following problem

$$\begin{cases} (-\Delta)^{\mathrm{s}}\mathrm{u} = \mathrm{H}(\mathrm{u}-\alpha)\mathrm{f}(\mathrm{x},\mathrm{u}) & \text{in } \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \mathbb{R}^{\mathrm{N}}\backslash\mathbb{U}, \end{cases} \tag{1.7}$$

where $\alpha > 0$ and f is increasing in u and satisfy:
(F_1) there exists c_1, $c_2 > 0$ with $c_2 < \lambda_1$ such that

$$|\mathrm{f}(\mathrm{r})| \leq \mathrm{c}_1 + \mathrm{c}_2|\mathrm{r}|, \quad \text{for all } \mathrm{r} \in \mathbb{R}.$$

(F_2)

$$\frac{\mathrm{f}(\alpha)}{\alpha} > 2\lambda_1|\mathbb{U}|\frac{\int_{\mathbb{U}}\varphi_1^2 \mathrm{dx}}{\left(\int_{\mathbb{U}}\varphi_1 \mathrm{dx}\right)^2},$$

where φ_1 is an eigenfunction associated with the eigenvalue λ_1 corresponding to the eigenvalue problem

$$\begin{cases} (-\Delta)^{\mathrm{s}}\mathrm{u} = \lambda\mathrm{u} & \text{in } \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \mathbb{R}^{\mathrm{N}}\backslash\mathbb{U}. \end{cases}$$

It is remarkable that, (F_2) is a strong requirements have been set on the non-linear term f.

Inspired by the studies mentioned above, this paper aims to extend this class of problems in $\mathbb{F}^s$ and to study the existence and multiplicity of (weak) solutions for the problem (1.1) which is a generalization of the problem (1.7). To our surprise, there are no similar results about the existence and multiplicity of solutions for the elliptic problem in $\mathbb{F}^s$ with discontinuous nonlinearities. Hence, the main novelty of this study is that it takes into account an issue for the $a(\cdot)$-Laplacian operator with strongly nonlinear parts f that do not meet the Ambrosetti-Rabinowitz superlinearity requirement. Moreover, we do not impose any monotonicity or convexity restrictions on f. We will come across some additional difficulties, for example, the functional $\mathcal{E}_\mathrm{f}$ associated to the problem (1.1) is not C^1. Our proofs require more advanced computing skills. Our strategy consists in showing that the functional $\mathcal{E}_\mathrm{f}$ is locally Lipschitz (loc-lip) due to Clarke [11] and the Chang theory [10] for nondifferentiable functions. The present literature using variational techniques in combination with nonlocal analysis has generated a lot of attention. Additionally, the fact that nonlinearities can occur discontinuously in nature has also served as a driving force for this work.

This article is organized as follows: First, we give some initial basic results and definitions of Clarke generalized derivative and fractional Orlicz-Sobolev spaces, as well as some well-known lemma and assumptions. In Section 3, we state the main results and the related proof. In Section 4, we point out certain examples of functions A, φ and f for which the results of this article can be implemented.

2. Mathematical background and hypotheses.

Throughout this paper $\mathbf{N}$ is the set of all N-function, $s \in (0,1)$ and $\mathbb{X}$ is a reflexive real Banach space.

The usual norm (Luxemburg norm) on the Orlicz space $\mathbb{L}_A(\mathbb{U})$ is

$$\|\mathrm{u}\|_\mathrm{A} = \inf\Big\{\tau > 0 \,/ \int_\mathbb{U} \mathrm{A}\Big(\frac{\mathrm{u}}{\tau}\Big)\,\mathrm{dx} \le 1\Big\}.$$

Recall that,

$$\overline{\mathrm{A}}(\tau) := \sup_{r\ge 0}\{\tau r - \mathrm{A}(r)\}.$$

Also,

$$\overline{\mathrm{A}}\big(\varphi(\tau)\big) = \varphi(\tau) - \mathrm{A}(\tau) \le \mathrm{A}(2\tau). \tag{2.1}$$

In $\mathbb{L}_A(\mathbb{U})$ is well-know, the Young inequality

$$r\tau \le \overline{\mathrm{A}}(\tau) + \mathrm{A}(r) \ \text{ for all } r,\, \tau \ge 0. \tag{2.2}$$

Also, Hölder inequality holds

$$\int_\mathbb{U} |\mathrm{u}\,\mathrm{v}|\,\mathrm{dx} \le ||\mathrm{u}||_\mathrm{A}||\mathrm{v}||_{(\overline{\mathrm{A}})} \text{ for all } \mathrm{v} \in \mathbb{L}_{\overline{\mathrm{A}}}(\mathbb{U}) \text{ and } \mathrm{u} \in \mathbb{L}_\mathrm{A}(\mathbb{U}), \tag{2.3}$$

where $||.||_{(A)}$ is the Orlicz norm defined by

$$||\mathrm{u}||_{(\mathrm{A})} := \sup_{||\mathrm{v}||_{\overline{\mathrm{A}}}\le 1} \int_\mathbb{U} \mathrm{u(x)v(x)}\,\mathrm{dx}.$$

The notation $B \prec\prec \mathrm{A}$ means that, for every $\varepsilon > 0$,

$$\frac{\mathrm{B}(\varepsilon\tau)}{\mathrm{A}(\tau)} \to 0 \quad \text{as } \tau \to \infty.$$

Recall that $\mathrm{A}^* \in \mathbf{N}$ is defined by

$$(\mathrm{A}^*)^{-1}(\tau) = \int_0^\tau \frac{\mathrm{A}^{-1}(\mathrm{r})}{\mathrm{r}^{\frac{\mathrm{N+s}}{\mathrm{N}}}}\,\mathrm{dr} \qquad \text{for } \tau \ge 0,$$

where we mention that

$$(\mathrm{A}_0)\ \int_0^1 \frac{\mathrm{A}^{-1}(\mathrm{r})}{\mathrm{r}^{1+\frac{\mathrm{s}}{\mathrm{N}}}}\,\mathrm{dr} < \infty \quad \text{and} \quad (\mathrm{A}_\infty)\int_1^{+\infty} \frac{\mathrm{A}^{-1}(\mathrm{r})}{\mathrm{r}^{1+\frac{\mathrm{s}}{\mathrm{N}}}}\,\mathrm{dr} = +\infty.$$

Now, we introduce the fractional Orlicz-Sobolev spaces $\mathbb{F}^s$ (see [6]) by

$$\mathbb{F}^{\mathrm{s}} = \Big\{\mathrm{u} \in \mathbb{L}_{\mathrm{A}}(\mathbb{U}) : \mathcal{F}(\lambda \mathrm{u}) < \infty \text{ for some } \lambda > 0\Big\}$$

which is a Banach space with norm

$$||\mathrm{u}||_{\mathrm{s,A}} = [\mathrm{u}]_{\mathrm{s,A}} + ||\mathrm{u}||_{\mathrm{A}},$$

where $[\cdot]_{s,\mathrm{A}}$ is given by

$$[\mathrm{u}]_{\mathrm{s,A}} = \inf\Big\{\lambda > 0 : \mathcal{F}\Big(\frac{\mathrm{u}}{\lambda}\Big) \leq 1\Big\},$$

and $\mathcal{F} : \mathbb{F}^s \to \mathbb{R}$ is a function defined by

$$\mathcal{F}(\mathrm{u}) = \int_{\mathbb{U}\times\mathbb{U}} \mathrm{A}(|\mathrm{D}^{\mathrm{s}}\mathrm{u}|)\mathrm{d}\phi.$$

Now, let us state our knowledge framework

$$\mathbb{F}^{\mathrm{s}}_0 := \big\{\mathrm{u} \in \mathbb{F}^{\mathrm{s}}(\mathbb{R}^{\mathrm{N}}) : \mathrm{u} = 0 \text{ a.e } \mathbb{R}^{\mathrm{N}} \setminus \mathbb{U}\big\}.$$

with the norm $[\cdot]_{s,\mathrm{A}}$. Notice that, $(\mathbb{F}^s_0, [\cdot]_{s,\mathrm{A}})$ is a reflexive, separable Banach space, and we find in [6] that, if $\mathbb{U}$ is bounded then $[\cdot]_{s,\mathrm{A}}$ is a norm in $\mathbb{F}^s_0$ equivalent to $||\cdot||_{s,\mathrm{A}}$.
In $\mathbb{F}^s_0$ is well-know, the Poincaré inequality

$$||\upsilon||_{\mathrm{A}} \leq \tau[\upsilon]_{\mathrm{s,A}}, \quad \text{for all } \upsilon \in \mathbb{F}^{\mathrm{s}}_0, \tag{2.4}$$

where τ is a positive constant.

Lemma 2.1 *[5] Set $\xi_0(\tau) := \min\{\tau^{\mathrm{l}_\varphi}, \tau^{\mathrm{n}_\varphi}\}$ and $\xi_1(\tau) := \max\{\tau^{l_\varphi}, \tau^{n_\varphi}\}$. Then we have:*
1)

$$\mathcal{F}\left(\frac{\tau}{[\tau]_{\mathrm{s,A}}}\right) \leq 1, \quad \text{for all } \tau \in \mathbb{F}^{\mathrm{s}}_0 \setminus \{0\},$$

2)

$$\xi_0([\tau]_{\mathrm{s,A}}) \leq \mathcal{F}(\tau) \leq \xi_1([\tau]_{\mathrm{s,A}}), \quad \text{for all } \quad \tau \in \mathbb{F}^{\mathrm{s}}_0.$$

Let $\mathrm{u} \in \mathbb{X}$. If there exists $\mathrm{L} > 0$ depending on the neighborhood $\mathbb{W}$ of u, such that

$$|\mathrm{E}(\mathrm{u}) - \mathrm{E}(\mathrm{v})| \leq \mathrm{L}||\mathrm{u} - \mathrm{v}||_{\mathbb{X}}, \quad \text{for all } \mathrm{u}, \mathrm{v} \in \mathbb{W},$$

then we say that $\mathrm{E} : \mathbb{X} \to \mathbb{R}$ is loc-Lip.
The Clarke generalized derivative E^0 of E at $\mathrm{u} \in \mathbb{X}$ with respect to the direction $z \in \mathbb{X}$ is defined as

$$\mathrm{E}^0(\mathrm{u}, \mathrm{z}) := \limsup_{(\rho,\varrho)\to(\mathrm{u},0)} \frac{\mathrm{E}(\rho + \varrho \mathrm{z}) - \mathrm{E}(\rho)}{\varrho},$$

while the Clarke subdifferential of E at a point $\mathrm{u} \in \mathbb{X}$, denoted by $\partial_{\mathrm{C}}\mathrm{E}(\mathrm{u}) \subset \mathbb{X}^*$ is defined as

$$\partial_{\mathrm{C}}\mathrm{E}(\mathrm{u}) := \{\zeta \in \mathbb{X}^* : \mathrm{E}^0(\mathrm{u}, \mathrm{z}) \geq \langle \zeta, \mathrm{z}\rangle, \quad \text{a.e. } \mathrm{z} \in \mathbb{X}\}.$$

The loc-Lip functional E is called regular (at $\mathrm{u} \in \mathbb{X}$), if the usual directional derivative E' exists for any $z \in \mathbb{X}$ and we have

$$\mathrm{E}^0(\mathrm{u}, \mathrm{z}) = \mathrm{E}'(\mathrm{u}, \mathrm{z}) \quad \text{for all } z \in \mathbb{X}.$$

The following are some properties that are required to be true. (see [11])
- $\partial_C E(u) = \{E'(u)\}$, when $E \in C^1(\mathbb{X}, \mathbb{R})$.
- $\partial_C(\alpha E)(u) = \alpha\partial_C E(u)$, for all $\alpha \in \mathbb{R}$.
- $\partial_C(E_1 + E_2)(u) \subseteq \partial_C E_1(u) + \partial_C E_2(u)$.
- If $0 \in \partial_C E(c)$, then $c \in \mathbb{X}$ is a critical point of E.
- The function $\sigma_E(\upsilon) : \mathbb{X} \to \mathbb{R}$ defined by $\sigma_E(u) := \min_{\zeta\in\partial_C E(u)} ||\zeta||_{\mathbb{X}_0^*}$ is lower semicontinuous.

From the previous definition and Clarke's subdifferential definition, a point u in $\mathbb{X}$ is a critical point of E_f if and only if $\sigma_{E_f}(u) = 0$, or equivalently,

$$E^0(u, z) \geq 0, \quad \text{for all}\, z \in \mathbb{X}.$$

Notice that, for a loc-Lip functional $E : \mathbb{X} \to \mathbb{R}$, a sequence $\{u_n\} \subset \mathbb{X}$ is called (PS)-sequence if

$$\sigma_E(u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

In this paper, we assume that

$$(a_3) \qquad 1 < l_\varphi := \inf_{r>0} \frac{r\varphi(r)}{A(r)} \leq m_\varphi := \sup_{r>0} \frac{r\varphi(r)}{A(r)} < +\infty \quad \text{for all } r > 0.$$

(f_1): there exists $c_1, c_2 > 0$ with $c_2 < \lambda_1$ (λ_1 is the constant given in [4] Lemma 2.3) such that

$$|f(x, r)| \leq c_1 + c_2\varphi(|r|))\, \text{for all}\, r \in \mathbb{R}.$$

(f_2)

$$\lim_{r\to\infty} \frac{f(x, r)}{\varphi(|r|)} = \infty.$$

Remark 2.1 The condition (a_3) implies that $A \in \Delta_2$, that is for a certain constant $k > 0$,

$$A(2r) \leq k\, A(r), \quad \text{for all } r > 0.$$

The embedding below will be used in this paper:

$$\mathbb{F}_0^s \overset{cpt}{\hookrightarrow} \mathbb{L}_B(\mathbb{U}), \quad \text{if} \quad B \prec\prec A^*.$$

In particular, by Lemma 2.7 in [13], we have $A \prec\prec A^*$. Then

$$\mathbb{F}_0^s \overset{cpt}{\hookrightarrow} \mathbb{L}_A(\mathbb{U}), \tag{2.5}$$

3. Mountain pass geometry and Main results.

Our main result is proved by using the following theorem.

Theorem 3.1 *([21]) Let $u_1, u_2 \in \mathbb{X}$. Let $E : \mathbb{X} \to \mathbb{R}$ be a loc-Lip functional satisfying the (PS)-condition and $r \in (0, ||u_2 - u_1||_{\mathbb{X}})$ be such that*

$$\inf_{u\in\partial_C B(u_1,r)} E(u) \geq \max\{E(u_1), E(u_2)\}.$$

Then, E has a critical point $\hat{u} \in \mathbb{X}\backslash\{u_1, u_2\}$ such that

$$E(\hat{u}) = \inf_{\gamma\in\Gamma} \max_{\varrho\in[0,1]} I(\gamma(\varrho)) \quad \text{and} \quad E(\hat{u}) \geq \max\{E(u_1), E(u_2)\},$$

where Γ is given by $\Gamma := \{\gamma \in C([0, 1], \mathbb{X}) : \gamma(0) = u_1, \gamma(1) = u_2\}$.

Theorem 3.2 *([10]) Let E be a loc-Lip functional on $\mathbb{X}$ which is bounded from below. If E is satisfies (PS)-condition at the level $c = \inf_{\mathbb{X}} E \in \mathbb{R}$, then $0 \in \partial_C E(c)$.*

Recall that under (a_1)-(a_3), $(-\Delta)^s_{a(\cdot)}$ given in (1.2) is well defined between $\mathbb{F}_0^s$ and $(\mathbb{F}_0^s)^*$ and the following formula is provided (see [6], Theorem 6.12)

$$\langle \mathcal{F}'(\upsilon), z\rangle = \int_{\mathbb{U}\times\mathbb{U}} a\big(|D^s u|\big) D^s u D^s z \, d\phi = \langle(-\Delta)^s_{a(.)} u, z\rangle, \tag{3.1}$$

for all $u, z \in \mathbb{F}_0^s$.

Auxiliary problem.

Let us set $\psi(\mathrm{u}) := \mathrm{H}(\mathrm{u}-\alpha)\mathrm{f}(\mathrm{u})$. Our primary finding relates to the existence of potential solutions to the problem

$$\begin{cases} (-\Delta)^{\mathrm{s}}_{\mathrm{a}(\cdot)}\mathrm{u} = \psi(\mathrm{u}) & \text{in} \quad \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \mathbb{R}^{\mathrm{N}} \backslash \mathbb{U}. \end{cases} \tag{3.2}$$

We have the following theorem:

Theorem 3.3 *If* (a_1)*-*(a_3)*,* (f_1) *and* (f_2) *hold true, then the problem*

$$\begin{cases} (-\Delta)^{\mathrm{s}}_{\mathrm{a}(\cdot)}\mathrm{u} \in \beta(\mathrm{u}) & \text{in} \quad \mathbb{U}, \\ \mathrm{u} = 0 & \text{on } \mathbb{R}^{\mathrm{N}} \backslash \mathbb{U}, \end{cases} \tag{3.3}$$

has two different non-zero solutions in $\mathbb{F}_0$ *where* β *is given by*

$$\beta(\mathrm{r}) = \begin{cases} \psi(\tau) & \text{if} \quad \alpha \neq \tau, \\ [0, \mathrm{f}(\alpha)] & \text{if} \quad \alpha = \tau. \end{cases}$$

Remark 3.1 As indicated in the introduction, if the free boundary

$$\Lambda_\alpha = \{x \in \mathbb{U} : u(x) = \alpha\}$$

has zero measure, then the solutions of (3.3) are also solutions of the problem (3.2).

Definition 3.1 We say that $u \in \mathbb{F}_0$ is the weak solution for the problem (3.3) if

$$\int_{\mathbb{U}^2} \mathrm{a}\big(|\mathrm{D}^{\mathrm{s}}\mathrm{u}|\big)\mathrm{D}^{\mathrm{s}}\mathrm{u}\mathrm{D}^{\mathrm{s}}\mathrm{v}d\phi = \int_{\mathbb{U}} \mathrm{w}\mathrm{v}\mathrm{d}\mathrm{x},$$

where $\mathrm{w} \in \beta(\mathrm{u})$.

Let us consider the energy functional $\mathcal{E}_f : \mathbb{F}_0 \to \mathbb{R}$

$$\mathcal{E}_{\mathrm{f}}(\mathrm{u}) := \mathcal{F}(\mathrm{u}) - \mathcal{J}(\mathrm{u}) = \int_{\mathbb{R}^{2\mathrm{n}}} \mathrm{A}\big(|\mathrm{D}^{\mathrm{s}}\mathrm{u}|\big)\, \mathrm{d}\phi - \int_{\mathbb{R}^{\mathrm{n}}} \Psi(\mathrm{u}(\mathrm{x}))\mathrm{d}\mathrm{x}, \quad \mathrm{u} \in \mathbb{F}_0, \tag{3.4}$$

where $\mathcal{J}(\mathrm{u}) = \displaystyle\int_{\mathbb{R}^{\mathrm{n}}} \Psi(\mathrm{u})\,\mathrm{d}\mathrm{x}$ and $\Psi(\mathrm{u}) = \displaystyle\int_0^{\mathrm{u}} \psi(\mathrm{r})\,\mathrm{d}\mathrm{r}$.

Lemma 3.1 *Let* $(\mathrm{u}_{\mathrm{n}}) \in \mathbb{F}_0$*. Assume that* (a_1)*-*(a_3)*,* (f_1)*,* (f_2)*,* $\mathcal{E}_{\mathrm{f}}(\mathrm{u}_{\mathrm{n}})$ *is bounded and* $\sigma_{\mathcal{E}_{\mathrm{f}}}(\mathrm{u}_{\mathrm{n}}) \to 0$ *as* $n \to \infty$*. Then, there exists* $\mathrm{u} \in \mathbb{F}_0$ *such that, up to a subsequence,* $\mathrm{u}_{\mathrm{n}} \to \mathrm{u}$ *in* $\mathbb{F}_0$*.*

Proof: In view of (f_2), we get, for each $\mathrm{K} > 0$, a constant $\mathrm{C}_{\mathrm{K}} > 0$ exists such that

$$\mathrm{f}(\mathrm{x}, \tau) \geq \mathrm{K}\varphi(|\tau|) - \mathrm{C}_{\mathrm{K}}, \quad \text{for all} \quad \mathrm{x} \in \mathbb{U},\ \tau > 0. \tag{3.5}$$

Hence,

$$\begin{aligned} \Psi(\mathrm{t}) = \int_0^{\mathrm{t}} \psi(\tau)\mathrm{d}\tau \quad &= \int_0^{\mathrm{t}} \mathrm{H}(\tau-\alpha)\mathrm{f}(\tau)\mathrm{d}\tau \\ &= \int_0^{\alpha} \mathrm{H}(\tau-\alpha)\mathrm{f}(\tau)\mathrm{d}\tau + \int_\alpha^{\mathrm{t}} \mathrm{H}(\tau-\alpha)\mathrm{f}(\tau)\mathrm{d}\tau \\ &\geq \int_\alpha^{\mathrm{t}} \mathrm{H}(\tau-\alpha)\mathrm{f}(\tau)\mathrm{d}\tau. \end{aligned}$$

Using (3.5) and (a_2), we infer

$$\Psi(\mathrm{t}) \geq (\mathrm{K}\varphi(\alpha) - \mathrm{C_K}) \int_\alpha^{\mathrm{t}} d\tau = (\mathrm{K}\varphi(\alpha) - \mathrm{C_K})(\mathrm{t} - \alpha). \tag{3.6}$$

Claim 1: The sequence $(\mathrm{u_n})$ is bounded in $\mathbb{F}_0$.
Let $\mathrm{w_n} \in \partial_{\mathrm{C}} \mathcal{J}(\mathrm{u_n}) \subset [0, \mathrm{f}(\alpha)]$ such that

$$\sigma_{\mathcal{E}_{\mathrm{f}}}(\mathrm{u_n}) \to 0, \quad \text{as} \quad \mathrm{n} \to \infty. \tag{3.7}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\Big| \langle \mathcal{F}'(\mathrm{u_n}), \zeta \rangle - \int_{\mathbb{U}} \mathrm{w_n} \zeta dx \Big| \leq \frac{1}{\mathrm{n}}, \text{ for all } \quad \mathrm{n} \geq \mathrm{n}_0, \ \zeta \in \mathbb{F}_0.$$

In particular $(\zeta = \mathrm{u_n})$, we have

$$\Big| - \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle + \int_{\mathbb{U}} \mathrm{w_n u_n} dx \Big| \leq \frac{1}{\mathrm{n}}, \text{ for all } \quad \mathrm{n} \geq \mathrm{n}_0.$$

Hence,

$$-\frac{1}{2\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle + \frac{1}{2\mathrm{m}_\varphi} \int_{\mathbb{U}} \mathrm{w_n u_n} dx \leq \frac{2}{\mathrm{m}_\varphi \mathrm{n}}, \text{ for all } \quad \mathrm{n} \geq \mathrm{n}_0. \tag{3.8}$$

Now, due the fact that $\mathcal{E}_{\mathrm{f}}(\mathrm{u_n})$ is bounded, we have a constant $\mathrm{L} > 0$ such that

$$\Big| \mathcal{F}(\mathrm{u_n}) - \int_{\mathbb{U}} \Psi(\mathrm{u_n}) dx \Big| \leq \mathrm{L}.$$

Hence,

$$\mathcal{F}(\mathrm{u_n}) - \int_{\mathbb{U}} \Psi(\mathrm{u_n}) dx \leq \mathrm{L}.$$

Using (a_3),

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle - \int_{\mathbb{U}} \Psi(\mathrm{u_n}) dx \leq \mathrm{L}. \tag{3.9}$$

Adding (3.8) and (3.9) to obtain

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle - \int_{\mathbb{U}} \Psi(\mathrm{u_n}) dx + \frac{1}{2\mathrm{m}_\varphi} \int_{\mathbb{U}} \mathrm{w_n u_n} dx \leq \frac{2}{\mathrm{m}_\varphi \mathrm{n}} + \mathrm{L}.$$

Using (3.6) in the last inequality, we have

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle - \int_{\mathbb{U}} \Big((\mathrm{K}\varphi(\alpha) - \mathrm{C_K})(\mathrm{u_n} - \alpha) \Big) dx + \frac{1}{2\mathrm{m}_\varphi} \int_{\mathbb{U}} \mathrm{w_n u_n} dx \leq \frac{2}{\mathrm{m}_\varphi \mathrm{n}} + \mathrm{L},$$

take $\mathrm{K} = \frac{\mathrm{f}(\alpha) + \mathrm{C_K}}{\varphi(\alpha)} > 0$, then

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle \leq \int_{\mathbb{U}} \Big(\mathrm{f}(\alpha) - \frac{1}{2\mathrm{m}_\varphi} \mathrm{w_n} \Big) \mathrm{u_n} dx + \mathrm{C}_1 + \frac{2}{\mathrm{m}_\varphi \mathrm{n}}.$$

Since, $\mathrm{m}_\varphi > 1$, and $\mathrm{w_n} \in [0, \mathrm{f}(\alpha)]$, then

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle \leq \mathrm{C}_2 \int_{\mathbb{U}} \mathrm{u_n} dx + \mathrm{C}_1 + \frac{2}{\mathrm{m}_\varphi \mathrm{n}}.$$

Using Hölder inequality and Poincarré inequality, we deduce

$$\frac{1}{\mathrm{m}_\varphi} \langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} \rangle \leq \mathrm{C}_3 [\mathrm{u_n}]_{\mathrm{s,A}} + \mathrm{C}_1 + \frac{2}{\mathrm{m}_\varphi \mathrm{n}}.$$

Again, by (a_3) and Lemma (2.1), we show that

$$\frac{1}{\mathrm{m}_\varphi}[\mathrm{u_n}]_{\mathrm{s,A}}^{\mathrm{l}_\varphi} \leq \mathrm{C}_3[\mathrm{u_n}]_{\mathrm{s,A}} + \mathrm{C}_1 + \frac{2}{\mathrm{m}_\varphi \mathrm{n}},$$

where C_1, C_2 and C_3 are a given constant. Therefore $\mathrm{u_n}$ is bounded in $\mathbb{F}_0$.
Claim 2: $\mathrm{f}(\mathrm{u_n})$ is bounded in $\mathrm{L}_{\overline{\mathrm{A}}}(\mathbb{U})$.
In view of (f_2), (2.1) and the fact that $\mathrm{A} \in (\Delta_2)$, we get

$$\int_{\mathbb{U}} \overline{\mathrm{A}}\Big(|\mathrm{f}(\mathrm{x}, \mathrm{u_n})|\Big)\mathrm{dx} \leq \int_{\mathbb{U}} \overline{\mathrm{A}}\Big(\mathrm{c}_1 + \mathrm{c}_2\varphi(|\mathrm{u_n}|)\Big)\mathrm{dx} \leq \mathrm{c}_4 + \mathrm{c}_5 \int_{\mathbb{U}} \mathrm{A}\Big(2|\mathrm{u_n}|\Big)\mathrm{dx}.$$

Hence, according to $\mathrm{A} \in (\Delta_2)$ and $\mathrm{u_n}$ bounded in $\mathrm{L_A}(\mathbb{U})$ we prove the Claim 2.
Claim 3: $\mathrm{u_n}$ converge strongly to u in $\mathbb{F}_0$.
Now, due to $\mathbb{F}_0$ is a reflexive space and Claim 1, we have

$$\mathrm{u_n} \to \mathrm{u} \quad \text{weakly in} \quad \mathbb{F}_0,$$

and the fact that, $\mathbb{F}_0^s \overset{\mathrm{cpt}}{\hookrightarrow} \mathbb{L}_A(\mathbb{U})$, up to a subsequence, we have

$$\mathrm{u_n} \to \mathrm{u} \quad \text{strongly in} \quad \mathrm{L_A}(\mathbb{U}). \tag{3.10}$$

Also, observing that

$$\partial_{\mathrm{C}}\mathcal{E}_{\mathrm{f}}(\mathrm{u_n}) = \partial_{\mathrm{C}}\mathcal{F}(\mathrm{u_n}) - \partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}),$$

where $\partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}) \subset [0, \mathrm{f}(\alpha)]$. Since, $\sigma_{\mathcal{E}_\mathrm{f}}(\mathrm{u_n}) \to 0$ as $n \to \infty$, there exists $\{\xi_n\} \subset (\mathbb{F}_0)^*$ such that

$$\xi_\mathrm{n} \in \partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}).$$

We easily see that

$$\begin{aligned} \Big|\langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big| &= \Big|\langle \partial_{\mathrm{C}}\mathcal{E}_{\mathrm{f}}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle + \langle \partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big| \\ &\leq \Big|\langle \partial_{\mathrm{C}}\mathcal{E}_{\mathrm{f}}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big| + \Big|\langle \partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big|. \end{aligned}$$

According to (3.7), we have

$$\Big|\langle \partial_{\mathrm{C}}\mathcal{E}_{\mathrm{f}}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big| \leq ||\zeta_\mathrm{n}||_{\mathbb{F}_0^*}||\mathrm{u_n} - \mathrm{u}||_{\mathrm{s,A}} \to 0, \quad \text{as} \quad \mathrm{n} \to \infty. \tag{3.11}$$

Also, by Hölder inequality, (3.10) and the fact that ξ_n is bounded in $\mathrm{L}_{\overline{\mathrm{A}}}(\mathbb{U})$, we have

$$\Big|\langle \partial_{\mathrm{C}}\mathcal{J}(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle\Big| = \Big|\int_{\mathbb{U}} \xi_\mathrm{n}(\mathrm{u_n} - \mathrm{u})\mathrm{dx}\Big| \leq ||\xi_\mathrm{n}||_{\overline{\mathrm{A}}}||\mathrm{u_n} - \mathrm{u}||_{\mathrm{A}} \to 0, \quad \text{as} \quad \mathrm{n} \to \infty. \tag{3.12}$$

Using (3.11) and (3.12), we get to

$$\limsup_{\mathrm{n}\to\infty}\langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle \leq 0.$$

The following Lemma finishes the proof of Lemma (3.1)

Lemma 3.2 *[7] If $\mathrm{u_n} \rightharpoonup \mathrm{u}$ in $\mathbb{F}_0$ and $\limsup\langle \mathcal{F}'(\mathrm{u_n}), \mathrm{u_n} - \mathrm{u}\rangle \leq 0$. Then $\mathrm{u_n} \to \mathrm{u}$ in $\mathbb{F}_0$.*

□

Proposition 3.1 *Under (a_1)-(a_3) and (f_1), there exists a global minimum $\mathrm{u} \neq 0$.*

Proof: For $u \in \mathbb{F}_0$. By (3.4) and Lemma (2.1), we have

$$\mathcal{E}_f(u) = \mathcal{F}(u) - \mathcal{J}(u) \geq \xi_0([u]_{s,A}) - \int_{\mathbb{U}} \Psi(u)dx. \tag{3.13}$$

In other hand, by (f_1), we have

$$\begin{aligned}
\Psi(t) = \int_0^t \psi(\tau)d\tau \quad &= \int_0^t H(\tau-\alpha)f(\tau)d\tau \\
&= \int_0^\alpha H(\tau-\alpha)f(\tau)d\tau + \int_\alpha^t H(\tau-\alpha)f(\tau)d\tau \\
&\leq \int_\alpha^t H(\tau-\alpha)f(\tau)d\tau \\
&\leq \int_\alpha^t (c_1 + c_2\varphi(\tau))d\tau \\
&\leq c_1 \int_\alpha^t d\tau + c_2 \int_\alpha^t \varphi(\tau)d\tau \\
&\leq c_1(t-\alpha) + c_2 \int_0^t \varphi(\tau)d\tau \\
&\leq c_1 t + c_1 A(t).
\end{aligned} \tag{3.14}$$

Combining (3.14) in (3.13), we infer

$$\begin{aligned}
\mathcal{E}_f(u) &\geq \mathcal{F}(u) - \int_{\mathbb{U}} \Psi(u)dx \\
&\geq \mathcal{F}(u) - \int_{\mathbb{U}} (c_1 u + c_2 A(u))dx \\
&\geq \mathcal{F}(u) - c_1 \int_{\mathbb{U}} u - c_2 \int_{\mathbb{U}} A(u)dx,
\end{aligned}$$

using Lemma 2.3 in [4], Hölder inequality and Poincaré inequality, we get

$$\begin{aligned}
\mathcal{E}_f(u) &\geq \mathcal{F}(u) - \tilde{c}_1||u||_A - \frac{c_2}{\lambda_1}\mathcal{F}(u) \\
&\geq \Big(1 - \frac{c_2}{\lambda_1}\Big)\mathcal{F}(u) - \tilde{c}_2[u]_{s,A},
\end{aligned}$$

using Lemma (2.1), we obtain

$$\mathcal{E}_f(u) \geq \Big(1 - \frac{c_2}{\lambda_1}\Big)\xi_0([u]_{s,A}) - \tilde{c}_2[u]_{s,A}.$$

Since, $c_2 < \lambda_1$, the functional $\mathcal{E}_f$ is coercive and bounded from below. Lemma 3.1 allows us to guarantee that $\mathcal{E}_f$ satisfies the (PS)-condition. Thus, we get to the conclusion that $\mathcal{E}_f$ will have a minimum in $\mathbb{F}_0$ using the Theorem 3.2. $\square$

Using the mountain pass theorem (Theorem 3.1), the second critical point can be established. We have

Proposition 3.2 *Assume (f_1) and (f_2), there exists a mountain pass critical point $v \neq 0$.*

In order to demonstrate this claim, we must confirm that the related functional $\mathcal{E}_f$ possesses the geometrical properties needed by the m.p.t. We have the following lemmas:

Lemma 3.3 *Let $s \in (0,1)$ and f satisfy the assumptions (f_1) and (f_2). Then, there exist $\tau > 0$ and $\varrho > 0$ such that for any $u \in \mathbb{F}_0$ with $||u||_{s,A} = \varrho$, we have $\mathcal{E}_f(u) \geq \tau$.*

Proof: Since,

$$\mathcal{E}_{\rm f}(\rm u) \geq \Big(1 - \frac{c_2}{\lambda_1}\Big)\xi_0([\rm u]_{s,A}) - \tilde{c}_2[\rm u]_{s,A}.$$

Thus,

$$\mathcal{E}_{\rm f}(\rm u) \geq \xi_0([\rm u]_{s,A})\Big(c_3 - \tilde{c}_2[\rm u]_{s,A}^{1-\theta}\Big).$$

where $c_3 := 1 - \frac{c_2}{\lambda_1}$ and $\theta := \begin{cases} l_\varphi & \text{if} \quad [\rm u]_{s,A} \geq 1, \\ m_\varphi & \text{if} \quad [\rm u]_{s,A} < 1. \end{cases}$

Let $\rm u \in \mathbb{F}_0$ with $||\rm u||_{s,A} = \varrho$. We see that there exists ϱ that verifies $\varrho > \Big(\tilde{c}_2/c_3\Big)^{\frac{1}{\theta-1}}$ and so there exists $\tau > 0$ such that

$$\mathcal{E}_{\rm f}(\rm u) \geq \xi_0(\varrho)(c_3 - \tilde{c}_2\varrho^{1-\theta}) := \tau.$$

□

Lemma 3.4 *If (f_2) holds true, then there exists $t \in \mathbb{R}$ such that $\mathcal{E}_{\rm f}(\rm tw_0) < 0$ for each $\rm w_0 \in \mathbb{F}_0\backslash\{0\}$.*

Proof: Recall that, the assumption (f_2) implies (3.6) for each $\rm K > 0$. Now, taking $\rm w_0 \in \mathbb{F}_0\backslash\{0\}$, by the estimation (3.6), we have

$$\begin{aligned} \mathcal{E}_{\rm f}(\rm tw_0) &= \mathcal{F}(\rm tw_0) - \mathcal{J}(\rm tw_0) = \mathcal{F}(\rm tw_0) - \int_{\mathbb{U}} \Psi(\rm tw_0)dx \\ &\leq \mathcal{F}(\rm tw_0) - (K\varphi(\alpha) - C_K)\int_{\mathbb{U}}(\rm tw_0 - \alpha)dx \\ &= \xi_1(|t|)\mathcal{F}(\rm w_0) - (K\varphi(\alpha) - C_K)|t|\int_{\mathbb{U}} w_0 dx + (K\varphi(\alpha) - C_K)\alpha|\mathbb{U}|. \end{aligned} \tag{3.15}$$

Take $\rm K = \frac{C_K}{2\varphi(\alpha)}$, then $\mathcal{E}_{\rm f}(\rm tw_0) < 0$ for $|t| \to 0$. □

The Ambrosetti-Rabinowitz mountain pass theorem, in Chang's version, serves as the foundation for the demonstration (see Theorem 3.1). Due to Lemmas 3.1, 3.3 and Lemma 3.4, we establish the existence of a further critical point $v \neq 0$ of the functional $\mathcal{E}_{\rm f}$.
Conclusion of the proof of Theorem 3.3: Combining Propositions 3.1 and 3.2 results in the proving of our Theorem 3.3.

4. Example.

In this section, we show some examples of functions φ, A and f for which the results of this paper can be applied.

For instance we can take the function $\varphi(\tau) := l_\varphi|\tau|^{l_\varphi - 1}\log(1 + |\tau|) + \frac{\tau^{l_\varphi}}{1+\tau}$, with $1 < p < \infty$. Then, $l_\varphi = p$, $n_\varphi = p + 1$ and

$$A(\tau) = |\tau|^{l_\varphi}\log(1 + |\tau|).$$

Hence, the conditions (a_3) is satisfied.
Now, let us check conditions (A_0) and (A_∞). Due to L'Hôpital's rule, we obtain

$$\begin{aligned} \lim_{\tau\to 0}\frac{A(\tau)}{\tau^{p+1}} &= \lim_{\tau\to 0}\frac{|\tau|^p\log(1 + |\tau|)}{\tau^{p+1}} \\ &= \lim_{\tau\to 0}\frac{\log(1+\tau)}{\tau} \\ &= \lim_{\tau\to 0}\frac{1}{\tau+1} \\ &= 1 \end{aligned}$$

We infer that $\mathrm{A} \sim \tau^{\mathrm{p}+1}$ near zero. Also, from remarks in [1], p.248, we deduce the condition (A_∞) holds true if and only if

$$\int_0^1 \frac{\mathrm{z}^{\frac{1}{\mathrm{p}+1}}}{\mathrm{z}^{1+\frac{\mathrm{s}}{\mathrm{N}}}}\,\mathrm{dz} < \infty, \qquad \text{or} \qquad \mathrm{s}(\mathrm{p}+1) < \mathrm{N}.$$

Moreover, by the change of variable $\mathrm{z} = \mathrm{A}(\tau)$, we have

$$\int_1^\tau \frac{\mathrm{A}^{-1}(\mathrm{z})}{\mathrm{z}^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{dz} = \int_{\mathrm{A}^{-1}(1)}^{\mathrm{A}^{-1}(\tau)} \frac{\tau\varphi(\tau)}{\mathrm{A}(\tau)}(\mathrm{A}(\tau))^{-\mathrm{s}/\mathrm{N}}\mathrm{d}\tau. \tag{4.1}$$

A simple calculation yields

$$\lim_{\tau\to\infty}\frac{\tau\varphi(\tau)}{\mathrm{A}(\tau)} = \lim_{\tau\to\infty}\left(\frac{\mathrm{p}\tau^{\mathrm{p}}\log(1+\tau)}{\tau^{\mathrm{p}}\log(1+\tau)} + \frac{\frac{\tau^{\mathrm{p}}}{1+\tau}}{\tau^{\mathrm{p}}\log(1+\tau)}\right) = \lim_{\tau\to\infty}\left(\mathrm{p} + \frac{1}{(1+\tau)\log(1+\tau)}\right) = \mathrm{p}. \tag{4.2}$$

and

$$\lim_{\tau\to\infty}\mathrm{A}(\tau) = \lim_{\tau\to\infty}\tau^{\mathrm{p}}\log(1+\tau) = \infty. \tag{4.3}$$

Relations (4.1), (4.2) and (4.3) yield

$$\int_1^\infty \frac{\mathrm{A}^{-1}(\mathrm{z})}{\mathrm{z}^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{dz} = \infty.$$

Equivalently, we can write

$$\infty = \int_1^\infty \frac{\mathrm{A}^{-1}(\mathrm{z})}{\mathrm{z}^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{dz} = \lim_{\tau\to\infty}\int_{\mathrm{A}^{-1}(1)}^{\mathrm{A}^{-1}(\tau)} \frac{\tau\varphi(\tau)}{\mathrm{A}(\tau)^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{d}\tau$$

Using (a_3) we infer that

$$\begin{aligned}\lim_{\tau\to\infty}\int_{\mathrm{A}^{-1}(1)}^{\mathrm{A}^{-1}(\tau)} \frac{\tau\varphi(\tau)}{\mathrm{A}(\tau)^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{d}\tau &\le \lim_{\tau\to\infty}\frac{1}{\mathrm{n}_\varphi}\int_{\mathrm{A}^{-1}(1)}^{\mathrm{A}^{-1}(\tau)} \frac{\mathrm{A}(\tau)}{\mathrm{A}(\tau)^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{d}\tau \\ &= \lim_{\tau\to\infty}\frac{1}{\mathrm{n}_\varphi}\int_{\mathrm{A}^{-1}(1)}^{\mathrm{A}^{-1}(\tau)} \frac{\mathrm{d}\tau}{\mathrm{A}(\tau)^{\frac{\mathrm{s}}{\mathrm{N}}}}.\end{aligned}$$

Hence,

$$\int_{\mathrm{A}^{-1}(1)}^\infty \frac{\mathrm{d}\tau}{\mathrm{A}(\tau)^{\frac{\mathrm{s}}{\mathrm{N}}}} = \infty.$$

Equivalently,

$$\int_{\mathrm{A}^{-1}(1)}^\infty \frac{\mathrm{d}\tau}{\log(1+\tau)^{\frac{\mathrm{s}}{\mathrm{N}}}\tau^{\frac{\mathrm{ps}}{\mathrm{N}}}} = \infty.$$

Due to,

$$\log(1+\mathrm{x}) \le \mathrm{x}, \quad \text{for all} \quad \mathrm{x} > 0,$$

we obtain,

$$\frac{1}{\log(1+\tau)^{\frac{\mathrm{s}}{\mathrm{N}}}\tau^{\frac{\mathrm{ps}}{\mathrm{N}}}} \ge \frac{1}{\tau^{\frac{\mathrm{s}(\mathrm{p}+1)}{\mathrm{N}}}}$$

Since $\mathrm{s}(\mathrm{p}+1) < \mathrm{N}$, we find

$$\int_{\mathrm{A}^{-1}(1)}^\infty \frac{\mathrm{d}\tau}{\tau^{\frac{\mathrm{s}(\mathrm{p}+1)}{\mathrm{N}}}} = \infty.$$

which concludes that

$$\int_1^\infty \frac{\mathrm{A}^{-1}(\mathrm{z})}{\mathrm{z}^{\frac{\mathrm{N}+\mathrm{s}}{\mathrm{N}}}}\mathrm{dz} = \infty.$$

So, (A_∞) is satisfied.

Let us define the function f by $(\mathrm{x},\mathrm{t}) \mapsto |\mathrm{x}|\left(\mathrm{q}|\mathrm{t}|^{\mathrm{q}-1}\log(1+|\mathrm{t}|)+\dfrac{\mathrm{t}}{1+\mathrm{t}}\right)$ with $\mathrm{q}>\mathrm{p}$. Obviously (f_2) holds. Indeed,

$$
\begin{aligned}
\lim_{\tau\to\infty}\frac{\mathrm{f}(\mathrm{x},\tau)}{\varphi(\tau)} &= |\mathrm{x}| \lim_{\tau\to\infty}\frac{\mathrm{q}|\tau|^{\mathrm{q}-1}\log(1+|\tau|)+\dfrac{\mathrm{t}}{1+\tau}}{\mathrm{q}|\tau|^{\mathrm{q}-1}\log(1+|\tau|)+\dfrac{\tau}{1+\tau}}\\
&= |\mathrm{x}| \lim_{\tau\to\infty}\frac{\mathrm{q}\tau^{\mathrm{q}-1}\log(1+\tau)\left(1+\dfrac{\tau}{\mathrm{q}\tau^{\mathrm{q}-1}\log(1+\tau)(1+\tau)}\right)}{\mathrm{p}\tau^{\mathrm{p}-1}\log(1+\tau)\left(1+\dfrac{\tau}{\mathrm{p}\tau^{\mathrm{p}-1}\log(1+\tau)(1+\tau)}\right)}\\
&= \frac{\mathrm{q}|\mathrm{x}|}{\mathrm{p}}\lim_{\tau\to\infty}\left(\tau^{\mathrm{q}-\mathrm{p}}\times\frac{1+\dfrac{\tau}{\mathrm{q}\tau^{\mathrm{q}-1}\log(1+\tau)(1+\tau)}}{1+\dfrac{\tau}{\mathrm{p}\tau^{\mathrm{p}-1}\log(1+\tau)(1+\tau)}}\right).
\end{aligned}
$$

Hence

$$\lim_{\tau\to\infty}\frac{\tau}{\mathrm{q}\tau^{\mathrm{q}-1}\log(1+\tau)(1+\tau)}=0 \tag{4.4}$$

and

$$\lim_{\tau\to\infty}\frac{\tau}{\mathrm{p}\tau^{\mathrm{p}-1}\log(1+\tau)(1+\tau)}=0. \tag{4.5}$$

Due to (4.4), (4.5) and the fact that $\mathrm{p}<\mathrm{q}$, we concludes $\lim_{\tau\to\infty}\dfrac{\mathrm{f}(\mathrm{x},\tau)}{\varphi(\tau)}=\infty$.

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