Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.66650

Generalized Solutions of the Cauchy Problem Involving Φ-Caputo Fractional Derivatives

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ABSTRACT: The main objective of this paper is to extend the Φ-Caputo fractional derivative in the Colombeau algebra of generalized functions, we study also the existence and uniquness of solutions for fractional differential equations involving Φ-Caputo fractional dirivative in the extended Colombeau algebras. As application, our theoretical result has been illustrated by providing a suitable example.

Key Words: Φ-Fractional integral and derivative, generalized function, Φ-Caputo fractional derivatives, extended Colombeau algebras.

Contents

1. Introduction

Due to its huge applications in different fields such as physics, chemistry, engineering, finance and other sciences, fractional calculus has become an indispensable branch of mathematics. As an extension of the traditional integer calculus, which has the properties of an infinity memory and is hereditary. The fractional calculus plays a crucial role to give a real modeling for many real-world phenomena, which pushes researchers to study its qualitative aspects, in order to show the exact results. The study of this theory of fractional calculus has developed considerably during the 19th and 20th centuries. To present a common expression for various approaches of the fractional derivative, Almida in [9] tried to introduce a function in the definition of the approach of Caputo and he succeeded in unifying the approaches of Caputo and that of Hadamard, this type of fractional derivative is called Φ-Caputo fractional derivative. This Fractional calculus has seen a great expansion. Because that it has diverse applications in many areas, as physics and technology, for instance [12,13,14].

To solve the distribution multiplication problem, an algebra of generalized functions denoted by $\mathcal{G}$ was introduced by Colombeau in 1982 [2,3]. The space of distributions $\mathcal{D}'$ is contained in the differential algebra $\mathcal{G}$. Additionally, in this algebra, nonlinear operations that are more general than multiplication make sense [1,4,6,7]. Therefore since nonlinear ODEs and PDEs with singularities frequently exist in life and science, the Colombeau algebra $\mathcal{G}$ is very convenient for solving them [10].

To introduce the fractional derivatives into the Colombeau algebra, Mirjana Stojanovic(see [8,11]) constructed a new generalized functions algebra symbolized by $\mathcal{G}^e$. This new algebra will be considered as an extension of the classical Colombeau algebra with the aim of extending all derivatives to non-integer ones denoted by $D^{q,\Phi}$, $q \in \mathbb{R}_+ \cup \{0\}$ and $\Phi \in C^n(\mathbb{R})$ such that $\Phi'(t) > 0$ for all $t \in \mathbb{R}$.

Aiming at resolving problems related to distribution multiplication and other non-linear operations with singularities, such as non-integer derivatives, Φ-Caputo fractional derivatives has been introduced into the extended Colombeau algebra of generalized functions.

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted January 12, 2023. Published March 16, 2023

In this paper, we will prove the existence and uniqueness of the Cauchy Problem in the generalized functions extension Colombeau Algebra $\mathcal{G}^e(\mathbb{R})$

$$\begin{cases} D^{q,\Phi}_{0^+}\mu(t) = \Psi(t,\mu(t)) \\ \mu(0) = \mu_0, \quad \mu_0 \in \tilde{\mathbb{R}}^n \end{cases}$$

With $\Psi \in \mathcal{G}^e_\xi(\mathbb{R}^+ \times \mathbb{R}^n)$ which can be specfied below, such that its gradient $\nabla_\mu \Psi$ is of L^∞-log-type and $\mu_0 \in \tilde{\mathbb{R}}^n$. If $\Phi(t) = t$, we obtain the Cauchy problem

$$\begin{cases} D^q\mu(t) = \Psi(t,\mu(t)), \\ \mu(0) = \mu_0, \quad \mu_0 \in \mathcal{G}^e(\mathbb{R}), \end{cases}$$

where $\Psi \in \mathcal{G}^e_\xi(\mathbb{R}^+ \times \mathbb{R}^n)$ such that its gradient $\nabla_\mu \Psi$ is of L^∞-log-type and $\mu_0 \in \tilde{\mathbb{R}}^n$. The proof details of the existence and uniqueness of this problem can be found in article [5] .

Next parts of the present paper are structured as follows: In section 2 some introductory concepts and proprieties on Φ-Caputo derivative are presented. Thereafter, the fundamental notions of the new expanded Colombeau theory to derivatives of arbitrary order are detailed. Section 3 concerns the regularization of Φ-Caputo derivatives. The existence and uniqueness of the cauchy problem in the new extension of the Colombeau algebra of generalized functions will be studied in section 4. Finally, the paper is concluded by an application.

2. Preliminairies

We start this section by introducing some necessary definition anad basic results required for further developments.

Definition 2.1. *[9] Let $q > 0$, $\sigma \in C^n(\mathbb{R})$ and $\Phi \in C^n(\mathbb{R})$ with $\Phi'(t) > 0$ for all $t \in \mathbb{R}$. The Φ- Caputo fractional integral of order q of a function σ can be defined as follows*

$$J^{q,\Phi}\sigma(t) = \frac{1}{\Gamma(q)}\int_0^t \Phi'(\xi)(\Phi(t) - \Phi(\xi)^{q-1}\sigma(\xi)d\xi.$$

Definition 2.2. *[9] The Φ-Caputo fractional derivative at order q of a function σ can be defined as follows*

$$D^{q,\Phi}\sigma(t) = \frac{1}{\Gamma(p-q)}\int_0^t \frac{\Phi'(\xi)\sigma^{[p]}_{\Phi,\epsilon}(\xi)}{(\Phi(t)-\Phi(\xi))^{q+1-p}}d\xi,$$

where $\sigma^{[p]}_\Phi(t) = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^p\sigma(t), \quad p = [q]+1$ and $[q]$ symbolizes the integer part of the real number q.

Proposition 2.3. *[9] Let $\mu : [a,b] \longrightarrow \mathbb{R}$ be a function.*

- *If $\mu \in C[a,b]$, then $D^{q,\Phi}_{0^+}I^{q,\Phi}_{0^+}\mu(t) = \mu(t)$.*
- *If $\mu \in C^n[a,b]$, then $I^{q,\Phi}_{0^+}D^{q,\Phi}_{0^+}\mu(t) = \mu(t) - \sum_{k=0}^{n-1}\frac{\mu^{[k]}_\Phi(0)}{k!}(\Phi(t)-\Phi(0))^k$.*

3. Main results

Lets Ω be an open subset of $\mathbb{R}$. We denote by $K \subset\subset \Omega$ a compact subset of Ω. Recall the definition of the extension of Colombeau algebras in a sense of extension of the entire derivatives to the fractional ones. Let $\mathcal{E}^e(\Omega)$ an algebra of all sequences $(\mu_\epsilon)_{\epsilon>0}$ of real valued smooth functions $C^\infty(\Omega)$.

$$\mathcal{E}^e_M(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega) : \forall K \subset\subset \Omega, \forall q \in \mathbb{R}_+ \cup \{0\}, \exists N \in \mathbb{N}, \sup_{x\in K}|D^{q,\Phi}\mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\},$$

is an algebra and

$$\mathcal{N}^e(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in \mathcal{E}^e_M(\Omega) : \forall K \subset\subset \Omega, \forall q \in \mathbb{R}_+ \cup \{0\}, \forall b \in \mathbb{N}, \ \sup_{x\in K} |D^{q,\Phi}\mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^b)\Big\}$$

is an ideal therein, where $D^{q,\Phi}$ is the Φ-Caputo fractional derivative for $p-1<q<n$, $p\in\mathbb{N}$. The extention of Colombeau algebra is the factor set

$$\mathcal{G}^e(\Omega) = \mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega).$$

Similary, we define the extended Colombeau algebra of tempered generalized functions is the factor set

$$\mathcal{G}^e_\xi(\Omega) = \mathcal{E}^e_\xi(\Omega)/\mathcal{N}^e_\xi(\Omega),$$

where

$$\mathcal{E}^e_{M,\xi}(\Omega) = \Big\{\mu_\epsilon \in \mathcal{E}^e(\Omega); \forall K \subset\subset \Omega, \forall q \in \mathbb{R}_+ \cup \{0\}, \exists N \in \mathbb{N}, \ \sup_{t\in\Omega}(1+|t|)^{-N}|D^{q,\Phi}\mu_\epsilon(t)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\}$$

and

$$\mathcal{N}^e_\xi(\Omega) = \Big\{\mu_\epsilon \in \mathcal{E}^e(\Omega) \forall K \subset\subset \Omega, \forall q \in \mathbb{R}_+ \cup \{0\}, \forall b \in \mathbb{N}, \ \sup_{t\in\Omega}(1+|t|)^{-N}|D^{q,\Phi}\mu_\epsilon(t)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{b})\Big\}.$$

For embed Φ-Caputo fractional derivative into Colombeau algebra we use the regularization with delta sequence in order to obtain moderateness

$$\tilde{D}^{q,\Phi}\sigma = \frac{1}{\Gamma(p-q)}\int_0^t \frac{\Phi'(\xi)\sigma^{[p]}_{\Phi,\epsilon}(\xi)}{(\Phi(t)-\Phi(\xi))^{q+1-p}}d\xi * \varphi_\epsilon(t), \tag{3.1}$$

where φ_ϵ is a delta sequence. We have

$$\begin{aligned}
|\tilde{D}^{q,\Phi}\sigma_\epsilon - D^{q,\Phi}\sigma_\epsilon| &= |D^{q,\Phi}\sigma_\epsilon * \varphi_\epsilon(t) - D^{q,\Phi}\sigma_\epsilon(t)| \\
&= |\int D^{q,\Phi}\sigma_\epsilon(t-s)\varphi_\epsilon(s)ds - D^{q,\Phi}\sigma_\epsilon(t)| \\
&= |\int D^{q,\Phi}\sigma_\epsilon(t-s)\frac{1}{\epsilon}\varphi(\frac{s}{\epsilon})ds - D^{q,\Phi}\sigma_\epsilon(t)| \\
&= |\int D^{q,\Phi}\sigma_\epsilon(t-\epsilon u)\varphi(u)du - \int D^{q,\Phi}\sigma_\epsilon(t)\varphi(u)du| \\
&= |\int \left(D^{q,\Phi}\sigma_\epsilon(t-\epsilon u) - D^{q,\Phi}\sigma_\epsilon(t)\right)\varphi(u)du| \\
&= |\int D^{q,\Phi}\left(\sigma_\epsilon(t-\epsilon u) - \sigma_\epsilon(t)\right)\varphi(u)du| \\
&= \mathcal{O}_{\epsilon\to 0}(\epsilon^q).
\end{aligned}$$

$$\begin{aligned}
\tilde{D}^{q,\Phi}\sigma_\epsilon = D^{q,\Phi}\sigma_\epsilon * \varphi_\epsilon(t) &\le \frac{1}{\Gamma(n-q)}\left(\int_0^t \frac{\Phi'(\xi)\sigma^{[n]}_{\Phi,\epsilon}(\xi)}{(\Phi(t)-\Phi(\xi))^{q+1-n}}d\xi\right) * \varphi_\epsilon(t) \\
&\le \frac{1}{\Gamma(p-q)}\sup_{t\in[0,T]}\left\{\left|\int_0^t \frac{\Phi'(\xi)\sigma^{[p]}_{\Phi,\epsilon}(\xi)}{(\Phi(t)-\Phi(\xi))^{q+1-p}}d\xi\right| \|\varphi_\epsilon(t)\|\right\} \\
&\le \frac{1}{\Gamma(p-q)}\frac{C}{\epsilon}\sup_{\xi\in[0,T]}\left\{\left|\sigma^{[p]}_{\Phi,\epsilon}(\xi)\right|\right\}\frac{(\Phi(T)-\Phi(0))^{p-q}}{p-q} \\
&\le C_{T,q}\epsilon^{-N}, \quad N\in\mathbb{N}
\end{aligned}$$

$$\begin{aligned}
|(\tilde{D}^{q,\Phi)}\sigma_\epsilon)'| = |(D^{q,\Phi}\sigma_\epsilon * \varphi_\epsilon)'(t)| = |D^{q,\Phi}\sigma_\epsilon(t) * \varphi'_\epsilon(t)| &\le \frac{C}{\epsilon}\sup_{t\in[0,T]}|D^{q,\Phi}\sigma_\epsilon(t)| \\
&\le C\epsilon^{-N}, \quad N\in\mathbb{N}.
\end{aligned}$$

To prove the moderatness of higher fractionnal derivatives we use the property of semigroup of fractionnal differentiation. Let $0<\gamma<1$, we have

$$D^{\gamma,\Phi}(D^{q,\Phi}\sigma_\epsilon) = D^{\gamma+q,\Phi}\sigma_\epsilon.$$

Then

$$D^{\gamma,\Phi}(\tilde{D}^{q,\Phi}\sigma_\epsilon) = D^{\gamma,\Phi}(D^{q,\Phi}\sigma_\epsilon * \varphi_\epsilon) = D^{\gamma+q,\Phi}\sigma_\epsilon * \varphi_\epsilon,$$

where

$$\begin{aligned}
(D^{\gamma+q,\Phi}\sigma_\epsilon) * \varphi_\epsilon(t) &= \frac{1}{\Gamma(p-q)}\int_0^t \frac{\Phi'(\xi)\sigma^{[p]}_{\Phi,\epsilon}(\xi)}{(\Phi(t)-\Phi(\xi))^{q+1-p}}d\xi * \varphi_\epsilon(t) \\
&\le \frac{1}{\Gamma(n-q)}\sup_{\xi\in[0,T]}\{|\sigma^{[p]}_{\Phi,\epsilon}(\xi)|\}|\frac{(\Phi(t)-\Phi(0))^{p-(q+\gamma)}}{p-(q+\gamma)} * \varphi_\epsilon(t)| \\
&\le \frac{1}{\Gamma(p-q)}\sup_{\xi\in[0,T]}\{|\sigma^{[p]}_{\Phi,\epsilon}(\xi)|\}\frac{C}{\epsilon}\sup_{\xi\in[0,T]}\{\frac{1}{|\Phi'(\xi)|}\}\frac{(\Phi(T)-\Phi(0))^{p-(q+\gamma)+1}}{p-(q+\gamma)+1} \\
&\le C_{T,q,\gamma}\epsilon^{-N}.
\end{aligned}$$

Now, we study the existence and uniqueness of the following Cauchy problem

$$\begin{cases} D^{q,\Phi}_{0^+}\mu(t) = \Psi(t,\mu(t)), \\ \mu(0) = \mu_0, \quad \mu_0 \in \tilde{\mathbb{R}}^n, \end{cases} \tag{3.2}$$

in the extension Colombeau algebra $\mathcal{G}^e(\mathbb{R})$ of generalized functions where $\Psi \in \mathcal{G}^e_\xi(\mathbb{R}^+ \times \mathbb{R}^n)$ such that its gradient $\nabla_\mu\Psi$ is of L^∞-log-type and $\mu_0 \in \tilde{\mathbb{R}}^n$.

Definition 3.1. *A generalized function $\mu \in \mathcal{G}(\mathbb{R}\times\mathbb{R}^n)$is called L^∞-*log*-type, if for some representative there exist $N \in \mathbb{N}$ for every compact $K \subset \mathbb{R}$, such that*

$$\sup_{t\in K}\sup_{x\in\mathbb{R}^n}|\mu_\epsilon(t,x)| = \mathcal{O}_{\epsilon\to 0}(\ln(\epsilon^{-N})). \tag{3.3}$$

Theorem 3.2. *Let $\Psi \in \mathcal{G}^e_\xi(\mathbb{R}^+\times\mathbb{R}^n)$ and $\nabla_\mu F$ is L^∞-*log*-type. Then the problem (3.2) has a unique solution in $\mathcal{G}^e(\mathbb{R}^n)$ the extended Colombeau algebra, for given $\mu_0 \in \tilde{\mathbb{R}}^n$.*

Proof. Consider the integral solution of Cauchy problem (**3.2**)

$$\begin{aligned}
\mu_\epsilon(t) &= \mu_{0\epsilon} + I^{q,\Phi}_{0^+}\Psi_\epsilon(s,\mu_\epsilon(s)) \\
&= \mu_{0,\epsilon} + \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\Psi_\epsilon(s,\mu_\epsilon(s))ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\mu_\epsilon(t) = \mu_{0,\epsilon} &+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\Psi_\epsilon(s,0)ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\int_0^1 \nabla_\mu\Psi_\epsilon(s,\lambda\mu_\epsilon(s))d\lambda\mu_\epsilon(s)ds
\end{aligned}$$

$$\begin{aligned}
|\mu_\epsilon(t)| \le |\mu_{0,\epsilon}| &+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|\Psi_\epsilon(s,0)|ds \\
&+ \frac{|\nabla_\mu\Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\mu_\epsilon(s)ds \\
&\le |\mu_{0,\epsilon}| \\
&+ \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\Psi_\epsilon| + \frac{|\nabla_\mu\Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\mu_\epsilon(s)ds.
\end{aligned}$$

By the Gronwall inequality, we get

$$|\mu_\epsilon(t)| \leq \left(|\mu_{0,\epsilon}| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\Psi_\epsilon|\right) \exp\left(\frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)ds\right)$$
$$\leq \left(|\mu_{0,\epsilon}| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\Psi_\epsilon|\right) \exp\left(\frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\nabla_\mu \Psi_\epsilon|\right).$$

As $(\mu_{0,\epsilon})_\epsilon \in \mathcal{E}_M^e(\mathbb{R})$ and $\nabla_\mu \Psi$ is L^∞-log-type, then there exist $N \in \mathbb{N}$ such that

$$|\mu_\epsilon| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

Now, we pass to the first derivative. We know that

$$D_{0^+}^{q,\Phi}\mu_\epsilon(t) = \Psi_\epsilon\left(t, \mu_\epsilon(t)\right).$$

Then,

$$D_{0^+}^{q,\Phi}\mu'_\epsilon(t) = \frac{d}{dt}\Psi_\epsilon\left(t, \mu_\epsilon(t)\right).$$

The first approximation to Ψ_ϵ yields

$$D_{0^+}^{q,\Phi}\mu'_\epsilon(t) = \frac{d}{dt}\Psi_\epsilon\left(t, 0\right) + |\nabla_\mu \Psi_\epsilon|\mu'_\epsilon(t) + N_\epsilon(t),$$

where N_ϵ is the negligible part. Then,

$$\mu'_\epsilon(t) = \mu'_\epsilon(0) + \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\frac{d}{ds}\Psi_\epsilon(s,0)ds$$
$$+ \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)\mu'_\epsilon(s)ds$$
$$+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)N_\epsilon(s)ds.$$

Thus,

$$|\mu'_\epsilon(t)| \leq |\mu'_\epsilon(0)| + \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|\frac{d}{ds}\Psi_\epsilon(s,0)|ds$$
$$+ \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|\mu'_\epsilon(s)|ds$$
$$+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|N_\epsilon(s)|ds.$$

Then,

$$|\mu'_\epsilon(t)| \leq |\mu'_\epsilon(0)| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\frac{d}{dt}\Psi_\epsilon| + \frac{(\Phi(t)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon|$$
$$+ \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|\mu'_\epsilon(s)|ds.$$

By the Gronwall inequality, we cab obtain

$$|\mu'_\epsilon(t)| \leq (|\mu'_\epsilon(0)| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\frac{d}{dt}\Psi_\epsilon|$$
$$+ \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon|)\exp(\frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)ds)$$
$$\leq \left(|\mu'_\epsilon(0)| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\frac{d}{dt}\Psi_\epsilon| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon|\right) \exp\left(\frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\nabla_\mu \Psi_\epsilon|\right).$$

As $(\mu_{0,\epsilon})_\epsilon \in \mathcal{E}^e_M(\mathbb{R}^n)$ and $\nabla_\mu \Psi$ is L^∞-log-type, then there exist $N \in \mathbb{N}$ such that

$$|\mu'_\epsilon| = \mathcal{O}_{\epsilon \to 0}\left(\epsilon^{-N}\right).$$

Similary for the other entire derivatives.

Now, the same is proved for the fractional derivatives. We take the fractional derivative of the equation (3.2). Let $\gamma \in \mathbb{R}^+$, we have

$$D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) = D^{\gamma,\Phi}_{0^+} I^{q,\Phi}_{0^+} \Psi_\epsilon\left(t, \mu_\epsilon(t)\right).$$

So, we have to discuss two cases: The first case if $\gamma \geq q$, we have

$$\begin{aligned} D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) &= D^{\gamma-q+q,\Phi}_{0^+} I^{q,\Phi}_{0^+} \Psi_\epsilon\left(t, \mu_\epsilon(t)\right) \\ &= D^{\gamma-q,\Phi}_{0^+} \Psi_\epsilon\left(t, \mu_\epsilon(t)\right). \end{aligned}$$

Without loss of generality we take $0 < \gamma - q < 1$. The same holds for $p - 1 < q < p$, $p \in \mathbb{N}$. The first approximation to Ψ_ϵ yields

$$D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) = D^{\gamma-q,\Phi}_{0^+}\Psi_\epsilon(t, 0) + |\nabla_\mu \Psi_\epsilon| D^{\gamma-q,\Phi}_{0^+}\mu_\epsilon(t) + N_\epsilon(t),$$

where $N\epsilon(t)$ is the negligible part. Then, we have

$$\begin{aligned} D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) &= \frac{1}{\Gamma(\gamma-q)} \int_0^t (\Phi(t) - \Phi(s))^{\gamma-q-1} \Phi'(s) \frac{d}{ds} \Psi_\epsilon(s, 0) ds \\ &+ \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(\gamma-q)} \int_0^t (\Phi(t) - \Phi(s))^{\gamma-q-1} \Phi'(s) \mu'_\epsilon(s) ds + N_\epsilon(t) \\ &\leq \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)} \left|\frac{d}{ds}\Psi_\epsilon\right| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)} |\nabla_\mu \Psi_\epsilon||\mu'_\epsilon| + |N_\epsilon|. \end{aligned}$$

By previous step and as $\nabla_\mu \Psi$ is L^∞-log-type, then there exists $N \in \mathbb{N}$ such that

$$\left|D^{\gamma,\Phi}_{0^+}\mu_\epsilon\right| = \mathcal{O}_{\epsilon \to 0}\left(\epsilon^{-N}\right).$$

The second case if $q \geq \gamma$, we have

$$\begin{aligned} D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) &= D^{\gamma,\Phi}_{0^+} I^{q-\gamma+\gamma,\Phi}_{0^+} \Psi_\epsilon(t, \mu_\epsilon(t)) \\ &= I^{q-\gamma,\Phi}_{0^+} \Psi_\epsilon(t, \mu_\epsilon(t)) \\ &= \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-\gamma-1} \Phi'(s) \Psi_\epsilon(t, \mu_\epsilon(s)) ds. \end{aligned}$$

Then,

$$\begin{aligned} D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) &= \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-\gamma-1} \Phi'(s) \Psi_\epsilon(s, 0) ds \\ &+ \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-\gamma-1} \int_0^1 \nabla_\mu \Psi_\epsilon\left(s, \lambda\mu_\epsilon\right) d\lambda \, \mu_\epsilon(s) ds. \end{aligned}$$

And this,

$$\begin{aligned} D^{\gamma,\Phi}_{0^+}\mu_\epsilon(t) &= \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-\gamma-1} \Phi'(s) \Psi_\epsilon(s, 0) ds \\ &+ \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-\gamma-1} \int_0^1 \nabla_\mu \Psi_\epsilon(s, \lambda\mu_\epsilon) d\lambda \, \mu_\epsilon(s) ds \end{aligned}$$

$$\begin{aligned}|D_{0^+}^{\gamma,\Phi}\mu_\epsilon(t)| &\leq \frac{1}{\Gamma(q-\gamma)}\int_0^t (\Phi(t)-\Phi(s))^{q-\gamma-1}\Phi'(s)|\Psi_\epsilon(s,0)|ds \\ &+ \frac{\nabla_\mu\Psi_\epsilon|}{\Gamma(q-\gamma)}\int_0^t (\Phi(t)-\Phi(s))^{q-\gamma-1}\Phi'(s)|\mu_\epsilon(s)|ds \\ &\leq \frac{(\Phi(T)-\Phi(0))^{q-\gamma}}{(q-\gamma)\Gamma(q-\gamma)}|\Psi_\epsilon| + \frac{(\Phi(T)-\Phi(0))^{q-\gamma}}{(q-\gamma)\Gamma(q-\gamma)}|\nabla_\mu\Psi_\epsilon||\mu_\epsilon|.\end{aligned}$$

By previous step and as $\nabla_\mu\Psi$ is L^∞-log-type, then there exists $N\in\mathbb{N}$ such that

$$\left|D_{0^+}^{\gamma,\Phi}\mu_\epsilon\right| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{-N}\right).$$

To obtain uniqueness, assume that the Cauchy problem **(3.2)** has two solutions μ, y with representatives $(\mu_\epsilon)_{\epsilon>0}, (y_\epsilon)_{\epsilon>0}$. Then, we have

$$\begin{cases} D_{0^+}^{q,\Phi}(\mu_\epsilon(t)-y_\epsilon(t)) = \Psi_\epsilon(t,\mu_\epsilon(t)) - \Psi_\epsilon(t,y_\epsilon(t)) + n_\epsilon(t), \\ \mu_\epsilon(0)-y_\epsilon(0) = n_{0,\epsilon}, \end{cases} \tag{3.4}$$

where $(n_\epsilon)_\epsilon \in \mathcal{N}^e(\mathbb{R}^n), (n_{0,\epsilon})_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$. Similar arguments as above imply that $(\mu_\epsilon - y_\epsilon)_{\epsilon>0}$ belongs to $\mathcal{N}^e(\mathbb{R}^n)$.

Consider the integral solution of the equation **(3.4)**

$$\begin{aligned}\mu_\epsilon(t)-y_\epsilon(t) &= n_{0\epsilon} + I_{0^+}^{q,\Phi}\Psi_\epsilon(s,\mu_\epsilon(s) - \Psi_\epsilon(t,y_\epsilon(t)) + I_{0^+}^{q,\Phi}(n_\epsilon(t)) \\ &= n_{0,\epsilon} + \frac{1}{\Gamma(q)}\int_0^t \Big[\Phi(t)-\Phi(s))^{q-1}\Phi'(s)(\Psi_\epsilon(s,\mu_\epsilon(s)) - \Psi_\epsilon(s,y_\epsilon(s))\Big]ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)n_\epsilon(s)ds.\end{aligned}$$

The first approximation to Ψ_ϵ yields

$$\begin{aligned}\mu_\epsilon(t)-y_\epsilon(t) = n_{0,\epsilon} &+ \frac{\nabla\Psi_\epsilon}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)(\mu_\epsilon(s)-y_\epsilon(s))ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)N_\epsilon(s)ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)n_\epsilon(s)ds,\end{aligned}$$

where $(N_\epsilon)_\epsilon$ is the negligible part.

$$\begin{aligned}|\mu_\epsilon(t)-y_\epsilon(t)| \leq |n_{0,\epsilon}| &+ \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|n_\epsilon| \\ &+ \frac{|\nabla_\mu\Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)|\mu_\epsilon(s)-y_\epsilon(s)|ds.\end{aligned}$$

By the Gronwall inequality, we get

$$\begin{aligned}|\mu_\epsilon(t)-y_\epsilon(t)| \leq \Big(|n_{0,\epsilon}| &+ \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon| \\ &+ \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|n_\epsilon|\Big)\exp\Big(\frac{|\nabla_\mu\Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t)-\Phi(s))^{q-1}\Phi'(s)ds\Big).\end{aligned}$$

Then,

$$|\mu_\epsilon(t)-y_\epsilon(t)| \leq \left(|n_{0,\epsilon}| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|N_\epsilon| + \frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|n_\epsilon|\right)\exp\left(\frac{(\Phi(T)-\Phi(0))^q}{q\Gamma(q)}|\nabla_\mu\Psi_\epsilon|\right).$$

As $\nabla_\mu \Psi$ is L^∞-log-type, $(n_\epsilon)_\epsilon, (N_\epsilon)_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$ and $n_{0,\epsilon} \in \mathcal{N}^e(\mathbb{R}^n)$

$$|\mu_\epsilon - y_\epsilon| = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^q\right) \quad \forall q \in \mathbb{N}.$$

For the first derivative, we have

$$D_{0^+}^{q,\Phi}(\mu'_\epsilon(t) - y'_\epsilon(t)) = \frac{d}{dt}\Psi_\epsilon(t, x(t)) - \frac{d}{dt}\Psi_\epsilon(t, y(t)) + \frac{d}{dt}n_\epsilon(t).$$

The first approximation to Ψ_ϵ yields

$$D_{0^+}^{q,\Phi}(\mu'_\epsilon(t) - y'_\epsilon(t)) = |\nabla_\mu \Psi_\epsilon|(\mu'_\epsilon(t) - y'_\epsilon(t)) + \frac{d}{dt}n_\epsilon(t) + N_\epsilon(t),$$

where $(N_\epsilon)_\epsilon$ is the negligible part. Then

$$\begin{aligned}
\mu'_\epsilon(t) - y'_\epsilon(t) &= n_{0,\epsilon} + \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q}\int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)(\mu'_\epsilon(s) - y'_\epsilon(s))ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)\frac{d}{ds}n_\epsilon(s)ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)N_\epsilon(s)ds.
\end{aligned}$$

Then,

$$\begin{aligned}
|\mu'_\epsilon(t) - y'_\epsilon(t)| \leq |n_{0,\epsilon}| &+ \frac{(\Phi(t) - \Phi(0))^q}{q\Gamma(q)}|\frac{d}{ds}n_\epsilon| + \frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|N_\epsilon| \\
&+ \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)|\mu'_\epsilon(s) - y'_\epsilon(s)|ds.
\end{aligned}$$

By the Gronwall inequality

$$\begin{aligned}
|\mu'_\epsilon(t) - y'_\epsilon(t)| &\leq \Big(|n_{0,\epsilon}| + \frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|\frac{d}{ds}n_\epsilon| + \frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|N_\epsilon|\Big) \\
&\quad \exp\Big(\frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)}\int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)ds\Big) \\
&\leq \Big(|n_{0,\epsilon}| + \frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|\frac{d}{ds}n_\epsilon| + \frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|N_\epsilon|\Big) \\
&\quad \exp\Big(\frac{(\Phi(T) - \Phi(0))^q}{q\Gamma(q)}|\nabla_\mu \Psi_\epsilon|\Big).
\end{aligned}$$

As $\nabla_\mu \Psi$ is L^∞-log-type, $(n_\epsilon)_\epsilon, (N_\epsilon)_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$ and $(n_{0,\epsilon})_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$

$$|\mu'_\epsilon - y'_\epsilon| = \mathcal{O}_{\epsilon\to 0}(\epsilon^q) \quad \forall q \in \mathbb{N}.$$

Now we prove the same for the fractional derivative. Take the fractional derivative of the problem (3.4). Let $\gamma \in \mathbb{R}^+$, we have

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = D_{0^+}^{\gamma,\Phi} I_{0^+}^{q,\Phi}(\Psi_\epsilon(t, \mu_\epsilon(t)) - \Psi_\epsilon(t, y_\epsilon(t))) + D_{0^+}^{\gamma,\Phi} I_{0^+}^{q,\Phi}(n_\epsilon(t)).$$

If $\gamma \geq q$, we have

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = D_{0^+}^{\gamma-q+q,\Phi} I_{0^+}^{q,\Phi}(\Psi_\epsilon(t, \mu_\epsilon(t)) - \Psi_\epsilon(t, y_\epsilon(t))) + D_{0^+}^{\gamma-q+q,\Phi} I_{0^+}^{q,\Phi}(n_\epsilon(t)),$$

then

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = D_{0^+}^{\gamma-q,\Phi}(\Psi_\epsilon(t, \mu_\epsilon(t)) - \Psi_\epsilon(t, y_\epsilon(t))) + D_{0^+}^{\gamma-q,\Phi}(n_\epsilon(t)).$$

Without loss of generalisity we take $0 < \gamma - q < 1$.
The same for $p-1 < q < p$. The first approximation to Ψ_ϵ yiels

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = \frac{|\nabla_\mu \Psi_\epsilon|}{\Gamma(q)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)|\mu'_\epsilon(s) - y'_\epsilon(s)|ds + \frac{1}{\Gamma(\gamma - q)} \int_0^t (\Phi(t) - \Phi(s))^{\gamma-q-1}\Phi'(s)n_\epsilon(s)ds + \frac{1}{\Gamma(\gamma - q)} \int_0^t (\Phi(t) - \Phi(s))^{\gamma-q-1}\Phi'(s)N_\epsilon(s)ds,$$

where $(N_\epsilon)_\epsilon$ is the negligible part. Then,

$$|D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)|) \leq \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|\nabla_\mu \Psi_\epsilon||\mu'_\epsilon(t) - y'_\epsilon(t)| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|n_\epsilon| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|N_\epsilon|.$$

By previous step and as $\nabla_\mu \Psi$ is L^∞-log-type, $(n_\epsilon)_\epsilon, (N_\epsilon)_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$ and $(n_{0,\epsilon})_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$

$$|D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t))| = \mathcal{O}_{\epsilon\to 0}(\epsilon^q) \quad \forall q \in \mathbb{N}.$$

If $q \geq \gamma$, we have

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = D_{0^+}^{\gamma,\Phi} I_{0^+}^{\gamma+q-\gamma,\Phi}(\Psi_\epsilon(t,\mu_\epsilon(t)) - \Psi_\epsilon(t, y_\epsilon(t))) + D_{0^+}^{\gamma,\Phi} I^{\gamma+q-\gamma,\Phi}(n_\epsilon(t)) = I_{0^+}^{q-\gamma,\Phi}(\Psi_\epsilon(t,\mu_\epsilon(t)) - \Psi_\epsilon(t, y_\epsilon(t))) + I_{0^+}^{q-\gamma,\Phi}(n_\epsilon(t)),$$

then

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)(\Psi_\epsilon(s,\mu_\epsilon(s) - \Psi_\epsilon(s, y_\epsilon(s)))ds + \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)n_\epsilon(s)ds.$$

Then there exist a negligible function $(N_\epsilon)_\epsilon$ such that

$$D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t)) = \frac{\nabla_\mu \Psi_\epsilon}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)(\mu_\epsilon(t) - y_\epsilon(t))ds + \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)N_\epsilon(s)ds + \frac{1}{\Gamma(q-\gamma)} \int_0^t (\Phi(t) - \Phi(s))^{q-1}\Phi'(s)n_\epsilon(s)ds \leq \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|\nabla_\mu \Psi_\epsilon||\mu_\epsilon(t) - y_\epsilon(t)| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|N_\epsilon| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|n_\epsilon|.$$

And this

$$|D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t))| \leq \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|\nabla_\mu \Psi_\epsilon||\mu_\epsilon(t) - y_\epsilon(t)| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|N_\epsilon| + \frac{(\Phi(T) - \Phi(0))^{\gamma-q}}{(\gamma-q)\Gamma(\gamma-q)}|n_\epsilon|.$$

By previous step and as $\nabla_\mu \Psi$ is L^∞-log-type, $(n_\epsilon)_\epsilon, (N_\epsilon)_\epsilon \in \mathcal{N}^e(\mathbb{R}^n)$

$$|D_{0^+}^{\gamma,\Phi}(\mu_\epsilon(t) - y_\epsilon(t))| = \mathcal{O}_{\epsilon\to 0}(\epsilon^q) \quad \forall q \in \mathbb{N}.$$

This complete the proof. □

4. Application

Consider the system of ODE involving Φ-Caputo fractional derivatives in the extended colombeau algebra.

$$\begin{cases} \mu'(t) = \Psi(t,\mu(t)) + D^{q,\Phi}\delta(t), \\ \mu(t_0) = \mu_0, \quad \mu_0 \in \mathcal{G}^e(\mathbb{R}), \end{cases} \tag{4.1}$$

where $0 < q < 1$. We regularize the equation **(4.1)**, we have

$$\mu_\epsilon'(t) = \Psi_\epsilon(t,\mu_\epsilon(t)) + D^{q,\Phi}\varphi_\epsilon(t) * \varphi_\epsilon(t).$$

The integral form of this equation is

$$|\mu_\epsilon'(t)| \leq |\mu_{0,\epsilon}| + \int_0^t |\Psi(0,s)|ds + \int_0^t |\int_0^1 (\nabla_\mu \Psi_\epsilon(s,\lambda\mu_\epsilon(s)))d\lambda)||\mu_\epsilon(s)|ds + \int_0^t |D^{q,\Phi}\varphi_\epsilon(\xi) * \varphi_\epsilon(\xi)|d\xi.$$

Let's prove the moderateness of the last term. We have

$$\begin{aligned}
\int_0^t |D^{q,\Phi}\varphi_\epsilon(\xi) * \varphi_\epsilon(\xi)|d\xi| &\leq C\int_0^t \sup_{\tau\in supp\varphi_\epsilon} |D^{q,\Phi}\varphi_\epsilon(\xi-\tau)|d\xi \\
&\leq C\frac{1}{\Gamma(m-q)}\int_0^t\int_0^\xi |\frac{\Phi'(\xi)\ \varphi_\epsilon'(s)}{(\Phi(\xi)-\Phi(s))^q}|d\xi \\
&\leq \frac{C_{q,m}}{\epsilon}\int_0^t\left(\int_0^{\xi\epsilon}\frac{\Phi'(m\epsilon)\varphi'(m)}{(\Phi(\xi)-\Phi(\epsilon m))^q}dm\right)d\xi \\
&\leq \frac{C_{q,m}}{\epsilon}\int_0^t \sup_{m\in[0,T]}|\varphi'(m)|\left(\int_0^{\xi\epsilon}\frac{\Phi'(m\epsilon)}{(\Phi(\xi)-\Phi(\epsilon m))^q}dm\right)d\xi \\
&\leq \frac{C_{q,m,\varphi'}}{\epsilon}\int_0^t\frac{(\Phi(\xi)-\Phi(0))^{1-q}}{1-q}d\xi \\
&\leq \frac{C_{q,m,\varphi'}}{\epsilon}\sup_{\xi\in[0,T]}\{\frac{1}{|\Phi'(\xi)|}\}|\int_0^t\frac{(\Phi(\xi)-\Phi(0))^{1-q}\Phi'(\xi)}{1-q}|d\xi \\
&\leq \frac{C_{q,m,\varphi'}}{\epsilon(1-q)}\sup_{\xi\in[0,T]}\{\frac{1}{|\Phi'(\xi)|}\}\frac{(\Phi(T)-\Phi(0))^{2-q}}{2-q} \\
&\leq C_\epsilon^{-N}, \quad \exists N\in\mathbb{N}.
\end{aligned}$$

Let $0 < \gamma < 1$, we have

$$|\tilde{D}^{\gamma,\Phi}(\mu'(t))| \leq |\tilde{D}^{\gamma,\Phi}\Psi_\epsilon(t,0)(t)| + |\ln\epsilon||\tilde{D}^{\gamma,\Phi}(\mu_\epsilon(t))| + |\tilde{D}^{\gamma,\Phi}(D^{q,\Phi}\varphi_\epsilon * \varphi_\epsilon(t))|. \tag{4.2}$$

Let's prove the moderatness of the fractional part in sup-norm, we have

$$\begin{aligned}
\tilde{D}^{\gamma,\Phi}(D^{q,\Phi}\varphi_\epsilon * \varphi_\epsilon) &= \frac{1}{\Gamma(1-\gamma)}\left(\int_0^t \Phi'(\xi)\frac{(D^{q,\Phi}\varphi_\epsilon * \varphi_\epsilon)'(\xi)}{(\Phi(t)-\Phi(\xi))^\gamma}d\xi\right) * \varphi_\epsilon(t) \\
&= \frac{1}{\Gamma(1-\gamma)}\left(\int_0^t \Phi'(\xi)\frac{D^{q,\Phi}\varphi_\epsilon * \varphi_\epsilon'(\xi)}{(\Phi(t)-\Phi(\xi))^\gamma}d\xi\right) * \varphi_\epsilon(t) \\
&\leq \frac{C}{\Gamma(1-q)}\int_0^t \Phi'(\xi)\frac{\sup_{\xi\in[0,T]}\{|D^{q,\Phi}\varphi_\epsilon(\xi)|\}}{(\Phi(t)-\Phi(\xi))^\gamma}d\xi * \varphi_\epsilon(t) \\
&\leq \frac{C}{\Gamma(1-\gamma)\Gamma(1-q)}\frac{1}{\epsilon^2}\sup_{\xi\in[0,T]}\{|\varphi(\xi)|\}\int_0^t\frac{(\Phi(\xi)-\Phi(0)^{1-q}}{(\Phi(t)-\Phi(\xi))^\gamma}d\xi * \varphi_\epsilon(t) \\
&\leq C_{T,q,\gamma,\varphi,\Phi'}\epsilon^{-2-\gamma}.
\end{aligned}$$

The other parts of equation (4.2) are moderate. Integrating from 0 to t with previously calculated part, we obtain

$$|\tilde{D}^{\gamma,\Phi}(\mu_\epsilon(t))| \leq |\tilde{D}^{\gamma,\Phi}(\mu_{0\epsilon}| + \int_0^t |\tilde{D}^{\gamma,\Phi}(\Psi_\epsilon(s,0)|ds + |\ln\epsilon| \int_0^t |\tilde{D}^{\gamma,\Phi}(\mu_\epsilon(t))|d\xi + T\frac{C}{\epsilon^{2+\gamma}}.$$

By the Gronwall inequality

$$|\tilde{D}^{\gamma,\Phi}(\mu_\epsilon(t))| \leq C\epsilon^{-N}\exp(-T\ln\epsilon) \leq C\epsilon^{-N}, \quad \exists N>0, t\in\mathbb{R}_+\cup\{0\}.$$

If $p-1<\gamma<p$, the similar process can be used for higher fractional derivatives.
To prove uniqueness, we assume that the equation (4.1) has tow solutions μ, y with representatives $(\mu_\epsilon)_\epsilon$, $(y_\epsilon)_\epsilon$ and their difference $w_\epsilon(t)$. By subtraction of these two equations, we obtain

$$\| D^{\gamma,\Phi}(w'_\epsilon(t)| \leq |\ln\epsilon||D^{\gamma,\Phi}w_\epsilon(t)|.$$

Integrating on the interval $[0,t)$, $t<T$, $T>0$, we find

$$|D^{\gamma,\Phi}w_\epsilon(t)| \leq |D^{\gamma,\Phi}w_0| + |\ln\epsilon| \int_0^t |D^{\gamma,\Phi}w_\epsilon(\xi)|.$$

Since $|D^{\gamma,\Phi}w_0|=0$. By the Gronwall inequality, $|D^{\gamma,\Phi}w_\epsilon(t)|\leq 0$ and

$$\begin{aligned}|D^{\gamma,\Phi}w_\epsilon(t)| &\leq \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{|w'_\epsilon(\xi)|}{(\Phi(t)-\Phi(\xi))^\gamma}d\xi \\ &\leq \sup_{t\in[0,T]}|w'_\epsilon(t)|\frac{(\Phi(T)-\Phi(0))^{1-\gamma}}{1-\gamma}.\end{aligned}$$

Then $|w'_\epsilon(t)|\approx 0$, $|w_\epsilon(t)|\approx|w_\epsilon(0)|=0$. And this $\mu_\epsilon(t)\approx y_\epsilon(t)$, and uniqueness follows for $0<\gamma<1$. The same way we prove for $p-1<\gamma<p$, for all $p\in\mathbb{N}$.

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