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h-Ricci soliton and Gradient h-Ricci soliton on para-Kenmotsu manifold *

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ABSTRACT: The main objective of current paper is to examine the h-Ricci soliton and gradient h-Ricci soliton on a para-Kenmotsu manifold when h has a definite signal. Firstly, we show that h-Ricci soliton on the present manifold is Einstein whenever the potential vector field V is contact and if the potential vector field V is collinear with the Reeb vector field ξ, then the manifold is η-Einstein manifold. Next, we prove that a η-Einstein para-Kenmotsu metric as an h-Ricci soliton reduces to Einstein manifold. Finally, we show that a similar result occurs in the case of gradient h-Ricci soliton.

Key Words: h-Ricci soliton, gradient h-Ricci soliton, para-Kenmotsu manifold, Einstein manifold, η-Einstein manifold.

Contents

1. Introduction

In 1982, Richard S. Hamilton [11] introduced the Ricci soliton, a natural generalization of Einstein manifold. Given a one-parametric family of metrics $g(t)$ on a smooth Riemannian manifold M^n defined on an interval I contained in R, denoting by $Ric_{g(t)}$ is Ricci tensor of the metric $g(t)$, the equation of Ricci flow is

$$\frac{d}{dt}g(t) = -2Ric_{g(t)}. \tag{1.1}$$

On a smooth manifold M^n along with the Riemannian metric g, the Ricci soliton is a triplet (g, V, λ), where V is a vector field known as potential vector field and λ is a real scalar satisfying the equation

$$\frac{1}{2}(L_V g)(X, Y) + Ric(X, Y) = \lambda g(X, Y), \tag{1.2}$$

for every vector fields X and Y on M^n, where $L_V g$ denote the Lie-derivative of g along the direction of the vector field V and Ric denotes the Ricci tensor corresponds to the metric g. Ricci soliton is a self-similar solution of Ricci flow defined by the geometric evolution equation (1.1) with the initial condition $g(0) = g$. A Ricci soliton is said to be expanding, steady and shrinking, corresponding to λ is negative, zero and positive, respectively.

Further, Pigola et.al., [13] introduced the almost soliton. Suppose the vector field V is gradient of a smooth function u on M^n, i.e., $V = \triangle u$, where $\triangle$ denotes the gradient operator. We say that the Ricci soliton is a gradient Ricci soliton and the function u is called potential function. For the gradient Ricci soliton, equation (1.2) takes the form

$$Hess\ u + Ric = \lambda g, \tag{1.3}$$

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where $Hess$ denotes the Hessian operator corresponding to the Riemannian connection ∇ of g.

An h-almost Ricci soliton is a generalization of almost Ricci soliton as in [1,13]. These are the solitons on a complete Riemannian manifold (M^n, g) with a vector field X on M^n, a soliton function $\lambda : M^n \to R$ and a function $h : M^n \to R$, which are smooth and satisfy the equation

$$Ric + \frac{h}{2} L_X g = \lambda g. \tag{1.4}$$

In the above equation, if λ is constant, it is called an h-Ricci soliton. Suppose $L_X g = L_{\nabla u} g$ for some smooth function $u : M^n \to R$, then we call the soliton as gradient h-almost Ricci soliton with the potential function u. In this case, the fundamental equation (1.4) can be written as

$$Ric + h\, Hess\, u = \lambda g. \tag{1.5}$$

The equation is also known as the Ric-Hessian equation. The almost h-Ricci soliton is expanding, steady or shrinking if λ is negative, zero or positive on M^n respectively and it is undefined if λ has no definite sign.

If $L_X g = cg$, i.e., X is a homothetic conformal vector field for some constant c, then h-almost soliton (M^n, g, X, h, λ) is said to be trivial. Otherwise, it is non-trivial. Moreover, 1-almost Ricci soliton is just a Ricci soliton, and 1-Ricci solitons are traditional Ricci solitons with constant λ. We can see that h has definite signal if either $h > 0$ or $h < 0$ on M^n.

The concept of h-almost Ricci solitons was first introduced by Gomes et al., [8]. They showed that compact non-trivial h-almost Ricci soliton on a manifold of dimension less than three with h having a definite signal and constant scalar curvature is isometric to a standard sphere with potential function well-determined and also gave the characterization for a special class of gradient h-Ricci solitons.

Further, Gabin Yun et al. [6] proved that, if a manifold M^n is Bach-flat and $\frac{dh}{du} > 0$, where u is the potential function of V, then the manifold is either Einstein or rigid. Further, they showed that if the dimension of a manifold is four, then the metric g is locally conformally flat.

Later Faraji [7] gave the complete classification of h-almost Ricci solitons with concurrent potential vector fields. Also, they obtained the condition on a submanifold of a Riemannian h-almost Ricci soliton to be an h-almost Ricci soliton. Finally, they classified h-almost Ricci soliton on a Euclidean hypersurface with $\lambda = h$.

Keneyuki and Williams [12] first introduced the odd-dimensional, almost para-contact structure with a pseudo-Riemannian metric, an associated structure of the para-Hermitian metric. Later the notion of the para-Kenmotsu manifold was introduced by Weyezko [9]. This structure is related to the Kenmotsu manifold in para-contact geometry.

The properties of Ricci soliton in Kenmotsu manifold studied by the authors De and Fatemah [4]. Later in [2] and [3], Patra studied the Ricci solitons in para contact geometry and they also studied Ricci soliton, Ricci almost soliton on para-Kenmotsu manifold. Based on the above literature study, we are motivated to study the h-Ricci solitons and gradient h-Ricci solitons on para-Kenmotsu manifolds.

2. Preliminaries

A (2n+1)-dimensional smooth manifold M is said to have an almost para-contact manifold, if it admits a structure with vector field ξ, (1,1)-tensor field ϕ and 1-form η satisfying the following conditions:

i) $\phi^2 = -I + \eta \otimes \xi$,

ii) $\eta(\xi) = 1$,

iii) on $2n$-dimensional distribution $D = ker(\eta)$, ϕ induces an almost para complex structure $\mathcal{H}$ with $\mathcal{H}^2 \equiv I$, where D^+ and D^- are the subbundles of $\mathcal{H}$ having dimension n each, corresponding to the eigenvalues +1 and -1, respectively.

By the definition of para contact structure, we have $\phi(\xi) = 0$, $\eta \cdot \phi = 0$ and $rank(\phi)$=2n. An almost para-contact structure is said to be normal if and only if the (1,2) type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically on M, where

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{2.1}$$

for all vector fields X and Y on M. Suppose M is an almost para-contact manifold endowed with the almost para contact structure (ϕ, ξ, η) admiting the pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

holds for all vector fields X and Y on M. Then M is called a compatible metric manifold.

In an almost para contact manifold the following condition

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

holds, then the manifold is called an almost para-Kenmotsu manifold. The para-Kenmotsu manifolds are the almost para-Kenmotsu manifolds with the torsion tensor $N_\phi(X, Y)$ zero identically on M. i.e., normal almost para-Kenmotsu manifolds are para-Kenmotsu manifolds.

On a $2n+1$-dimensional para-Kenmotsu manifold M, the following properties hold:

$$\begin{aligned}
\phi\xi &= 0, \quad \eta \otimes \phi = 0, \quad \nabla_X \xi = X - \eta(X)\xi, && (2.4)\\
(\nabla_X \eta)(Y) &= g(X, Y) - \eta(X)\eta(Y), \quad Q\xi = -2n\xi, && (2.5)\\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, && (2.6)\\
R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, && (2.7)\\
(L_\xi g)(X, Y) &= 2[g(X, Y) - \eta(X)\eta(Y)], && (2.8)
\end{aligned}$$

for any vector fields X, Y on M, where Q denotes the Ricci operator associated with the Ricci tensor Ric defined by $Ric(X, Y) = g(QX, Y)$ and R denotes the Riemannian curvature tensor.

A $(2n+1)$-dimensional Kenmotsu manifold is said to be η-Einstein, if there exist two smooth functions a and b satisfying the below condition:

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y). \tag{2.9}$$

If $b = 0$, it is clear that the η-Einstein manifold reduces to the Einstein manifold.
On contracting the above equation, we get $r = (2n+1)a + b$, where r denotes the scalar curvature of the manifold. Taking $Y = \xi$ in (2.9), we get $a + b = -2n$. On solving the preceding two equations, we get $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. By using these two values in (2.9), we obtain the Ricci curvature tensor as follows

$$Ric(X, Y) = \big(1 + \frac{r}{2n}\big)g(X, Y) - \big(2n + 1 + \frac{r}{2n}\big)\eta(X)\eta(Y). \tag{2.10}$$

3. h-Ricci soliton on para-Kenmotsu manifold

In this section we consider the metric g of $(2n+1)$-dimensional para-Kenmotsu manifold as a h-Ricci soliton.

Here we state an important Lemma, which will be used later in our work:

Lemma 3.1 *([14]) The Ricci operator on $(2n+1)$-dimensional para-Kenmotsu manifold satisfies the following:*

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{3.1}$$
$$(\nabla_\xi Q)X = -2QX - 4nX, \tag{3.2}$$

for an arbitrary vector field X on the manifold.

Theorem 3.1 *Let (M, ϕ, ξ, η, g) be a para-Kenmotsu manifold with g as an h-Ricci soliton, where h has a definite signal. If the potential vector field V is contact, then the soliton is expanding with V as strictly infinitesimal contact transformation and M is an Einstein manifold.*

Proof: Taking the covariant derivative of (1.4) along Z direction, we get

$$h(\nabla_Z L_V g)(X,Y) = -2\big\{(\nabla_Z Ric)(X,Y) - \left(\frac{Zh}{2}\right)(L_V g)(X,Y)\big\}. \tag{3.3}$$

From Yano [10], we have the commutation formula, given by

$$\big(L_V \nabla_Z g - \nabla_Z L_V g - \nabla_{[V,Z]} g\big)(X,Y) = -g\big((L_V\nabla)(X,Z),Y\big) - g\big((L_V\nabla)(Y,Z),X\big), \tag{3.4}$$

where g is metric compatible, then the above equation takes the form

$$(\nabla_Z L_V g)(X,Y) = g\big((L_V\nabla)(X,Z),Y\big) + g\big((L_V\nabla)(Y,Z),X\big), \tag{3.5}$$

for every vector fields X, Y and Z on M.
Since by knowing the fact that, $(L_V\nabla)(X,Y)$ is a symmetric tensor of type (1,2) and from the preceding equation, we obtain

$$2h\, g((L_V\nabla)(X,Z),Y) = h\big\{(\nabla_Z L_V g)(X,Y) + (\nabla_X L_V g)(Z,Y) - (\nabla_Y L_V g)(X,Z)\big\}, \tag{3.6}$$

for all vector fields X, Y and Z on M.
On substituting (3.3) in (3.6), we obtain

$$\begin{aligned} h^2 g\big((L_V\nabla)(X,Z),Y\big) &= -h\big\{(\nabla_Z Ric)(X,Y) + (\nabla_X Ric)(Z,Y) - (\nabla_Y Ric)(X,Z)\big\} \\ &\quad -\frac{h}{2}\big\{(Zh)(L_V g)(X,Y) + (Xh)(L_V g)(Z,Y) \\ &\quad -(Yh)(L_V g)(X,Z)\big\}, \end{aligned} \tag{3.7}$$

for arbitrary vector fields X, Y and Z on M.
Taking $Z = \xi$ in the above equation, we get

$$\begin{aligned} h^2 g\big((L_V\nabla)(X,\xi),Y\big) &= 2h\big\{Ric(X,Y) + 2ng(X,Y)\big\} + (\xi h)\big\{Ric(X,Y) - \lambda g(X,Y)\big\} \\ &\quad +(\lambda + 2h)\{(Yh)\eta(X) - (Xh)\eta(Y)\}, \end{aligned} \tag{3.8}$$

for all arbitrary vector fields X and Y on M.
Taking $(Xh) = (\xi h)\eta(X)$, the above equation reduces to the following

$$h^2\big(L_V\nabla\big)(X,\xi) = (2h + \xi h)QX + \big\{4nh - \lambda(\xi h)\big\}X, \tag{3.9}$$

for all vector fields X on M. On differentiating (3.9) along the arbitrary vector field Y, we obtain

$$\begin{aligned} h^2(\nabla_Y L_V\nabla)(X,\xi) &= -2h(Yh)(L_V\nabla)(X,\xi) - h^2\big\{(L_V\nabla)(X,Y) \\ &\quad -(L_V\nabla)(X,\xi)\eta(Y)\big\} + \big\{2(Yh) + \nabla_Y(\xi h)\big\}QX \\ &\quad +(2h + \xi h)(\nabla_Y Q)X + \big\{4n(Yh) - \lambda(\nabla_Y \xi h)\big\}X. \end{aligned} \tag{3.10}$$

Again from Yano [10], we have the following commutation formula

$$h^2(L_V R)(X,Y)\xi = h^2\big\{(\nabla_X L_V\nabla)(Y,\xi) - (\nabla_Y L_V\nabla)(X,\xi)\big\}. \tag{3.11}$$

Taking into account of (3.10), the above equation takes the form

$$\begin{aligned} h^2(L_V R)(X,Y)\xi &= -2h\big\{(Xh)(L_V\nabla)(Y,\xi) - (Yh)(L_V\nabla)(X,\xi)\big\} \\ &\quad +h^2\big\{(L_V\nabla)(Y,\xi)\eta(X) - (L_V\nabla)(X,\xi)\eta(Y)\big\} \\ &\quad +\big\{2(Xh + X(\xi h))\big\}QY - \big\{2(Yh) + Y(\xi h)\big\}QX \\ &\quad +(2h + \xi h)\big\{(\nabla_X Q)Y - (\nabla_Y Q)X\big\} \\ &\quad +\big\{4n(Xh) - \lambda X(\xi h)\big\}Y - \big\{4n(Yh) - \lambda Y(\xi h)\big\}X, \end{aligned} \tag{3.12}$$

for every arbitrary vector fields X, Y on M.
Noting $(Xh) = (\xi h)\eta(X)$, differentiate the preceeding equation along the vector field Y, and taking inner product with ξ, we get $Y(\xi h) = 0$ for all vector field Y on M, which implies that (ξh) is constant on M. Considering this fact in (3.12) and taking into account of (3.9), we get

$$\begin{aligned} h^2(L_V R)(X,\xi)\xi &= -2h(Xh)(L_V\nabla)(\xi,\xi) + 2h(\xi h)(L_V\nabla)(X,\xi) \\ &\quad -2(\xi h)QX + (\lambda - 2h)(\xi h)X - (\lambda + 2h)(\xi h)\eta(X)\xi. \end{aligned}$$

Because, h has the definite signal and taking the inner product with ξ in the preceding equation gives

$$(L_V R)(X,\xi)\xi = 0, \tag{3.13}$$

for all vector fields X on M.
On taking the Lie derivative of $g(\xi,\xi) = 1$ and employing $Q\xi = -2n\xi$, we get

$$h\,\eta(L_V\xi) = -2n - \lambda. \tag{3.14}$$

Plugging $Y = \xi$ in (1.4) and by straight forward computation we have

$$\frac{h}{2}\big\{(L_V\eta)(X) - g(X, L_V\xi)\big\} = (\lambda + 2n)\eta(X), \tag{3.15}$$

for all vector fields X on M. Substituting $Y = \xi$ in (2.7) and taking the Lie derivative along the potential vector field V, we obtain

$$h(L_V R)(X,\xi)\xi = -2(2n + \lambda)\big(X - \eta(X)\xi\big), \tag{3.16}$$

for all vector fields X on M. Unifying (3.15) and (3.16), we get

$$\lambda = -2n. \tag{3.17}$$

Substituting (3.16) in (3.14) and since h has a definite signal, we obtain

$$\eta(L_V\xi) = 0. \tag{3.18}$$

By our hypothesis, the potential vector field V is contact, and therefore there must be a smooth function f, such that $L_V\xi = f\xi$. On taking the inner product with ξ and comparing it with the previous equation, we obtain $f = 0$, which leads to $L_V\xi$=0.
The use of equation (3.17) in (3.15) and the fact that is non-zero yields

$$(L_V\eta)(X) = 0, \tag{3.19}$$

for all vector fields X on M. Thus, the vector field V is strictly infinitesimal contact.
We also have from ([10]), the commutation formula

$$(L_V\nabla)(X,Y) = L_V\nabla_X Y - \nabla_X L_V Y - \nabla_{[V,X]}Y. \tag{3.20}$$

Taking $Y = \xi$ in the previous equation and knowing the fact $(L_V\xi) = 0$, (3.19) provide

$$(L_V\nabla)(X,\xi) = 0. \tag{3.21}$$

Now comparision of (3.9) and (3.21) gives

$$(2h + \xi h)QX + \big(4nh - \lambda(\xi h)\big)X = 0. \tag{3.22}$$

On contracting (3.22) and substituting $\lambda = -2n$, it reduces to

$$\big(r + 2n(2n+1)\big)\big(\xi h + 2h\big) = 0.$$

If $(\xi h) = -2h$, then the covariant derivative of the above along the Reeb vector field ξ, give $\xi(\xi h) = 4h$. However, we know that (ξh) is constant and h is non-vanishing non-constant function on M, which is absurd. Hence $r = -2n(2n+1)$. Thus from (3.22), we have M is an Einstein manifold.
Therefore the h-Ricci soliton is trivial with the soliton constant $\lambda = -2n$ and the potential vector field is Killing. □

Theorem 3.2 *Let (M, ϕ, ξ, η, g) be a para-Kenmotsu manifold that admits a non-trivial h-Ricci soliton with a definite signal for h. If the potential vector field V is collinear with the Reeb vector field ξ, then M is an η-Einstein manifold.*

Proof: Since V is collinear with the Reeb vector field ξ, there exists a smooth function μ such that

$$V = \mu\xi. \tag{3.23}$$

On differentiating (3.23) along the arbitrary vector field V on M, we obtain

$$\nabla_X V = (X\mu)\xi + \mu\big(X - \eta(X)\xi\big), \tag{3.24}$$

for all vector fields X on M. By virtue of equation (3.24), the equation (1.4) reduces to

$$\begin{aligned} &2Ric(X,Y) + h\big\{(X\mu)\eta(Y) + 2\mu g(X,Y) - 2\mu\eta(X)\eta(Y)\\ &+(Y\mu)\eta(X)\big\} - 2\lambda g(X,Y) = 0, \end{aligned} \tag{3.25}$$

for all vector fields X and Y on M. Substituting $X = Y = \xi$ in the last equation, we deduce

$$h(\xi\mu) = (\lambda + 2n). \tag{3.26}$$

Again substituting $X = \xi$ in equation (3.25) and using (3.26), we have

$$h(Y\mu) = (\lambda + 2n)\eta(Y), \tag{3.27}$$

for all vector fields Y on M. Taking into account of (3.27), the equation (3.25) reduces to

$$Ric(X,Y) + (\mu h - \lambda)g(X,Y) + (\lambda + 2n - \mu h)\eta(X)\eta(Y) = 0, \tag{3.28}$$

for all vector fields X and Y on M. Now contraction of the preceding equation gives $r = -2n(\mu h - \lambda + 1)$. Using this in (3.28) we obtain

$$Ric(X,Y) = \left(\frac{r}{2n} + 1\right) g(X,Y) - \left(\frac{r}{2n} + (2n+1)\right) \eta(X)\eta(Y),$$

for all vector fields X and Y on M, which show that M is an η-Einstein manifold. □

Theorem 3.3 *Let (M, ϕ, ξ, η, g) be a para-Kenmotsu manifold. If g is a h-Ricci soliton with h having a definite signal, and the soliton vector field V is contact, then M is Einstien manifold and V is strictly infinitesimal contact transformation.*

Proof: On recalling (2.10), the Ricci operator can be expressed as

$$QX = \left(1 + \frac{r}{2n}\right) X - \left(2n + 1 + \frac{r}{2n}\right) \eta(X), \tag{3.29}$$

for all vector fields X on M. On differentiating (3.29) along an arbitrary vector field Y and again contracting along the vector field Y, we obtain,

$$\frac{(n-1)}{2n}(Xr) = \left(-\frac{\xi r}{2n} + 2n\big(2n + 1 + \frac{r}{2n}\big)\right) \eta(X),$$

for all vector fields X on M. Now setting $X = \xi$, the forgoing equation gives

$$\xi r = 4n\left(2n + 1 + \frac{r}{2n}\right).$$

Making use of last two equations, one can deduce

$$Xr = 4n\left(2n + 1 + \frac{r}{2n}\right) \eta(X) \;\; or \;\; Dr = 4n\left(2n + 1 + \frac{r}{2n}\right) \xi. \tag{3.30}$$

From the equations (1.4) and (2.10), we have

$$\frac{h}{2}(L_V g)(X,Y) = \Big(\lambda - 1 - \frac{r}{2n}\Big) g(X,Y) + \Big(2n+1+\frac{r}{2n}\Big)\eta(X)\eta(Y), \tag{3.31}$$

for all vector fields X and Y on M. On differentiating (3.31) along the arbitrary vector field Z and making use of (3.30), we ultimatly obtain

$$\begin{aligned}\frac{h^2}{2}(\nabla_Z L_V g)(X,Y) &= -\Big(\lambda - 1 - \frac{r}{2n}\Big)(Zh)g(X,Y) - \Big(2n+1+\frac{r}{2n}\Big)(Zh)\eta(X)\eta(Y)\\ &\quad + h\Big(2n+1+\frac{r}{2n}\Big)\big\{g(X,Z)\eta(Y) + g(Z,Y)\eta(X)\\ &\quad -2g(X,Y)\eta(Z)\big\},\end{aligned} \tag{3.32}$$

for all vector fields X, Y and Z on M.
Again from the Yano's commutation formula, we have

$$(L_V\nabla_X g - \nabla_X L_V g - \nabla_{[V,X]}g)(Y,Z) = -g\big((L_V\nabla)(X,Y),Z\big) - g\big((L_V\nabla)(X,Z),Y\big),$$

for all vector fields X, Y and Z on M.
Now by a simple calculation and by knowing the fact that $(L_V\nabla)$ is a symmetric tensor of type $(1,2)$, we deduce

$$h^2 g\big((L_V\nabla)(X,Y),Z\big) = \frac{h^2}{2}\big\{(\nabla_X L_V g)(Y,Z) + (\nabla_Y L_V g)(Z,X) - (\nabla_Z L_V g)(X,Y)\big\}, \tag{3.33}$$

for all vector fields X, Y and Z on M. Making use of (3.32) in (3.33) and taking $(Xh) = (\xi h)\eta(X)$, we obtain

$$\begin{aligned}h^2(L_V\nabla)(X,Y) &= \Big(\lambda - 1 - \frac{r}{2n}\Big)\big\{g(X,Y)Dh - (Yh)X - (Xh)Y\big\}\\ &- \Big(2n+1+\frac{r}{2n}\Big)\big\{2h\eta(X)Y + 2h\eta(Y)X + (Xh)\eta(Y)\xi\big\},\end{aligned} \tag{3.34}$$

for all vector fields X and Y on M. On taking covariant differentiation of (3.34) along an arbitrary vector field Z, we have

$$\begin{aligned}2h(Zh)(L_V\nabla)(X,Y) &= -\frac{Zr}{2n}\big\{g(X,Y)Dh - (Yh)X - (Xh)Y\big\}\\ &\quad + \Big(\lambda - 1 - \frac{r}{2n}\Big) g(X,Y)\nabla_z Dh - \frac{Z_r}{2n}\big\{2h\eta(X)Y + 2h\eta(Y)X\\ &\quad + (Xh)\eta(X)\xi\big\} - \Big(2n+1+\frac{r}{2n}\Big)\big\{2(Zh)\eta(X)\eta(Y)\\ &\quad + 2h(\nabla_Z\eta)(X)Y + 2(Zh)\eta(Y)X + 2h(\nabla_Z\eta)(Y)X\\ &\quad + (Xh)(\nabla_Z\eta)(Y)\xi + (Xh)\eta(Y)(\nabla_Z\xi)\big\} - h^2(\nabla_Z L_V\nabla)(X,Y).\end{aligned} \tag{3.35}$$

We know that

$$(L_V R)(X,Y)Z = (\nabla_X L_V\nabla)(Y,Z) - (\nabla_Y L_V\nabla)(X,Z) \tag{3.36}$$

for all vector fields X, Y and Z on M. Making use of (2.4), (2.5) and (3.35), the above equation reduces to,

$$\begin{aligned} h^2(L_V R)(X,Z)Y &= -2h(Xh)(L_V\nabla)(Z,Y) + 2h(Zh)(L_V\nabla)(X,Y) \\ &\quad -\frac{Xr}{2n}\{g(Z,Y)Dh - (Yh)Z\} + \frac{Zr}{2n}\{g(X,Y)Dh - (Yh)X\} \\ &\quad + \left(\lambda - 1 - \frac{r}{2n}\right)\{g(Z,Y)\nabla_X Dh - g(X,Y)\nabla_Z Dh\} \\ &\quad -2h\{\frac{Xr}{2n}\eta(Y)Z - \frac{Zr}{2n}\eta(Y)X\} \\ &\quad - \left(2n + 1 + \frac{r}{2n}\right)\{2(Xh)\eta(Y)Z + 2hg(Y,X)Z \\ &\quad +(Zh)g(Y,X)\xi + (\xi h)\eta(Z)\eta(Y)X - 2(Zh)\eta(Y)X \\ &\quad -2hg(Y,Z)X + 2h\eta(Z)\eta(Y)X - (Xh)g(Y,Z)\xi \\ &\quad -(\xi h)\eta(X)\eta(Y)Z - 2h\eta(X)\eta(Y)Z\}. \end{aligned} \tag{3.37}$$

Substituting ξ for X and Y and taking $(Xh) = (\xi h)\eta(X)$ or $Dh = (\xi h)\xi$, the above equation yields, $h^3 g((L_V R)(X,\xi),\xi,\xi) = 0$. Since $h \neq 0$, we have

$$(L_V R)(X,\xi)\xi = 0. \tag{3.38}$$

Now Lie-differentiating $g(\xi,\xi) = 1$ and using (3.31), we obtain

$$2h\eta(L_V\xi) = -h(\lambda + 2n). \tag{3.39}$$

Plugging $Y = \xi$ in (1.2), we get

$$\frac{h}{2}(L_V\eta)(X) = (\lambda + 2n)\eta(X). \tag{3.40}$$

Again, substitute ξ for Y in (2.6), to obtain $R(X,\xi)\xi = \eta(X)\xi - X$. Operating Lie derivative along the potential vector field V and using (3.39) and (3.40), the above equation gives,

$$\frac{h}{2}(L_V R)(X,\xi)\xi = -(2n + \lambda)\nabla_X\xi. \tag{3.41}$$

Comparing (3.38) and (3.41), we obtain $\lambda = -2n$.
From (3.39), we have $\eta(L_V\xi) = 0$. Since V is contact, we have a smooth function f on M, such that $L_V\xi = f\xi$.
Taking the inner product of the last equation with ξ gives $f = 0$ and $L_V\xi = 0$. With this information in (3.40), we obtain $L_V\eta = 0$, which implies V is strictly infinitesimal contact transformation.
Substituting $Y = \xi$ and using $L_V\xi = 0$ and $L_V\eta = 0$ in (3.20), we have

$$(L_V\nabla)(X,\xi) = 0. \tag{3.42}$$

Plugging $X = \xi$ in (3.34) and comparing with (3.42), we have

$$\left(2n + 1 + \frac{r}{2n}\right)(2n + 1 - 4n)\,\eta(X) = 0.$$

If $r \neq -2n(2n+1)$, then for $h = 1$, we get $2n = 3$, which is absurd for all $n > 1$. Hence $r = -2n(2n+1)$. Substituting this in (3.29), we have $Ric(X,Y) = -2ng(X,Y)$, for all vector fields X and Y on M. Therefore, M is an Einstein manifold. □

4. Gradient h-Ricci soliton on par-Kenmotsu manifold

Theorem 4.1 *Let M be a para-Kenmotsu manifold with the para contact structure (ϕ,ξ,η,g). If the metric g admits the gradient almost h-Ricci soliton, then M is Einstein manifold with Einstien constant $-2n$; otherwise, the potential vector field V is collinear with the Reeb vector field on some open set in M.*

Proof: Let g represent gradient h-Einstein soliton on the para-Kenmotsu manifold. From (1.5), we have

$$h\nabla_Y Du = \lambda Y - QY. \tag{4.1}$$

On differentiating the above equation along an arbitrary vector field X, we have

$$h\nabla_X\nabla_Y Du + (\nabla_X h)\nabla_Y Du = (X\lambda)Y + \lambda(\nabla_X Y) - (\nabla_X Q)Y - Q(\nabla_X Y). \tag{4.2}$$

From (4.2), we compute R as follows:

$$\begin{aligned} hR(X,Y)Du &= (X\lambda)Y - (Y\lambda)X - (\nabla_X Q)Y + (\nabla_Y Q)X \\ &\quad -(Xh)\nabla_Y Du + (Yh)\nabla_X Du. \end{aligned} \tag{4.3}$$

Substituting ξ for X in (4.3) and taking inner product with X, we get

$$\begin{aligned} hg(R(\xi,Y)Du,X) &= (\xi\lambda)g(X,Y) - (Y\lambda)\eta(X) + Ric(X,Y) \\ &\quad +2ng(X,Y) - (\xi h)g(\nabla_Y Du,X) + (Yh)g(\nabla_\xi Du,X). \end{aligned} \tag{4.4}$$

In view of (2.7), the above equation reduces to

$$hg\big(R(\xi,Y)Du,X\big) = h\big\{(\xi u)g(X,Y) - (Yu)\eta(X)\big\}. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\begin{aligned} (\xi\lambda)g(X,Y) - (Y\lambda)\eta(X) + Ric(X,Y) + 2ng(X,Y) - (\xi h)g(\nabla_Y Du,X) \\ + (Yh)g(\nabla_\xi Du,X) = h\big\{(\xi u)g(X,Y) - (Yu)\eta(X)\big\}. \end{aligned} \tag{4.6}$$

Plugging ξ for X, the above equation reduces to

$$(\xi h)\eta(Y) - (Y\lambda) + (Yh)g(\nabla_\xi Du,\xi) - (\xi h)g(\nabla_Y Du,\xi) = h\big\{(\xi u)\eta(Y) - (Yu)\big\}. \tag{4.7}$$

Using the Poincare Lemma, $g(\nabla_X Du,Y) = g(\nabla_Y Du,X)$, the preceding equation becomes

$$(\xi h)Y = (Yh)\xi. \tag{4.8}$$

Use of (4.8) in (4.7) gives

$$\xi(\lambda - hu)\eta(Y) = Y(\lambda - hu), \;\; D(\lambda - hu) = \xi(\lambda - hu)\eta. \tag{4.9}$$

By the use of (4.9) in (4.6) and making use of (4.8), we obtain

$$Ric(X,Y) + 2ng(X,Y) = \xi(\lambda - hu)\big\{\eta(X)\eta(Y) - g(X,Y)\big\}. \tag{4.10}$$

On tracing the above equation, we have

$$\xi(\lambda - hu) = \Big(2n + 1 + \frac{r}{2n}\Big). \tag{4.11}$$

Now we notice from (4.10), that

$$Ric(X,Y) = \Big(1 + \frac{r}{2n}\Big)\,g(X,Y) + \Big(2n + 1 + \frac{r}{2n}\Big)\,\eta(X)\eta(Y). \tag{4.12}$$

Here we substitute $Y = Du$ in the foregoing equation to obtain

$$Ric(X,Du) = \Big(1 + \frac{r}{2n}\Big)\,Xu + \Big(2n + 1 + \frac{r}{2n}\Big)\,(\xi u)\eta(X). \tag{4.13}$$

Suppose X and $Y \in ker\eta$. If we take $(Xh) = (\xi h)\eta(X)$ in (4.3), we obtain

$$hR(X,Y)Du = (X\lambda)Y - (Y\lambda)X - (\nabla_X Q)Y + (\nabla_Y Q)X. \tag{4.14}$$

Contracting the above equation along X, we obtain

$$hRic(Y, Du) = -2n(Y\lambda) + \frac{Yr}{2}. \tag{4.15}$$

Appllying the operator d on (4.9) and combined use of the facts that $d^2 = 0$ and $d\eta = 0$, then from (4.11), we obtain $-dr \wedge \eta = 0$. By the property $2(\omega \wedge \eta) = \omega \otimes \eta - \eta \otimes \omega$, the last equation reduces to

$$dr(X)\eta(Y) - dr(Y)\eta(X) = 0. \tag{4.16}$$

Plugging ξ for Y in (4.16), we have $Xr = (\xi r)\eta(X)$ and tracing (3.2), we obtain $\xi r = -2(r + 2n(2n+1))$. Solving the last two equations, we get

$$Xr = -2\big(r + 2n(2n+1)\big)\eta(X), \ \ or \ \ Dr = -2\big(r + 2n(2n+1)\big)\eta. \tag{4.17}$$

Unifying the equations (4.13) and (4.15), we find

$$\Big(1 + \frac{r}{2n}\Big) Yu + \Big(2n + 1 + \frac{r}{2n}\Big) (\xi u)\eta(Y) = -2n(Y\lambda) + \frac{Yr}{2}. \tag{4.18}$$

For a vector field Y in the distribution $D_\eta = ker\eta$, we have

$$4n^2(Y\lambda) + h(r + 2n)(Yu) = 0. \tag{4.19}$$

Invoking (4.9) and (4.11) in (4.19), we obtain

$$(4n^2 - r - 2n)(Yu) = 0. \tag{4.20}$$

From this we conclude that

$$\big(r + 2n(2n+1)\big)\big(Du - (\xi u)\xi\big) = 0. \tag{4.21}$$

If $r = -2n(2n+1)$, then from equation (4.12), we have M is an Einstein manifold with Einstein constant $-2n$.
If $r \neq -2n(2n+1)$ on some open set θ of M, then $Du = (\xi u)\xi$ on the open set θ. This completes the proof.

□

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