Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.66744

Bi-amalgamations of semiclean rings

Loubna Es-Salhi and Mohamed Chhiti*

ABSTRACT: This paper establishes necessary and sufficient conditions for a bi-amalgamation to inherit the semiclean (resp. UU, resp. periodic) property. Our results generalize previous studies on amalgamations and, generate examples which enrich literature with new and original families of rings satisfying the above mentioned-properties.

Key Words: Bi-amalgamated algebras along ideals, semiclean rings, UU rings, periodic rings.

Contents

1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital.
In 1977, W. K. Nicholson [25] introduced the concept of clean rings as a subclass of exchange rings. He defined a ring R to be clean if for every $a \in R$ there is u a unit in R and e an idempotent in R such that $a = u + e$. Over the last ten to fifteen years there has been an explosion of interest in this class of rings as well as the many generalizations and variations. In 2013, Diesl [15] introduced a new class of rings and called it nil-clean rings. A ring R is said to be nil-clean if every element of R can be written as the sum of a nilpotent and an idempotent. Nil-clean rings are clean, the converse fails, but holds if all units are unipotent. This fact suggest that the class of rings whose all units are unipotent could be studied independently. These rings were introduced by Călugăreanu [4], and were called UU rings. A ring R is said to be periodic if for each $r \in R$, $\{r^n\,;\, n \in \mathbb{N}\}$ is a finite set, equivalently, for every $r \in R$ there exist distinct positive integers n and m such that $r^n = r^m$. This definition is a particular case of the class of rings which are neither commutative nor unitary. Nil-clean rings are periodic. Furthermore, it was proved in [26, Theorem 3] that a ring R is periodic if and only if each element r in R can be written as $r = n + p$, where n is a nilpotent and $p^k = p$ for some positive integer $k \geq 2$. In 2003, Ye [28] introduced a new generalization of clean rings, he defined a ring R to be semiclean if every element of R can be written as the sum of a unit and a periodic element in R. Clean rings are semiclean. Moreover, in 2001, Han and Nicholson [19] showed that the group ring $\mathbb{Z}_{(7)}G$ is not clean, where $\mathbb{Z}_{(7)} = \{m/n \mid m, n \in \mathbb{Z} \textit{ and } gcd(7, n) = 1\}$ and G is a cyclic group of order 3, while Ye [28] proved that $\mathbb{Z}_{(p)}G$ is semiclean for any prime p and for any cyclic group G of order 3. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j')|a \in A, (j, j') \in J \times J'\}$$

called *the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g).*
This construction was introduced and studied by Kabbaj, Louartiti and Tamekkante in [21] as a natural generalization of duplications [6,10,13,14,24,27] and amalgamations [11,12,17].
In [21], the authors showed how these bi-amalgamations arise as pullbacks. Given $f : A \to B$ and $g : A \to C$ two ring homomorphisms and J and J' be two ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$, the bi-amalgamation is determined by the following pullback:

* Corresponding author

Submitted January 19, 2023. Published May 20, 2025
2010 *Mathematics Subject Classification:* Primary 13D05, 13D02, 13B02.

$$\begin{array}{ccc} A \bowtie^{f,g} (J,J') & \xrightarrow{\mu_1} & f(A)+J \\ \mu_2 \downarrow & & \downarrow \alpha \\ g(A)+J' & \xrightarrow{\beta} & \frac{A}{I_0} \end{array}$$

that is

$$R = \alpha \times_{\frac{A}{I_0}} \beta.$$

where μ_1 and μ_2 are the surjection morphisms induced from the canonical surjections of $(f(A)+J) \times (g(A)+J')$ into $f(A)+J$ and $g(A)+J'$, respectively, and $\alpha(f(a)+j) = \bar{a}$ and $\beta(g(a)+j') = \bar{a}$, for each $a \in A$ and $j, j' \in J \times J'$. That is

$$A \bowtie^{f,g} (J,J') = \alpha \times_{\frac{A}{I_0}} \beta.$$

The interest of these bi-amalgamations resides, partly, in their ability to cover several basic constructions in commutative algebra, including classical pullbacks (e.g., $D+M, A+XB[X], A+XB[[X]]$, etc.), Nagata's idealizations, and Boisen-Sheldon's CPI-extensions [3].
Given a ring homomorphism $f : A \to B$ and an ideal J of B, the bi-amalgamation $A \bowtie^{f,i} (J, f^{-1}(J))$ coincides with the amalgamated algebra (introduced and studied by D'Anna, Finocchiaro and Fontana in [11,17]) as the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

When $A = B$ and $f = id_A$, the amalgamated $A \bowtie^{id_A} I$ is called the amalgamation duplication of the ring A along the ideal I and denoted $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [10,14,13]). This construction can be seen as a bi-amalgamation as follows:

$$A \bowtie I = A \bowtie^{id_A, id_A} (I, I)$$

Given a ring A and an A-module E, the set $R := A \propto E$ of pairs (a,x) with pairwise addition and multiplication given by $(a,e)(c,d) = (ac, ad+ec)$ is called the trivial ring extension of A by E (also called idealization of E over A). Considerable work, part of it summarized in Glaz's book [18] and Huckaba's book [20], has been concerned with trivial ring extensions. Let $i : A \hookrightarrow R$ be the canonical embedding. After identifying E with $0 \propto E$, E becomes an ideal of B. According to [11, Remark 2.8]. It is non straightforward but it is known that $A \propto E$ coincides with $A \bowtie^i E$.
In [2] and [22], the authors studied the transfer of periodic (resp. UU, resp. semiclean) property to amalgamations.
This was our motivation to ask the same questions for bi-amalgamated algebras.
In this paper, the ideal of all nilpotent elements (respectively, the mulitiplicative group of units, the set of idempotents elements, the set of potents elements, the Jacobson radical) of the ring R is denoted by $Nil(R)$ (respectively, $U(R)$, $Idem(R)$, $E_\infty(R)$, $Rad(R)$).
At this point, we recall some necessary definitions which will be used in the next section.

Definition 1.1 *1) A ring R is said to be periodic if for each $r \in R$, $\{r^n ; n \in \mathbb{N}\}$ is a finite set, equivalently, for every $r \in R$ there exist distinct positive integers n and m such that $r^n = r^m$.*
2) A ring R is called a UU ring if all units are unipotent, that is $U(R) = Nil(R)+1$, i.e., each unit can be presented as $b+1$, where $b \in Nil(R)$.
3) An element $r \in R$ is said to be (uniquely) semiclean if r can be written (uniquely) in the form $r = u+p$ where $u \in U(R)$ and $p \in Per(R)$. The ring R is said to be (uniquely) semiclean if every element of R is (uniquely) semiclean.

2. Mains results

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. All along this section, $R := A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). We begin with the first result:

Theorem 2.1 *Under the above notation, the following assertions are equivalent:*

1. *R is periodic,*
2. *$f(A) + J$ and $g(A) + J'$ are periodic,*
3. *$(f(A) + J) \times (g(A) + J')$ is periodic.*

Proof: (1)⇒(2): Assume that R is periodic. Using [21, Proposition 4.1 (2)], $f(A)+J$ and $g(A)+J'$ are homomorphic images of R. Then by [2, Proposition 2.1], they are periodic rings.
(2)⇒(3): It follows directly from [2, Lemma 2.7 (2)].
(3)⇒(1): Assume that $(f(A) + J) \times (g(A) + J')$ is a periodic ring. Using [2, Lemma 2.7 (1)], R is a periodic ring since it is a subring of $(f(A) + J) \times (g(A) + J')$. □

The following corollary is an immediate application of Theorem 2.1.

Corollary 2.1 *Under the notation of Theorem 2.1, assume that f and g are surjective. If A is a periodic ring then so is R.*

Proof: Clearly, B and C are periodic rings, then so are $f(A) + J$ and $g(A) + J'$ since f and g are surjective. Consequently, the result follows directly from Theorem 2.1. □

The next result investigates the transfer of the periodic property in bi-amalgamated algebras, in case $J \times J'$ contains periodic elements.

Proposition 2.1 *Under the above notation, assume that f (or g) is injective. Then the following statements are equivalent:*

1. *R is periodic,*
2. *A is periodic and $J \times J' \subseteq Per(B) \times Per(C)$.*

Proof: (1)⇒(2): Assume that R is a periodic ring and let $a \in A$. There are distincts positive integers m and n such that $(f(a), g(a))^m = (f(a), g(a))^n$. Then $f(a^m) = f(a^n)$ and $g(a^m) = g(a^n)$. It follows that $a^m = a^n$ since f (or g) is injective. Therefore A is a periodic ring. Now, consider $(j, j') \in J \times J'$. Since R is a periodic, there are distincts positive integers m and n such that $(j, j')^n = (j, j')^m$. It follows that $j^n = j^m$ and $j'^n = j'^m$, and so $J \times J' \subseteq Per(B) \times Per(C)$, as desired.
(2)⇒(1): Let $a \in A$ and $(j, j') \in J \times J'$. Since A is a periodic ring, then $f(A)$ and $g(A)$ are periodic rings, equivalently, $\{f(a^n) \mid n \in \mathbb{N}\}$ and $\{g(a^n) \mid n \in \mathbb{N}\}$ are finite sets. On the other hand, $\{j^n \mid n \in \mathbb{N}\}$ and $\{j'^n \mid n \in \mathbb{N}\}$ are also finite sets since J and J' satisfied the periodic-like property. Then, there exists a positive integer p such that $\{f(a^n) \mid n \in \mathbb{N}\} = \{1, f(a), \ldots, f(a^p)\}$, $\{g(a^n) \mid n \in \mathbb{N}\} = \{1, g(a), \ldots, g(a^p)\}$, $\{j^n \mid n \in \mathbb{N}\} = \{1, j, \ldots, j^p\}$ and $\{j'^n \mid n \in \mathbb{N}\} = \{1, j', \ldots, j'^p\}$. Let n be a positive integer, by the binomial theorem we get $(f(a) + j)^n = \sum_{m=0}^{n} \binom{n}{m} f(a^m) j^{n-m}$. Thus $(f(a) + j)^n = \sum_{0 \le k,l \le p} r_{k,l} f(a^k) j^l$, where $r_{k,l}$ is a positive integer for each $0 \le k, l \le p$. Let q be the characteristic of $f(A)$, by using [22, Lemma 2.2] we have $q \neq 0$. Thus we can write $r_{k,l} = s_{k,l} q + t_{k,l}$, where $s_{k,l}$ and $t_{k,l}$ are positive integers and $0 \le t_{k,l} \le q - 1$. It follows that $(f(a) + j)^n = \sum_{0 \le k,l \le p} t_{k,l} f(a^k) j^l$ since $q f(a^k) = 0$. Consequently, $\{f(a + j)^n \mid n \in \mathbb{N}\}$ is a finite set. By similar arguments as above, we can show that $\{g(a + j')^n \mid n \in \mathbb{N}\}$ is a finite set, and thus $\{(f(a) + j, g(a + j')^n \mid n \in \mathbb{N}\}$ is also a finite set. Consequently, R is a periodic ring, as desired. □

The following corollary is a particular case of Proposition 2.1.

Corollary 2.2 *Under the notation of Theorem 2.1, assume that A is a periodic ring. If ($J \subseteq Nil(B)$ or $J \subseteq Idem(B)$ or $J \subseteq E_\infty(B)$) and ($J' \subseteq Nil(C)$ or $J' \subseteq Idem(C)$ or $J' \subseteq E_\infty(C)$), then R is a periodic ring.*

Proof: Note that every nilpotent (resp., idempotent, resp., potent) element is periodic. Then the proof follows directly from Proposition 2.1. □

Theorem 2.1 and Proposition 2.1 recover known results for the special case of amalgamated algebras and duplications, as recorded in the following corollaries.

Corollary 2.3 *([2, Theorem 2.6] and [23, Theorem 2.6])*
Let $f : A \to B$ and let J be an ideal of B. Then the following statements are equivalant:

1. *$A \bowtie^f J$ is periodic,*
2. *A and $f(A) + J$ are periodic,*
3. *$A \times (f(A) + J)$ is periodic,*
4. *A is periodic and $J \subseteq Per(B)$.*

Proof: (1) ⇔ (2) ⇔ (3): Obvious.
(1) ⇔ (4): Recall that $A \bowtie^f J = A \bowtie^{i,f} (f^{-1}(J), J)$. Thus, using Proposition 2.1, $A \bowtie^f J$ is periodic if and only if

1. A is periodic,
2. $J \subseteq Per(B)$ and $f^{-1}(J) \subseteq Per(A)$.

Since A is periodic the condition $f^{-1}(J) \subseteq A$ is evident. Hence, $A \bowtie^f J$ is periodic if and only if A is periodic and $J \subseteq Per(B)$. □

Corollary 2.4 *([23, Corollary 2.8])*
Let A be a ring and I an ideal of A. Then $A \bowtie I$ is periodic if and only if so is A.

First, as an illustrative example for Theorem 2.1, we provide a non-nil-clean periodic ring which arises as a bi-amalgamation.

Example 2.1 Let $A := \mathbb{Z}/2\mathbb{Z}$, $B := \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$, $C := \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $J := 0\times\mathbb{Z}/3\mathbb{Z}$ and $J' := 0\times\mathbb{Z}/4\mathbb{Z}$. Consider the two ring homomorphisms defined by: $f(a) = (a, 0)$ and $g(a) = (a, 0)$ for all $a \in A$. Obviously, $f^{-1}(J) = g^{-1}(J') = 0$. Then,

1. R is periodic .
2. R is not nil-clean.

Proof: (1) Clearly, $f(A) + J = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = B$ and $g(A) + J' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = C$ are periodic rings since they are finite product of periodic rings. Hence, by using Theorem 2.1, it follows that R is periodic.
(2) By [8, Theorem 2.1], R is not nil-clean since $f(A) + J$ is not nil-clean. □

The following example illustrates corollary 2.2.

Example 2.2 Let $(A_1, \mathfrak{m}_1)$ be a periodic ring with ${\mathfrak{m}_1}^2 = 0$. Let $(A, \mathfrak{m}) := (A_1 \propto E_1)$ be the trivial ring extension of A_1 by a nonzero $A_1/\mathfrak{m}_1$-vector space E_1. (For instance, $A_1 := \mathbb{Z}/4\mathbb{Z}$, $\mathfrak{m}_1 := 2\mathbb{Z}/4\mathbb{Z}$ and $E_1 := \mathbb{Z}/2\mathbb{Z}$). Let $B := A \propto E$ be the trivial ring extension of A by a nonzero $A/\mathfrak{m}$-vector space E. Let

$$f : \begin{array}{ccc} A & \to & B \\ (a, e) & \mapsto & ((a, e), 0) \end{array}$$

be an injective ring homomorphism and $J := \mathfrak{m} \propto E$ be the maximal ideal of B. Let $C := A_1$ and let

$$g: \begin{array}{ccc} A & \rightarrow & C \\ (a,e) & \mapsto & a \end{array}$$

be a surjective ring homomorphism and $J' := \mathfrak{m}_1$ be the maximal ideal of C. Obviously, $f^{-1}(J) = g^{-1}(J') = \mathfrak{m}_1 \propto E_1$. Then R is periodic.

Proof: By [22, Theorem 2.1], A is periodic. One can easily check that $J^2 = 0$ and $J'^2 = 0$. Hence, by using Corollary 2.2, it follows that R is periodic. □

The next example illustrates corollary 2.3.

Example 2.3 Let (p,q) be a pair of distinct prime integers, $R = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, and let E be an R-module. Let $A = R^{\mathbb{N}}$ be the set of all sequences of elements of R. Consider the natural ring homomorphisms $f: A \hookrightarrow A \propto E$ and let $J := 0 \propto E$. Then, $A \bowtie^f J$ is a periodic ring.

Proof: By [22, Example 2.3], A and $f(A) + J = A \propto E$ are periodic rings. Consequently, $A \bowtie^f J$ is a periodic ring in view of Corollary 2.3.

□

Theorem 2.2 *Under the above notation, we have:*

1. *If $f(A) + J$ and $g(A) + J'$ are UU rings, then so is R.*
2. *Assume that $J' \subseteq Nil(C)$. Then, R is UU if and only if $f(A) + J$ is UU.*
3. *Assume that $J \subseteq Nil(B)$. Then, R is UU if and only if $g(A) + J'$ is UU.*

Proof: (1) Assume that $f(A) + J$ and $g(A) + J'$ are UU rings. According to [21, Proposition 3.1], the bi-amalgamation is detemined by the following pullback

$$\begin{array}{ccc} R & \xrightarrow{p_B} & f(A)+J \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle \alpha} \\ g(A)+J' & \xrightarrow{\beta} & \frac{A}{I_0} \end{array}$$

that is

$$R = \alpha \times_{\frac{A}{I_0}} \beta.$$

Clearly, $P_B(R) = f(A) + J$ and $P_C(R) = g(A) + J'$. By using [2, Theorem 4.2 (3)], it follows that R is UU, as desired.

(2) Assume that R is UU and $J' \subseteq Nil(C)$. Thus, in view of [9, page 451, (2.3)], $\frac{R}{0 \times J'}$ is also UU since we have $0 \times J' \subseteq Rad(R)$. Then in accordance with [21, Proposition 4.1 (2)], it follows that $f(A) + J$ is UU. The sufficiency follows at once by [9, Theorem 2.4 (1)] and [21, Proposition 4.1 (2)] taking into acount that $0 \times J'$ is a nil ideal of R.

(3) The proof is similar to (2) above.

□

For the special case of amalgamations, we have:

Corollary 2.5 *Let $f: A \rightarrow B$ be a ring homomorphism and J an ideal of B. Then,*

1. *If A and $f(A) + J$ are UU, then $A \bowtie^f J$ is UU.*
2. *$A \bowtie^f J$ is UU if and only if so is A and $J \subseteq Nil(B)$.*

Remark 2.1

1. Assume that f (or g) is injective. If R is UU, then A is also UU.
2. If $f(A)+J$ and $g(A)+J'$ are UU rings, then so is R, by Theorem 2.2 (1). But the converse is not true in general. A counter-example (for the special case of amalgamated algebras) is given below.

Example 2.4 Let $A := \mathbb{Z}/2\mathbb{Z}$, $B := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, $f : A \to B$ be the ring homomorphism such that $f(a) = (a, 0)$. Clearly $\mathbb{Z}/2\mathbb{Z}$ is UU. If we set $J := 0 \times 2\mathbb{Z}/8\mathbb{Z}$, then $J \subseteq Nil(B)$, and thus $A \bowtie^f J$ is UU (by Corollary 2.5), but $f(A) + J = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is not UU since $\mathbb{Z}/8\mathbb{Z}$ is not UU (by [9, page 451, (2.2)]).

Next, as an illustrative example for Corollary 2.5, we provide a non-nil-clean UU ring arising as an amalgamation.

Example 2.5 Let A be the $\mathbb{F}_2$-algebra generated by x, y with the single relation $x^2 = 0$, E un A-module (For instance, $E := \mathbb{F}_2[y]$) and, $B := A \propto E$. Consider $f : A \to B$ defined by $f(a) = (a, 0)$ and $J := (F_2x + xAx) \propto E$ an ideal of B. Then,

1. $A \bowtie^f J$ is UU.
2. $A \bowtie^f J$ is not nil-clean.

Proof: (1) In view of [9, Example 2.5], A is a UU ring and J is a nil ideal of B since $\mathbb{F}_2x + xAx$ is a nil ideal of A. Then $A \bowtie^f J$ is a UU ring (by Corollary 2.5(2)).
(2) It is easy to check that $y - y^2$ is not nilpotent in A. Then A is not nil clean and so $A \bowtie^f J$ is not nil clean (by [1, Theorem 2.1]).

□

Proposition 2.2 *Under the above notation, assume that for every $u \in U(A)$, $p \in Per(A)$ and $(j, j') \in J \times J'$, either $(f(u)+j, g(u)+j') \in U(B \times C)$ (for example, if $J \times J' \subseteq Rad(B \times C)$) or $(f(p)+j, g(p)+j') \in Per(B) \times Per(C)$ (for example, if $J \times J' \subseteq Per(B) \times Per(C)$). If A is semiclean, then R is semiclean. The converse holds if f (or g) is injective.*

Proof: Let $r = (f(a) + j, g(a) + j') \in R$ and write $a = u + p$ where $u \in U(A)$ and $p \in Per(A)$. Then if $(f(u) + j, g(u) + j') \in U(B \times C)$, r is written as: $r = (f(p), g(p)) + (f(u) + j, g(u) + j')$ and if $(f(p) + j, g(p) + j') \in Per(B \times C)$, r is written as: $r = (f(p) + j, g(p) + j') + (f(u), g(u))$. Then, the conclusion follows directly from the assumptions. Conversely, assume that R is semiclean and f is injective. Let $a \in A$. we can write $(f(a), g(a)) = (f(p), g(p)) + (f(u), g(u))$ where, $(f(p), g(p))$ and $(f(u), g(u))$ are respectively, periodic and invertible elements in R. Thus, there exists $v \in U(A)$ such that $(f(u), g(u))(f(v), g(v)) = (1, 1)$. Which implies that $uv = 1$ since f is injective. Consequently, u is a unit in A. Furthermore, $(f(p), g(p))$ is periodic, then there ar distincts positive integers m and n such that $(f(p), g(p))^n = (f(p), g(p))^m$. Which implies that $p^n = p^m$ since f is injective. Thus, p is periodic. Finally, a is a semiclean element, writing that $a = u + p$. Consequently, A a semiclean ring. □

For the special case of amalgamations, we obtain:

Corollary 2.6 *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B.*

1. *Assume that $J \subseteq Rad(B)$. Then $A \bowtie^f J$ is a semiclean ring if and anly if A is a semiclean ring.*
2. *Assume that $J \subseteq Per(B)$. Then $A \bowtie^f J$ is a semiclean ring if and anly if A is a semiclean ring.*

For duplications, we have:

Corollary 2.7 *Let A be a ring and I an ideal of A such that $I \subseteq Rad(A)$. Then, $A \bowtie I$ is semiclean if and only if A is semiclean.*

The following result is an immediate application of Corollary 2.6 (1) and Corollary 2.7.

Corollary 2.8 *Under the notation of Corollary 2.6 and Corollary 2.7, assume that* $(B, \mathfrak{m}_B)$ *and* $(A, \mathfrak{m}_A)$ *are local rings. Then:*

1. $A \bowtie^f J$ *is a semiclean ring if and anly if* A *is a semiclean ring.*

2. $A \bowtie I$ *is semiclean if and only if* A *is semiclean.*

Proof: This follows from Corollary 2.6 and Corollary 2.7 and the fact that $J \subseteq \mathfrak{m}_B = Rad(B)$ and $I \subseteq \mathfrak{m}_A = Rad(A)$. $\square$

In what follows, we show that the semicleanness of R depends to the choice of f and g.

Proposition 2.3 *Under the above notation, assume that* A *is semiclean and* $\frac{A}{I_0}$ *is uniquely semiclean. Then* R *is semiclean if and only if* $f(A)+J$ *and* $g(A)+J'$ *are semiclean.*

Proof: Assume that R is semiclean. In view of [21, Proposition 4.1 (2)], we have the following isomorphism of rings $\frac{R}{0\times J'} \cong f(A)+J$ and $\frac{R}{J\times 0} \cong g(A)+J'$. Then, $f(A)+J$ and $g(A)+J'$ are semiclean since every homomorphic image of a semiclean ring is semiclean [28, Lemma 2.1]. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are semiclean rings and consider $a \in A$ and $(j,j') \in J \times J'$. Since A is semiclean, we can write $a = u + p$ where $(u,p) \in U(A) \times Per(A)$. On the other hand, since $f(A)+J$ is semiclean, $f(a)+j = f(x)+k_1+f(y)+k_2$ with $(f(x)+k_1, f(y)+k_2) \in U(f(A)+J) \times Per(f(A)+J)$. It is clear that $\overline{f(x)}, \overline{f(u)} \in U(\frac{f(A)+J}{J})$ and $\overline{f(y)}, \overline{f(p)} \in Per(\frac{f(A)+J}{J})$, and we have $\overline{f(a)} = \overline{f(u)} + \overline{f(p)} = \overline{f(x)} + \overline{f(y)}$. Thus, $\overline{f(x)} = \overline{f(u)}$ and $\overline{f(y)} = (\overline{f(p)}$ since $\frac{f(A)+J}{J} \cong \frac{A}{I_0}$ is uniquely semiclean. Consider $\tilde{k_1}$, $\tilde{k_2} \in J$ such that $f(x) = f(u)+\tilde{k_1}$ and $f(y) = f(p)+\tilde{k_2}$. By similar argument to above, there exists $\bar{k'_1}$, $\bar{k'_2} \in J'$ such that $g(x') = g(u)+\bar{k'_1}$ and $g(y') = g(p)+\bar{k'_2}$. Thus we have, $(f(a)+j, g(a)+j') = (f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}_1+k'_1)+(f(p)+\tilde{k_2}+k_2, g(p)+\bar{k'_2}+k'_2)$. It is easy to see that, $(f(p)+\tilde{k_2}+k_2, g(p)+\bar{k'_2}+k'_2)$ is a Periodic element of R. It suffices to show that $(f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}_1+k'_1) \in U(R)$. Indeed, $(f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}+k'_1) \in U(f(A)+J\times g(A)+J')$ then there exists $(f(r)+\tilde{k}, g(s)+\bar{k}$ such that $((f(u)+\tilde{k_1}+k_1)(f(r)+\tilde{k}), (g(u)+\bar{k'_1}+k'_1)(g(s)+\bar{k'})) = (1,1)$. Thus, $(\overline{f(u)f(r)}, \overline{g(u)g(s)}) = (\bar{1},\bar{1})$. Then, $(\overline{f(r)}, \overline{g(s)}) = (\overline{f(u^{-1})}, \overline{g(u^{-1})})$. So, there exists $(\tilde{k_0}, \bar{k_0}) \in J \times J'$ such that $(f(r), g(s)) = (f(u^{-1})+\tilde{k_0}, g(u^{-1})+\bar{k_0})$. Hence, $(f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}+k'_1)((f(u^{-1})+\tilde{k_0}+\tilde{k}, g(u^{-1})+\bar{k_0}+\bar{k})) = (f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}+k'_1)(f(r)+\tilde{k}, g(s)+\bar{k}) = (1,1)$. It follows that, $(f(u)+\tilde{k_1}+k_1, g(u)+\bar{k'_1}+k'_1) \in U(R)$. Thus, R is semiclean.

$\square$

In Proposition 2.2, the condition that $(f(u)+j, g(u)+j') \in U(B\times C)$ (or $(f(p)+j, g(p)+j') \in Per(B)\times Per(C)$) for every $u \in U(A)$, $p \in Per(A)$ and $(j,j') \in J\times J'$, is a necessary condition for R to be semiclean.
The next example illustrates the case where none of the conditions in Proposition 2.2 is satisfied.

Example 2.6 Let G be a cyclic group of order 3 and p any prime number. set $A := \mathbb{Z}_{(p)}G$, $B = C := A[X]$, $J = J' := (X)$ and $f = g : A \hookrightarrow A[X]$. Then,

1. $\exists u\ U(A)$ and $(j,j') \in J \times J'$ such that $(f(u)+j, g(u)+j') \notin U(B\times C)$.

2. $\exists p\ Per(A)$ and $(j,j') \in J \times J'$ such that $(f(p)+j, g(p)+j') \notin Per(B)\times Per(C)$.

3. A is semiclean.

4. R is not semiclean.

Proof: (1) $(f(1)+X, g(1)+X) = (1+X, 1+X) \notin U(B\times C)$.
(2)$(f(0)+X, g(0)+X) = (X, X) \notin Per(B)\times Per(C)$.
(3) By [28, Theorem 3.1].
(4) By [28, Example 3.2], $f(A)+J = A[X]$ is not semiclean. Hence, R is not semiclean (by Proposition 2.3). □

As an illustrative example for Proposition 2.2, we provide a non-clean (resp. non-reduced, resp. non-local)-semiclean ring which arises as a bi-amalgamation.

Example 2.7 Let A be a semiclean ring that is not clean (For instance, $A := \mathbb{Z}_{(7)}G$, where G is a cyclic group of order 3), E a nonzero un A-module, $B := A \propto E$, and $C := A[[X]]$. Consider the natural ring homomorphisms $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ and let $J := 0 \propto E$. Clearly, $f^{-1}(J) = g^{-1}(0) = 0$. Then:

1. R is semiclean.

2. R is not clean (respectively, not local, not reduced).

Proof: (1) By Proposition 2.2, R is semiclean since $J\times J' \subseteq Rad(B\times C)$ and $A = \mathbb{Z}_{(7)}G$ is semiclean by [28, Theorem 3.1].
(2) By [5, Proposition 2.1], R is not clean since $f(A)+J = (A\propto 0)+(0\propto E) = A\propto E$ is not clean (by [7, Corollary 2.12]), since A is not clean by [19, Example 1]. However, R is not reduced by [21, Proposition 4.7 (d)] since $J\cap Nil(B) = J \neq 0$. Finaly R is not local since $f(A)+J = A\propto E$ is not local (by [21, Proposition 5.4]), since A is not local. □

The next examples illustrate Corollary 2.6.

Example 2.8 Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B. Set the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(0) \in A\}$ of the ring of power series $B[[X]]$. Then, $A + XB[[X]]$ is semiclean if and only if A is semiclean.

Proof: It follows from [11, Example 2.5] and Corollary 2.6. □

Example 2.9 Let T be a ring and $J \subseteq Rad(T)$ an ideal of T and let D be a subring of T such that $J\cap D = (0)$. The ring $D+J$ is semiclean if and only if D is semiclean.

Proof: It follows from [11, Proposition 5.1 (3)] and Corollary 2.6. □

Before we announce the last main result in this section, we recall the following lemma.

Lemma 2.1 *Let R be a commutative ring. Then R is periodic (resp. UU) if and only if $R/Nil(R)$ is periodic (resp. UU).*

Proof: Follows from [2, Proposition 2.1] (resp. [9, Theorem 2.4 (1)]).

□

Theorem 2.3 *Under the above notation: Set $\overline{A} = \frac{A}{Nil(A)}$, $\overline{B} = \frac{B}{Nil(B)}$, $\overline{C} = \frac{C}{Nil(C)}$, $\pi_B : B \to \overline{B}$, $\pi_C : C \to \overline{C}$, the canonical projections, $\overline{J} = \pi_B(J)$ and $\overline{J'} = \pi_C(J')$.*
Let $\overline{f} : \overline{A} \to \overline{B}$, $\overline{g} : \overline{A} \to \overline{C}$ be the ring homomorphisms defined by setting: $\overline{f}(\overline{a}) = \overline{f(a)}$ and $\overline{g}(\overline{a}) = \overline{g(a)}$.
Set $\overline{R} = \overline{A} \bowtie^{\overline{f},\overline{g}} (\overline{J}, \overline{J'})$. Then, $\overline{R}$ is periodic (resp. UU) if and only if R is periodic (resp. UU).

Proof:

Consider the map:

$$\begin{array}{cccc} \psi : & \frac{\frac{R}{Nilp(R)}}{(f(a)+j, g(a)+j')} & \to & \overline{R} \\ & & \mapsto & (\overline{f}(\overline{a})+\overline{j}, \overline{g}(\overline{a})+\overline{j'}) \end{array}$$

It is easy to show that ψ is a ring isomorphism. Hence, the desired result is obtained directly from Lemma 2.1.

□

Corollary 2.9 *Under the above notation, assume that $J \times J' \subseteq Nil(B \times C)$ and f (or g) is injective. Then R is periodic (resp. UU) if and only if A is periodic (resp. UU).*

Proof: Preserving the same notation of Theorem 2.3, assume that f is injective and $J \times J' \subseteq Nil(B \times C)$. It follows that $\overline{f}$ is injective and $\overline{J} = (0)$ and $\overline{J'} = (0)$. Thus, $\overline{R} \cong \overline{A}$ (by [21, Proposition 4.1 (3)]). Combining Lemma 2.1 and Theorem 2.3, we get the desired result. □

References

1. C. Bakkari and M. Es-Saidi, *Nil-clean property in amalgamated algebras along an ideal*, Annali dell'universita di ferrara, (2018).
2. C. Bakkari, M. Es-Saidi, N. Mahdou et al, *Extension of semiclean rings*, Czech Math J 72, 461-476, (2022).
3. M. B. Boisen and P. B. Sheldon, *CPI-extension: overrings of integral domains with special prime spectrum*, Can. J. Math. 29, 722-737, (1977).
4. G. Călugăreanu, *UU rings*, Carpathian J. Algebra Appl. 31, 157-163, (2015).
5. M. Chhiti and L. Es-Salhi, *Clean-like properties in bi-amalgamation algebras*, São Paulo Journal of Mathematical Sciences, 1-8, (2022).
6. M. Chhiti, M. Jarrar, S. Kabbaj and N. Mahdou, *Prüfer conditions in an amalgamated duplication of a ring along an ideal*, Comm. Algebra 43(1), 249-261, (2015).
7. M. Chhiti, N. Mahdou and M. Tamekkante, *Clean property in amalgamated algebras along an ideal*, Hacettepe J. Math. Stat. 44(1), 41-49, (2015).
8. M. Chhiti and S. Moindze, *Nil-clean and weakly nil-clean properties in Bi-Amalgamated algebras along ideals*, Palestine Journal of Mathematics 11(2), 551-560, (2022).
9. P. V. Danchev and T.Y. Lam, *Rings with unipotent units*, Publ. Math. Debrecen 88, 449-466, (2016).
10. M. D'Anna, *A construction of Gorenstein rings*, J. Algebra 306(2), 507-519, (2006).
11. M. D'Anna, C. A. Finacchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Comm Algebra and Aplications, Walter De Gruyter, 241-252, (2009).
12. M. D'Anna, C. A. Finacchiaro, and M. Fontana, *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Appl. Algebra 214, 1633-1641, (2010).
13. M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, Journal of Algebra and its Applications 6(3), 443-459, (2007).
14. M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. 45(2), 241-252, (2007).
15. A. J. Diesl, *Nil clean rings*, J. Algebra 383, 197-211, (2013).
16. A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory, a survey*, Moroccan Journal of Algebra and Geometry with Applications 1(1), 139-182, (2022).
17. C. Finocciaro and M. Fontana, *Prüfer-like conditions on an amalgamated algebra along an ideal*, Houston J. Math. 40(1), 63-79, (2014).
18. S. Glaz, *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics 1371, (1989).
19. J. Han and W. K. Nicholson, *Extensions of clean rings*, Communication in Algebra 29(6), 2589-2595, (2001).
20. J. A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York-Basel, 1998.
21. S. Kabbaj, K. Louartiti and M. Tamekkante, *Bi-amalgamated algebras along ideals*, J. Commut. Algebra, 65-87, (2017).

22. M. Kabbour, *Trivial ring extensions and amalgamations of periodic rings*, J. Commut. Algebra, 65-87, (2017).

23. D. Khurana, *Lifting potent elements modulo nil ideals*, Journal of Pure and Applied Algebra 225(11), 106762, (2021).

24. H. Maimani and S. Yassemi, *Zero-divisor graphs of amalgamated duplication of a ring along an ideal: the basic properties*, J. Pure Appl. Algebra 212(1), 168-174, (2008).

25. W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. 229, 278-279, (1977).

26. M. Ôhori, *On strongly π-regular rings and periodic rings*, Math. J. Okayama Univ. 27, 49-52, (1985).

27. J. Shapiro, *On a construction of Gorenstein rings proposed by M. D'Anna*, J. Algebra 323(4), 1155-1158, (2010).

28. Y. Ye, *Semiclean rings*, Comm. Alg. 31, 5609-5625, (2003).

Loubna Es-Salhi,
Department of Mathematics,
Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez,
Morocco.
E-mail address: loubna.essalhi@usmba.ac.ma

and

Mohamed Chhiti,
Faculty of Economics and Social Sciences of Fez, University S.M. Ben Abdellah Fez,
Morocco.
E-mail address: chhiti.med@hotmail.com