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Primal Topological Spaces

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ABSTRACT: The purpose of this paper is to introduce a new structure called primal. Primal is the dual structure of grill. Like ideal, the dual of filter, this new structure also generates a new topology named primal topology. We introduce a new operator using primal, which satisfies Kuratowski closure axioms. Mainly, we prove that primal topology is finer than the topology of a primal topological space. Also, we provide the structure of the base of primal topology and prove other fundamental results related to this new structure. Furthermore, we not only discuss some of this new structure's properties but also enrich it with many examples.

Key Words: Primal, grill, primal topological space, Kuratowski closure axioms, base.

Contents

1. Introduction

Topology is one of the branches of mathematics which is highly applicable [21]. Due to its applicability in both science and social science, several new ideas have been developed in topology with classical structures. Kuratowski introduced the idea of ideal from filter [22]. One may consider ideal as the dual of filter. Similarly, one of the classical structures of topology is grill. The definition of grill was introduced by Chóquet [15] in the year 1947. Thron [5] introduced proximity structures in grills. In 1977, Chattopadhyay and Thron [3] extended ideas of closure space with grills. Moreover, Chattopadhyay et al. [4] extended ideas of grills to study merotopic spaces. Since then, the structure of grills has been widely used in topology. Roy and Mukherjee [6,7,8] studied various topological properties with grills. Operators based on grills were introduced by Roy et al. [9], Nasef and Azzam [13], and many others. Modak [10,11] studied grill-filter space and related properties. Hosny [12] studied grill structures in δ-set. Cluster systems via grills were studied in [14]. Moreover, various advanced results with grills were studied in [16,17,18,19,20] and many others. But, it is important to note that the literature on grill structures is less in comparison to filter, ideal, etc. Moreover, interdisciplinary applications of grills are rare to be found. Janković and Hamlett [23] introduced a new topological space using ideal from a given topology of a topological space. Since primal is the dual structure of grill, thus we are motivated by Janković and Hamlett [23] to introduce a new topology using primal.

In this paper, we introduce the dual structure of a grill named 'primal'. Moreover, we introduce a new topology named 'primal topology' and study several fundamental properties. Moreover, this paper is extended to [1], where several new topological operators are introduced in primal topological spaces. In 1990, Isham [26] connected general topology with quantum topology. In this paper, we also observe some scopes of quantum behaviors of a new structure $(.)^\diamond$ introduced in Definition 4.1. Similarly, uncertain behaviors can be observed in Theorem 4.19 and results next to it. Best to our knowledge, this paper is the first paper in the literature of topology to introduce the dual structure of grill. Thus, we are confined to study fundamental results only related to primal. Also, Al-Omari et al. [2] have studied this new

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notion from different directions.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (briefly, X and Y) represent topological spaces unless otherwise stated. For any subset A of a topological space X, $cl(A)$ and $int(A)$ denote closure and interior of A, respectively. The powerset of a set X will be denoted by 2^X. The family of all open neighborhoods of a point x of a topological space (X,τ) is denoted by $\tau(x)$. Now, we procure the following definition of the grill:

Definition 2.1. *[15] A family $\mathcal{G} \subseteq 2^X$ is called a grill on X if $\mathcal{G}$ satisfies the following conditions:*

(1) $\emptyset \notin \mathcal{G}$,

(2) *if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$,*

(3) *if $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

Since primal is the dual structure of grill, thus we skip to add other results on grills, otherwise it will unnecessarily increase the volume of this paper.

3. Primal and a new topological structure

In this section, we introduce a new structure in topology. This new structure is called primal. Primal is the dual structure of grill. Now, we have the following definition of primal:

Definition 3.1. *Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is called a primal on X if it satisfies the following conditions:*

(i) $X \notin \mathcal{P}$,

(ii) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$,

(iii) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

Corollary 3.2. *Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is a primal on X if and only if it satisfies the following conditions:*

(i) $X \notin \mathcal{P}$,

(ii) if $B \notin \mathcal{P}$ and $B \subseteq A$, then $A \notin \mathcal{P}$,

(iii) if $A \notin \mathcal{P}$ and $B \notin \mathcal{P}$, then $A \cap B \notin \mathcal{P}$.

Now, let us consider the following two examples:

Example 3.3. *Let $X = \{a,b\}$. Then, all primals defined on X are $\mathcal{P}_1 = \emptyset$, $\mathcal{P}_2 = \{\emptyset, \{a\}\}$, $\mathcal{P}_3 = \{\emptyset, \{b\}\}$ and $\mathcal{P}_4 = \{\emptyset, \{a\}, \{b\}\}$.*

Example 3.4. *Let X be a non-empty set. It is not difficult to see that the family $\mathcal{P} = 2^X \setminus \{X\}$ is a primal on X, where 2^X denotes the powerset of X.*

Proposition 3.5. *Let X be a non-empty set. Then,*

a) the family $\mathcal{P}_1 = \{A \subseteq X : |A^c| \geq \aleph_0\}$ is a primal on X, where $\aleph_0$ is the lowest infinite cardinal number and $|A^c|$ is the cardinality of A^c.

b) the family $\mathcal{P}_2 = \{A \subseteq X : |A^c| > \aleph_0\}$ *is a primal on* X*, where* $\aleph_0$ *is the lowest infinite cardinal number and* $|A^c|$ *is the cardinality of* A^c*.*

Proof. $a)$ We will just show that $\mathcal{P}_1$ is a primal on X. It can be similarly shown that $\mathcal{P}_2$ is also a primal on X. If X is finite, then $\mathcal{P}_1 = \emptyset$ which is a primal. Now, let X be an infinite set.

(i) Since $|X^c| = |\emptyset| = 0 \not\geq \aleph_0$, we have $X \notin \mathcal{P}_1$.

(ii) Let $A \in \mathcal{P}_1$ and $B \subseteq A$. Then, $|A^c| \geq \aleph_0$ and $A^c \subseteq B^c$. Therefore, $|B^c| \geq |A^c| \geq \aleph_0$ which means $B \in \mathcal{P}_1$.

(iii) Let $A \notin \mathcal{P}_1$ and $B \notin \mathcal{P}_1$. Then, $|A^c| < \aleph_0$ and $|B^c| < \aleph_0$ and so $|(A \cap B)^c| = |A^c \cup B^c| \leq |A^c| + |B^c| < \aleph_0 + \aleph_0 = \aleph_0$. Hence, $A \cap B \notin \mathcal{P}_1$.

$b)$ The proof can be obtained similarly to the proof of (a). □

Theorem 3.6. *Let* $\mathcal{G}$ *be a grill on* X*. Then,* $\{A \subseteq X : A^c \in \mathcal{G}\}$ *is a primal on* X*.*

Proof. Let $\mathcal{G}$ be a grill on X and $\mathcal{P} = \{A \subseteq X : A^c \in \mathcal{G}\}$. Then, we are to show that $\mathcal{P}$ is primal.

(i) Since $\emptyset \notin \mathcal{G}$, thus $X \notin \mathcal{P}$.

(ii) Let $A \in \mathcal{P}$ and $B \subseteq A$. Then, $A^c \subseteq B^c$. Since $A^c \in \mathcal{G}$, thus $B^c \in \mathcal{G}$. Hence, $B \in \mathcal{P}$.

(iii) Let $A \cap B \in \mathcal{P}$. Then, $A^c \cup B^c = (A \cap B)^c \in \mathcal{G}$. Therefore, we get $A^c \in \mathcal{G}$ or $B^c \in \mathcal{G}$. Hence, $A \in \mathcal{P}$ or $B \in \mathcal{P}$. Thus, $\mathcal{P}$ is a primal on X. □

Theorem 3.7. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be two primals on* X*. Then,* $\mathcal{P} \cup \mathcal{Q}$ *is a primal on* X*.*

Proof. (i) Since $\mathcal{P}$ and $\mathcal{Q}$ are two primals on X, thus we get $X \notin \mathcal{P}$ and $X \notin \mathcal{Q}$. Hence, $X \notin \mathcal{P} \cup \mathcal{Q}$.

(ii) Let $A \in \mathcal{P} \cup \mathcal{Q}$ and $B \subseteq A$. Then, $A \in \mathcal{P}$ or $A \in \mathcal{Q}$. Then, $B \in \mathcal{P}$ or $B \in \mathcal{Q}$. It yields $B \in \mathcal{P} \cup \mathcal{Q}$.

(iii) Let $A \cap B \in \mathcal{P} \cup \mathcal{Q}$. Then, $A \cap B \in \mathcal{P}$ or $A \cap B \in \mathcal{Q}$. If $A \cap B \in \mathcal{P}$, then either $A \in \mathcal{P}$ or $B \in \mathcal{P}$. Again, if $A \cap B \in \mathcal{Q}$, then either $A \in \mathcal{Q}$ or $B \in \mathcal{Q}$. Then, obviously $A \in \mathcal{P} \cup \mathcal{Q}$ or $B \in \mathcal{P} \cup \mathcal{Q}$.

Thus, $\mathcal{P} \cup \mathcal{Q}$ is a primal on X. □

Corollary 3.8. *It is clear that* 2^n *primals can be written on a set with* n *elements.*

Remark 3.9. *The intersection of two primals defined on* X *need not to be a primal on* X *as shown by the following example:*

Example 3.10. *Let* $X = \{a, b\}$ *with primals* $\mathcal{P} = \{\emptyset, \{a\}\}$ *and* $\mathcal{Q} = \{\emptyset, \{b\}\}$ *on* X*. Then,* $\mathcal{P} \cap \mathcal{Q} = \{\emptyset\}$ *is not a primal on* X *since* $\{a\} \cap \{b\} = \emptyset \in \mathcal{P} \cap \mathcal{Q}$*, but neither* $\{a\} \in \mathcal{P} \cap \mathcal{Q}$ *nor* $\{b\} \in \mathcal{P} \cap \mathcal{Q}$*.*

Remark 3.11. *The family of sets formed by the intersection (union) of the elements of two primals need not to be a primal on* X *as shown by the following example:*

Example 3.12. *Let* $X = \{a, b\}$ *with primals* $\mathcal{P} = \{\emptyset, \{a\}\}$ *and* $\mathcal{Q} = \{\emptyset, \{b\}\}$ *on* X*.*

(a) *The family* $\mathcal{R} = \{P \cap Q : P \in \mathcal{P}$ *and* $Q \in \mathcal{Q}\} = \{\emptyset\}$ *is not a primal on* X *since* $\{a\} \cap \{b\} = \emptyset \in \mathcal{R}$ *but neither* $\{a\} \in \mathcal{R}$ *nor* $\{b\} \in \mathcal{R}$*.*

(b) *The family* $\mathcal{S} = \{P \cup Q : P \in \mathcal{P}$ *and* $Q \in \mathcal{Q}\} = 2^X$ *is not a primal on* X*, since* $X \in \mathcal{S}$*.*

Definition 3.13. *A topological space* (X, τ) *with a primal* $\mathcal{P}$ *on* X *is called a primal topological space and denoted by* $(X, \tau, \mathcal{P})$*.*

Example 3.14. *Let $\mathbb{R}$ be the set of all real numbers with the usual topology $\mathcal{U}$. Let $\mathcal{P} = \{A \subseteq \mathbb{R} : |A^c| \geq \aleph_0\}$. Then, $(\mathbb{R}, \mathcal{U}, \mathcal{P})$ is a primal topological space.*

Now, we define a new kind of topological operator based on primal. The new structure is given below.

4. The diamond operator

Definition 4.1. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $(\cdot)^\diamond : 2^X \to 2^X$ as $A^\diamond(X, \tau, \mathcal{P}) = \{x \in X : (\forall U \in \tau(x))(A^c \cup U^c \in \mathcal{P})\}$ for any subset A of X. We can also write $A^\diamond_{\mathcal{P}}$ as $A^\diamond(X, \tau, \mathcal{P})$ to specify the primal as per our requirements.*

Remark 4.2. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. For any subset A of X, $A^\diamond \subseteq A$ or $A \subseteq A^\diamond$ need not always to be true as shown by the following examples.*

Example 4.3. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. We consider a primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ on X. Now, if $A = \{a, b\}$, then $\{a, b\} = A \nsubseteq A^\diamond = \emptyset$.*

Example 4.4. *Let $X = \{a, b, c\}$ with the indiscrete topology. We consider a primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ on X. Now, if $A = \{c\}$, then $X = A^\diamond \nsubseteq A = \{c\}$.*

Theorem 4.5. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold for any two subsets A and B of X.*

(i) *if $A^c \in \tau$, then $A^\diamond \subseteq A$,*

(ii) $\emptyset^\diamond = \emptyset$,

(iii) $cl(A^\diamond) = A^\diamond$,

(iv) $(A^\diamond)^\diamond \subseteq A^\diamond$,

(v) *if $A \subseteq B$, then $A^\diamond \subseteq B^\diamond$,*

(vi) $A^\diamond \cup B^\diamond = (A \cup B)^\diamond$,

(vii) $(A \cap B)^\diamond \subseteq A^\diamond \cap B^\diamond$.

Proof. (i) Let $A^c \in \tau$ and $x \in A^\diamond$. Suppose that $x \notin A$. Then, $A^c \in \tau(x)$. Since $x \in A^\diamond$, $A^c \cup U^c \in \mathcal{P}$ for all $U \in \tau(x)$. Therefore, $X = A \cup A^c = (A^c)^c \cup A^c \in \mathcal{P}$. This contradicts with $X \notin \mathcal{P}$. Hence, $A^\diamond \subseteq A$.

(ii) It is obvious from (i), since $\emptyset^c \in \tau$.

(iii) We have always $A^\diamond \subseteq cl(A^\diamond)$. Conversely, let $x \in cl(A^\diamond)$ and $U \in \tau(x)$. Then, $U \cap A^\diamond \neq \emptyset$. Therefore, there exists $y \in X$ such that $y \in U$ and $y \in A^\diamond$. Then, we have $V^c \cup A^c \in \mathcal{P}$ for all $V \in \tau(y)$. Thus, we have $U^c \cup A^c \in \mathcal{P}$. This means that $x \in A^\diamond$. Hence, $cl(A^\diamond) \subseteq A^\diamond$. Consequently, we get $cl(A^\diamond) = A^\diamond$.

(iv) It is obvious from (i) and (iii).

(v) Let $A \subseteq B$ and $x \in A^\diamond$. Then, we have $A^c \cup U^c \in \mathcal{P}$ for all $U \in \tau(x)$. Thus, $B^c \cup U^c \in \mathcal{P}$ since $A \subseteq B$. Hence, $x \in B^\diamond$. Thus, $A^\diamond \subseteq B^\diamond$.

(vi) We can get from (v) that $A^\diamond \subseteq (A \cup B)^\diamond$ and $B^\diamond \subseteq (A \cup B)^\diamond$. Hence, $A^\diamond \cup B^\diamond \subseteq (A \cup B)^\diamond$. Conversely, let $x \notin A^\diamond \cup B^\diamond$. Then, $x \notin A^\diamond$ and $x \notin B^\diamond$. Then, there exist open sets U and V containing x such that $A^c \cup U^c \notin \mathcal{P}$ and $B^c \cup V^c \notin \mathcal{P}$. Put $W = U \cap V$. Hence, W is open containing x such that $A^c \cup W^c \notin \mathcal{P}$ and $B^c \cup W^c \notin \mathcal{P}$. Then, we have $(A \cup B)^c \cup W^c = (A^c \cap B^c) \cup W^c = (A^c \cup W^c) \cap (B^c \cup W^c) \notin \mathcal{P}$ since $\mathcal{P}$ is primal. This means that $x \notin (A \cup B)^\diamond$. Thus, $(A \cup B)^\diamond \subseteq A^\diamond \cup B^\diamond$.

(vii) It is obvious from (v). $\square$

Remark 4.6. *The inclusion given in (vii) of Theorem 4.5 need not to be reversible. It is shown in the following example:*

Example 4.7. *Let (X, τ) be indiscrete topological space, where $X = \{a, b, c\}$. We consider a primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ on X. Now, if $A = \{b\}$ and $B = \{c\}$, then we have $A^\diamond \cap B^\diamond = X \cap X = X \neq \emptyset = \emptyset^\diamond = (A \cap B)^\diamond$.*

Theorem 4.8. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If $A \in \tau$, then $A \cap B^\diamond \subseteq (A \cap B)^\diamond$.*

Proof. Let $A \in \tau$ and $x \in A \cap B^\diamond$. Therefore, $x \in A$ and $x \in B^\diamond$. Then, we have $B^c \cup U^c \in \mathcal{P}$ for all $U \in \tau(x)$. Since $A \in \tau$, we get $(A \cap B)^c \cup U^c = B^c \cup (A \cap U)^c \in \mathcal{P}$ for all $U \in \tau(x)$. This means that $x \in (A \cap B)^\diamond$. Thus, $A \cap B^\diamond \subseteq (A \cap B)^\diamond$. $\square$

Definition 4.9. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $cl^\diamond : 2^X \to 2^X$ as $cl^\diamond(A) = A \cup A^\diamond$, where A is any subset of X.*

Theorem 4.10. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. Then, the following statements hold:*

(i) $cl^\diamond(\emptyset) = \emptyset$,

(ii) $cl^\diamond(X) = X$,

(iii) $A \subseteq cl^\diamond(A)$,

(iv) *if* $A \subseteq B$, *then* $cl^\diamond(A) \subseteq cl^\diamond(B)$,

(v) $cl^\diamond(A) \cup cl^\diamond(B) = cl^\diamond(A \cup B)$,

(vi) $cl^\diamond(cl^\diamond(A)) = cl^\diamond(A)$.

Proof. Let $A, B \subseteq X$.

(i) Since $\emptyset^\diamond = \emptyset$, we have $cl^\diamond(\emptyset) = \emptyset \cup \emptyset^\diamond = \emptyset$.

(ii) Since $X \cup X^\diamond = X$, we have $cl^\diamond(X) = X$.

(iii) Since $cl^\diamond(A) = A \cup A^\diamond$, we have $A \subseteq cl^\diamond(A)$.

(iv) Let $A \subseteq B$. We get from (v) of Theorem 4.5 that $A^\diamond \subseteq B^\diamond$. Therefore, we have $A \cup A^\diamond \subseteq B \cup B^\diamond$ which means that $cl^\diamond(A) \subseteq cl^\diamond(B)$.

(v) It is obvious from the definition of the operator $cl^\diamond$ and (v) of Theorem 4.5.

(vi) It is obvious from (iii) that $cl^\diamond(A) \subseteq cl^\diamond(cl^\diamond(A))$. On the other hand, since $A^\diamond$ is closed in X, we have $(A^\diamond)^\diamond \subseteq A^\diamond$. Therefore,

$$\begin{array}{rcl} cl^\diamond(cl^\diamond(A)) & = & cl^\diamond(A) \cup (cl^\diamond(A))^\diamond \\ & = & cl^\diamond(A) \cup (A \cup A^\diamond)^\diamond \\ & = & cl^\diamond(A) \cup A^\diamond \cup (A^\diamond)^\diamond \\ & \subseteq & cl^\diamond(A) \cup A^\diamond \cup A^\diamond \\ & = & cl^\diamond(A) \end{array}$$

Thus, we have $cl^\diamond(cl^\diamond(A)) = cl^\diamond(A)$. $\square$

Corollary 4.11. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the operator $cl^\diamond : 2^X \to 2^X$ defined by $cl^\diamond(A) = A \cup A^\diamond$, where A is any subset of X, is a Kuratowski closure operator.*

Definition 4.12. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\tau^\diamond = \{A \subseteq X : cl^\diamond(A^c) = A^c\}$ is a topology on X induced by topology τ and primal $\mathcal{P}$. It is called primal topology on X. We can also write $\tau^\diamond_{\mathcal{P}}$ instead of $\tau^\diamond$ to specify the primal as per our requirements.*

Theorem 4.13. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the primal topology $\tau^\diamond$ is finer than τ.*

Proof. Let $A \in \tau$. Then, A^c is τ-closed in X. From (i) of Theorem 4.5, we get $(A^c)^\diamond \subseteq A^c$. Thus, $cl^\diamond(A^c) = A^c \cup (A^c)^\diamond \subseteq A^c$. Since $A^c \subseteq cl^\diamond(A^c)$ is always true for any subset A of X, we obtain $cl^\diamond(A^c) = A^c$. This means that $A \in \tau^\diamond$. Thus, we have $\tau \subseteq \tau^\diamond$. □

Theorem 4.14. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold:*

(i) if $\mathcal{P} = \emptyset$, then $\tau^\diamond = 2^X$,

(ii) if $\mathcal{P} = 2^X \setminus \{X\}$, then $\tau = \tau^\diamond$.

Proof. (i) We have always $\tau^\diamond \subseteq 2^X$. Now, let $A \in 2^X$. Since $\mathcal{P} = \emptyset$, we have $A^\diamond = \emptyset$ for any subset A of X. Therefore, $cl^\diamond(A^c) = A^c$. This means that $A \in \tau^\diamond$. Hence, $2^X \subseteq \tau^\diamond$. Thus, we have $\tau^\diamond = 2^X$.

(ii) We have always $\tau \subseteq \tau^\diamond$ from Theorem 4.13. Now, we will prove that $\tau^\diamond \subseteq \tau$. Let $A \in \tau^\diamond$. Then, $A^c \cup (A^c)^\diamond = A^c$ which means that $(A^c)^\diamond \subseteq A^c$. Now, let $x \notin (A^c)^\diamond$. Then, there exists $U \in \tau(x)$ such that $U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{P}$. Since $\mathcal{P} = 2^X \setminus \{X\}$, we obtain $U^c \cup A = X$ and so $U \cap A^c = \emptyset$. Therefore, $x \notin cl(A^c)$. Thus, we have $cl(A^c) \subseteq (A^c)^\diamond \subseteq A^c$. Hence, $cl(A^c) = A^c$. It means that A^c is τ-closed and so, $A \in \tau$. Thus, $\tau^\diamond \subseteq \tau$. Consequently, we have $\tau = \tau^\diamond$. □

Remark 4.15. *The converse of Theorem 4.14(ii) need not to be true. It is shown in the following example:*

Example 4.16. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a, b\}\}$ and $\mathcal{P} = 2^X \setminus \{X, \{a, b\}\}$. Simple calculations show that $\tau = \tau^\diamond$, but $\mathcal{P}$ is not equal to $2^X \setminus \{X\}$.*

Theorem 4.17. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. Then, the followings hold:*

(i) $A \in \tau^\diamond$ if and only if for all x in A, there exists an open set U containing x such that $U^c \cup A \notin \mathcal{P}$,

(ii) if $A \notin \mathcal{P}$, then $A \in \tau^\diamond$.

Proof. (i) Let $A \in \tau^\diamond$.

$$
\begin{aligned}
A \in \tau^\diamond &\Leftrightarrow cl^\diamond(A^c) = A^c \\
&\Leftrightarrow A^c \cup (A^c)^\diamond = A^c \\
&\Leftrightarrow (A^c)^\diamond \subseteq A^c \\
&\Leftrightarrow A \subseteq ((A^c)^\diamond)^c \\
&\Leftrightarrow (\forall x \in A)(x \notin (A^c)^\diamond) \\
&\Leftrightarrow (\forall x \in A)(\exists U \in \tau(x))(U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{P}).
\end{aligned}
$$

(ii) Let $A \notin \mathcal{P}$ and $x \in A$. Put $U = X$. Then, U is a τ-open set containing x. Since $A \notin \mathcal{P}$ and $U^c \cup A = A$, we have $U^c \cup A \notin \mathcal{P}$. From (i), we get $A \in \tau^\diamond$. □

Theorem 4.18. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\mathcal{B}_{\mathcal{P}} = \{T \cap P : T \in \tau$ and $P \notin \mathcal{P}\}$ is a base for the primal topology $\tau^\diamond$ on X.*

Proof. Let $B \in \mathcal{B}_{\mathcal{P}}$. Then, there exist $T \in \tau$ and $P \notin \mathcal{P}$ such that $B = T \cap P$. Since $\tau \subseteq \tau^{\diamond}$, we get $T \in \tau^{\diamond}$. On the other hand, from Theorem 4.17(ii), we have $P \in \tau^{\diamond}$. Therefore, $B \in \tau^{\diamond}$. Consequently, $\mathcal{B}_{\mathcal{P}} \subseteq \tau^{\diamond}$. Now, let $A \in \tau^{\diamond}$ and $x \in A$. Then, from Theorem 4.17(i), there exists $U \in \tau(x)$ such that $U^c \cup A \notin \mathcal{P}$. Now, let $B = U \cap (U^c \cup A)$. Hence, we have $B \in \mathcal{B}_{\mathcal{P}}$ such that $x \in B \subseteq A$. □

Theorem 4.19. *Let $(X, \tau, \mathcal{P})$ and $(X, \tau, \mathcal{Q})$ be two primal topological spaces. If $\mathcal{P} \subseteq \mathcal{Q}$, then $\tau^{\diamond}_{\mathcal{Q}} \subseteq \tau^{\diamond}_{\mathcal{P}}$.*

Proof. Let $A \in \tau^{\diamond}_{\mathcal{Q}}$. Then, $A^c \cup (A^c)^{\diamond}_{\mathcal{Q}} = A^c$ which means that $(A^c)^{\diamond}_{\mathcal{Q}} \subseteq A^c$. Now, let $x \notin A^c$. Then, we get $x \notin (A^c)^{\diamond}_{\mathcal{Q}}$ and so there exists $U \in \tau(x)$ such that $U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{Q}$. Since $\mathcal{P} \subseteq \mathcal{Q}$, we have $U^c \cup A \notin \mathcal{P}$. Therefore, $x \notin (A^c)^{\diamond}_{\mathcal{P}}$. Thus, $(A^c)^{\diamond}_{\mathcal{P}} \subseteq A^c$ and so, $cl^{\diamond}(A^c) = A^c \cup (A^c)^{\diamond}_{\mathcal{P}} = A^c$. Hence, $A \in \tau^{\diamond}_{\mathcal{P}}$. Consequently, we have $\tau^{\diamond}_{\mathcal{Q}} \subseteq \tau^{\diamond}_{\mathcal{P}}$. □

Theorem 4.20. *Let $f : X \to Y$ be a function and $\mathcal{P} \subseteq 2^X$. If $\mathcal{P}$ is a primal on X and f is bijective, then $\mathcal{Q} = \{f(P) : P \in \mathcal{P}\}$ is a primal on Y.*

Proof. (i) Suppose that $Y \in \mathcal{Q}$. Then, there exists $P \in \mathcal{P}$ such that $f(P) = Y$ and $P \subsetneq X$. Therefore, we have $f(P) = f(X) = Y$. Since f is injective, we obtain $P = X$ which is a contradiction.

(ii) Let $A \in \mathcal{Q}$ and $B \subseteq A$. Then, there exists $P_1 \in \mathcal{P}$ such that $A = f(P_1)$. Now, set $P_2 = f^{-1}(B) \cap P_1$. It is obvious that $P_2 \subseteq P_1$. Since $\mathcal{P}$ is a primal on X, $\mathcal{P}$ is downward closed and so we have $P_2 \in \mathcal{P}$. Also, $B = f(P_2)$. This means that $B \in \mathcal{Q}$.

(iii) Let $A \cap B \in \mathcal{Q}$. Then, there exists $P \in \mathcal{P}$ such that $A \cap B = f(P)$. Since f is injective, we have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = P$ i.e. $f^{-1}(A) \cap f^{-1}(B) \in \mathcal{P}$. Therefore, we have $f^{-1}(A) \in \mathcal{P}$ or $f^{-1}(B) \in \mathcal{P}$. Thus, $f(f^{-1}(A)) \in \mathcal{Q}$ or $f(f^{-1}(B)) \in \mathcal{Q}$. Since f is surjective, we get $A = f(f^{-1}(A)) \in \mathcal{Q}$ or $B = f(f^{-1}(B)) \in \mathcal{Q}$. □

Remark 4.21. *For a function $f : X \to Y$ and a primal $\mathcal{Q}$ on Y, the family $\mathcal{P} = \{f^{-1}(Q) : Q \in \mathcal{Q}\}$ need not to be a primal on X as shown by the following example.*

Example 4.22. *Consider $X = \{a, b\}$, $Y = \{1, 2\}$ and $\mathcal{Q} = \{\emptyset, \{1\}\}$. Define the function $f : X \to Y$ by $f(x) = 1$. Then, $\mathcal{Q}$ is a primal on Y but $\mathcal{P} = \{f^{-1}(Q) : Q \in \mathcal{Q}\} = \{\emptyset, X\}$ is not a primal on X.*

5. Conclusion

This paper introduced primal, which is the dual structure of grill. It is well known that the ideal, the dual structure of filter, is one of the highly useful notions in topology [23], summability theory [24], real analysis [25], etc. Thus, ideal inspired us to introduce primal. Here, we introduced two new operators using primal. One of these two operators satisfies Kuratowski closure axioms. Moreover, we introduced a topology named primal topology ($\tau^{\diamond}$), which is finer than any topology τ of primal topological space $(X, \tau, \mathcal{P})$. Later, we provided a structure of base for $\tau^{\diamond}$ and proved several fundamental results. It is important to note that primal structures and some related results showed quantum behaviors, i.e., we could not determine some properties as universally true. For example, one may refer to Examples 4.3 and 4.4. Moreover, Remark 4.21 showed some uncertain behavior of this new structure. Thus, we hope that we should study this new notion more deeply in general topology and other areas. If possible, we are looking forward to connecting this notion with some ideas of quantum world [26] from the perspective of general topology in the future.

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References

1. A. Al-Omari, S. Acharjee, M. Özkoç, *A new operator of primal topological spaces.* Mathematica, 65(88)(2), 175-183, (2023).

2. A. Al-Omari and M. H. Alqahtani, *Primal structure with closure operators and their applications.* Mathematics, 11(24), (2023), 4946.

3. K. C. Chattopadhyay, W. J. Thron, *Extensions of closure spaces.* Can. J. Math. 29(6), 1277-1286, (1977).

4. K. C. Chattopadhyay, O. Njastad, W. J. Thron, *Merotopic spaces and extensions of closure spaces.* Canad. J. Math. 4, 613-629, (1983).

5. W. J. Thron, *Proximity structures and grills.* Math. Ann. 206, 35-62, (1973).

6. B. Roy, M. N. Mukherjee, *On a typical topology induced by a grill.* Soochow Jour. Math. 33(4), 771-786, (2007).

7. B. Roy, M. N. Mukherjee, *Concerning topologies induced by principal grills.* An. Stiint. Univ. AL. I. Cuza Iasi. Mat. (N.S.), 55(2), 285-294, (2009).

8. B. Roy, M. N. Mukherjee, *On a type of compactness via grills.* Mat. Vesnik, 59, 113-120, (2007).

9. B. Roy, M. N. Mukherjee, S. K. Ghosh, *On a new operator based on a grill and its associated topology.* Arab Jour. Math. 14(1), 21-32, (2008).

10. S. Modak, *Topology on grill-filter space and continuity.* Bol. Soc. Paran. Mat. 31(2), 219-230, (2013).

11. S. Modak, *Grill-filter space.* Jour. Indian Math. Soc. 80(3-4), 313-320 (2013).

12. R. A. Hosny, *δ-sets with grill.* Int. Math. Forum, 7(43), 2107-2113, (2012).

13. A. A. Nasef, A. A. Azzam, *Some topological operators via grills.* Jour. Linear Top. Alg. 5(3), 199-204, (2016).

14. R. Thangamariappan, V. Renukadevi, *Topology generated by cluster systems.* Math. Vesnik, 67(03), 174-184, (2015).

15. G. Chóquet, *Sur les notions de filter et grille.* Comptes Rendus Acad. Sci. Paris, 224, 171-173, (1947).

16. A. A. Azzam, S. S. Hussein, H. Saber Osman, *Compactness of topological spaces with grills.* Italian. Jour. Pure. Appl. Math. 44, 198–207, (2020).

17. A. Talabeigi, *On the Tychonoff's type theorem via grills.* Bull. Iranian Math. Soc. 42(1), 37–41, (2016).

18. N. Boroojerdian, A. Talabeigi, *One-point λ-compactification via grills.* Iran. Jour. Sci. Tech. Trans. A: Sci. 41, 909–912, (2017).

19. M. N. Mukherjee, A. Debray, *On H-closed spaces and grills.* An. Stiint. Univ. AL. I. Cuza Iasi. Mat. (N.S.), 44, 1-25, (1998).

20. I. Lončar, *A Note on inverse systems of S(n)-closed spaces.* Sarajevo Jour. Math. 23(1), 117-130, (2015).

21. S. Willard, *General topology.* Courier Corporation, (2012).

22. K. Kuratowski, *Topology: Volume I.* Elsevier, (2014).

23. D. Janković, T. R. Hamlett, *New topologies from old via ideals.* American Math. Monthly 97(4), 295-310, (1990).

24. B. C. Tripathy, M. Sen, S. Nath, *On generalized difference ideal convergence in generalized probabilistic n-normed spaces.* Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 91(1), 29-34, (2021).

25. H. Albayrak, Ö. Ölmez, S. Aytar, *Some set theoretic operators preserving ideal Hausdorff convergence.* Real Anal. Exchange, 47(1), 179-190, (2022).

26. C. J. Isham, *An introduction to general topology and quantum topology.* In Physics, Geometry and Topology, Springer, Boston, MA. 129-189, (1990).

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