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New Result Concerning Nonlocal Conformable Fractional Differential Equations of Neutral Type with Measure of Noncompactness

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ABSTRACT: In this paper, we demonstrate the existence results of a solution for a class of initial value problems involving conformable derivatives with nonlocal conditions. The main result is established using a fixed point theorem attributed to Darbo-Sadovskii combined with semigroup theory.

Key Words: Fractional conformable derivative, sectoriel operator, nonlocal condition, analytic semigroupe, measure of noncompactness.

Contents

1. Introduction

Many dynamical processes in physics, biology, economics, and other areas of applications can be governed by abstract ordinary differential evolution equations of the following form:

$$\frac{d}{dt}\Big(z(t) - F\big(t, z(h_1(t))\big)\Big) = -Bz(t) + \varphi\big(t, z(h_2(t))\big) \tag{1.1}$$

Unfortunately, the classical derivative $\frac{d}{dt}$ appearing in equation (1.1) cannot model the dynamical processes with memory. Hence, in order to avoid this shortcoming of classical derivative, many authors try to replace the classical derivative by a fractional derivative [10] because fractional derivatives have been proved that they are a very good way to model many phenomena with memory in various fields of science and engineering [11,12,13]. In consequence, many researchers pay attention to form the best definition of fractional derivative. Recently, a novel definition named conformable fractional derivative is introduced in [12]. This new fractional derivative quickly becomes the subject of many contributions in several areas of science [1,2,15,18,19]. Here, we will investigate the same conformable fractional Cauchy problem with nonlocal condition. Specifically, we consider nonlocal fractional Cauchy problem with the from

$$\begin{cases} \frac{d^\xi}{dt^\xi}\Big(z(t) - F\big(t, z(h_1(t))\big)\Big) = -B\Big(z(t) - F\big(t, z(h_1(t))\big)\Big) + \varphi\big(t, z(h_2(t))\big) \\ z(0) + \phi(z) = z_0 \in Y = D(B^\xi), \end{cases} \tag{1.2}$$

where $\frac{d^\xi}{dt^\xi}$ is the conformable fractional derivative of $\xi \in (0,1)$. B is a sectoriel operator which generate a strongly analytic semi-groupe $\big(S(t)\big)$ on a Banach espace $(Y, \| \, \|)$, For more information on semigroup theory, we refer to [17]. The initial condition $z(0) + \phi(z) = z_0$ is a non-local condition this concept has become a hot topic in recent years. The degree of modeling has significantly improved as a result of their connection to classical problems, the thus making it more realistic. The nonlocal condition joined to the

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main equation instead of the classical initial condition is necessary to model well and write mathematically, physical phenomena like in electronics, in mechanics of materials, or in biomathematics in the way closest to the reality of many phenomena in multiple disciplines. This condition means that the initial condition depends on some future times. The nonlocal condition attracts the attention of many authors in several works [4,5,8,16]. We denote by $\mathcal{C}\big([0,\tau], D(B^{\xi}\big)$ the Banach space of continuous function from $[0,\tau]$ into the domain of the fractional power of B with the norm $\|v\|_{\xi} = \sup\limits_{s\in[0,\tau]} \|v(s)\|_{\xi} = \sup\limits_{s\in[0,\tau]} \|B^{\xi}v(s)\|$. The functions used in our study are $\psi : [0,\tau]\times D(B^{\xi}) \longrightarrow D(B^{\xi})$, $\varphi : [0,\tau]\times D(B^{\xi}) \longrightarrow D(B^{\xi})$ and $\phi : \mathcal{C}\big([0,\tau], D(B^{\xi})\big) \longrightarrow D(B^{\xi})$ and also $h_1, h_2 \in \mathcal{C}\big([0,\tau],[0,\tau]\big)$.

The structure of this paper is as follows. In Section 2, we review some preliminary facts about the conformable fractional derivative, semi-group theory, and the properties of the measure of noncompactness. Section 3 is dedicated to demonstrating the main result.

2. Preliminaries

In this section, we recall some concepts related to conformable fractional derivatives and some properties of the measure of noncompactness.

Definition 2.1 *[1]We define the conformable fractional derivative of z of order ξ at $t>0$ by*

$$\frac{d^{\xi}z(t)}{dt^{\xi}} = \lim_{h\to 0}\frac{z(t+he^{(\xi-1)t}) - z(t)}{h},$$

when this limit exists, we say that z is (ξ)-differentiable at t.
If z is (ξ)-differentiable and $\lim\limits_{t\to 0^+}\frac{d^{\xi}z(t)}{dt^{\xi}}$ exists, then define

$$\frac{d^{\xi}z(0)}{dt^{\xi}} = \lim_{t\to 0^+}\frac{d^{\xi}z(t)}{dt^{\xi}}.$$

The (ξ)-fractional integral of a function z is defined by

$$I^{\xi}(z)(t) = \int_0^t s^{\xi-1}z(s)ds.$$

Theorem 2.1 [12] *If z is a continuous function such that $z\in D(I^{\xi})$, then*

$$\frac{d^{\xi}\big(I^{\xi}(z(t))\big)}{dt^{\xi}} = z(t).$$

Definition 2.1 [1] *The fractional Laplace transform of order ξ of z is defined by*

$$\mathcal{L}_{\xi}(z(t))(\lambda) = \int_0^{+\infty} t^{\xi-1}e^{-\lambda\frac{t^{\xi}}{\xi}}z(t)dt.$$

The fractional Laplace transform of the conformable fractional derivative is given by the following proposition.

Proposition 2.1 [1] *if $z(t)$ is differentiable. Then,*

$$\mathcal{L}_{\xi}(\frac{d^{\xi}z(t)}{dt^{\xi}})(\lambda) = \lambda\mathcal{L}_{\xi}(z(t))(\lambda) - z(0).$$

Now, we provide some notions related to the fractional powers of an operator.

Definition 2.2 *[17] Let B be a sectoriel operator defined on a Banach space Y such that $\mathcal{R}e\sigma(B)>0$. For $\xi>0$, we denote by $B^{-\xi}$ the operator defined by:*

$$B^{-\xi} = \frac{1}{\Gamma(\xi)}\int_0^{+\infty} t^{\xi-1}S(t)dt.$$

Definition 2.3 *[17] Let B be a sectoriel operator defined on a Banach space Y, such that $\mathcal{R}e\sigma(B) > 0$. We define the family of operators $(B^\xi)_{\xi\geq 0}$ as follows: $B^0 = I_Y$ and for $\xi > 0$,*

$$B^\xi = (B^{-\xi})^{-1}, \qquad D(B^\xi) = Im(B^{-\xi}).$$

Theorem 2.2 *[17] If $(-B)$ is the infinitesimal generator of an analytic semi-group $(S(t))_{t\geq 0}$ and if $0 \in \rho(B)$, then:*

1. *$D(B^\xi)$ is a Banach space with the norm $\|z\|_\xi = \|B^\xi z\|$ for every $z \in D(B^\xi)$.*
2. *$S(t) : Y \longrightarrow D(B^\xi)$ for all $t > 0$ and $\xi \geq 0$.*
3. *For every $z \in D(B^\xi)$, we have $S(t)B^\xi z = B^\xi S(t)z$.*

Definition 2.4 *[17] $B : D(B) \subset Y \longrightarrow Y$ is said to be sectoriel operator of type (M, ω, θ) if there exists $M > 0$, $\omega \in \mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$ such that:*

1. *B is a linear and closed operator.*
2. *$\forall\lambda \notin w + S_\theta$, the resolvent $(\lambda I - B)^{-1}$ of B exists.*
3. *$\forall\lambda \notin w + S_\theta$, $|(\lambda I - B)^{-1}| \leq \frac{M}{|\lambda - w|}$, where $w + S_\theta := \{w + \lambda/\lambda \in \mathbb{C} \ \ avec \ \ |Arg(-\lambda)| < \theta\}$.*

Theorem 2.3 *[17] B is a densely sectoriel operator generates a strongly analytic semigroup $(S(t))_{t\geq 0}$. Moreover*

$$S(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\lambda I - B)^{-1}d\lambda,$$

with γ being a suitable path $\lambda \notin w + S_\theta$.

We conclude these preliminaries with the concept of a measure of noncompactness

Definition 2.5 [3, 7] *C is a bounded set in a Banach space Y, the Hausdorff measure of noncompactness σ is defined as*

$$\sigma(C) = \inf\{r > 0 : C \subset \cup_{i=1}^n B(x_i, r) \ ; \ x_i \in Y\}.$$

Definition 2.6 [7] *The operator $\Delta : D(\Delta) \subset Y \longrightarrow Y$ is said to be a σ-contraction if there exists a positive constant $K < 1$ such that $\sigma(\Delta(C)) \leq K\sigma(C)$ for any bounded closed subset $C \subseteq D(\Delta)$.*

Lemma 2.1 *[6,14] Let $C \subset Y$ be a bounded set; then, there exists a countable set $C_0 \subset C$ such that $\sigma(C) \leq 2\sigma(C_0)$.*

Lemma 2.2 [3, 7] *Let Y be a Banach space, and let $D, C \subset\subset Y$ be bounded. Then the following properties hold:*

1. *D is precompact if and only if $\sigma(D) = 0$;*
2. *$\sigma(D) = \sigma(\bar{D}) = \sigma\big(conv(D)\big)$, where $\bar{D}$ and $conv(D)$ denote the closure and convex of D,respectively;*
3. *$\sigma(D) \leq \sigma(C)$ where $D \subseteq C$;*
4. *$\sigma(D + C) \leq \sigma(D) + \sigma(C)$, where $D + C = \{x + y \ ; \ \ x \in D \ , \ y \in C\}$;*
5. *$\sigma(D \cup C) \leq max\{\sigma(D) \ , \ \sigma(C)\}$;*
6. *$\sigma(\lambda D) = |\lambda|\sigma(D)$;*
7. *If the operator $Q : D(Q) \subset Y \longrightarrow Z$ is Lipschitz continuous with $K \geq 0$, then we have $\rho\big(Q(C)\big) \leq K\sigma(C)$ for any subset $C \subseteq D(Q)$, where Z is another Banach space, and ρ represents the Hausdorff measure of noncompactness in Z.*

Lemma 2.3 *[9] Let $D_0 = \{y_n\} \subset \mathcal{C}$ be a countable set. Then,*

1. $\sigma\Big(D_0(t)\Big) = \sigma\big(\{y_n\}\big)$ *is Lebesgue integral on* $[0,\tau]$.

2. $\sigma\Big(\int_0^\tau D_0(s)ds\Big) \leq 2\int_0^\tau \sigma\Big(D_0(s)\Big)ds,$ *where* $\sigma\Big(\int_0^\tau D_0(s)ds\Big) = \sigma\Big(\{\int_0^\tau y_n(s)ds\}\Big)$.

Lemma 2.4 *[3] Let $D \subset \mathcal{C}$ be equicontinuous and bounded. Then,*

1. $\sigma\big(D(t)\big)$ *is continuous on* $[0,\tau]$.

2. $\sigma_c\big(D\big) = \max_{t\in[0,\tau]}\Big(\sigma\big(D(t)\big)\Big)$.

Lemma 2.5 *(Darbo-Sadovskii Theorem) [3, 7] Let $C \subset Y$ be a bounded, closed, and convex set. If $Q : C \longrightarrow C$ is a continuous and σ-contraction operator, then Q has at least one fixed point in C.*

3. Main results

Before presenting our main results, we introduce the following assumptions:

(H1) The function $\psi(t,.) : D(B^\xi) \longrightarrow D(B^\xi)$ is continuous and there exists a constant $L > 0$ such that

$$\|\psi(t_1,z) - \psi(t_2,z)\|_\xi \leq L\|t_1 - t_2\|, \quad \text{for all } t_1,t_2 \in [0,\tau] \quad \text{and for} \quad z \in D(B^\xi).$$

(H2) The function $\varphi(t,.) : D(B^\xi) \longrightarrow D(B^\xi)$ is continuous, and for all $r > 0$, there exists a function $\mu_r \in \mathcal{L}^\infty([0,a],\mathbb{R}^+)$ such that

$$\sup_{\|x\|\leq r} \|\varphi(t,z)\|_\xi \leq \mu_r(t).$$

(H3) $\varphi(.,z) : [0,\tau] \longrightarrow D(B^\xi)$ is continuous.

(H4) $\phi : \mathcal{C}\big([0,\tau],D(B^\xi)\big) \longrightarrow D(B^\xi)$ is continuous and compact.

(H5) There exist positive constants a and b such that $\|\phi(z)\|_\xi \leq a\|z\|_\xi + b$.

(H6) There exists a positive constant L_1 such that $\sigma\psi(t,D_0) \leq L_1\sigma(D_0)$ for any countable $D_0 \subset D(B^\xi)$ and $t \in [0,\tau]$.

(H7) There exists a positive constant L_2 such that $\sigma\varphi(t,D_0) \leq L_2\sigma(D_0)$ for any countable $D_0 \subset D(B^\xi)$, $t \in [0,\tau]$

Existence of mild solution

Lemma 3.1 *$z \in \mathcal{C}\big([0,\tau],D(B^\xi)\big)$ is an integral solution of fractional problem (1.2) if the function z is given by:*

$$\begin{aligned} z(t) &= \psi\big(t,z(h_1(t)\big) + S(\frac{t^\xi}{\xi})\Big(z_0 - \phi(z) - \psi\big(0,z(h_1(0)\big)\Big) \\ &+ \int_0^t s^{\xi-1}S(\frac{t^\xi - s^\xi}{\xi})\varphi(s,z(h_2(s)))ds. \end{aligned}$$

Proof: Applying the Laplace transform to equation (1.2), we obtain

$$\mathcal{L}_\xi\left(\frac{d^\xi}{dt^\xi}\left(z(t)-\psi\left(t,z\left(h_1\left(t\right)\right)\right)\right)\right)(\lambda)=-\mathcal{L}_\xi\left(B\left(z(t)-\psi(t,z(h_1(t)))\right)\right)(\lambda)+\mathcal{L}_\xi\left(\varphi(t,z(h_2(t)))\right)(\lambda).$$

Then,

$$\begin{aligned}\lambda\mathcal{L}_\xi\left(\big(z(t)-\psi\big(t,z(h_1(t))\big)\big)\right)(\lambda)-\big(z(0)-\psi\big(0,z(h_1(0))\big)\big) &= -B\mathcal{L}_\xi(z(t)-\psi(t,z(h_1(t)))(\lambda)\\ &+ \mathcal{L}_\xi(\varphi(t,z(h_2(t))))(\lambda).\end{aligned}$$

This implies that,

$$\big(\lambda+B\big)\mathcal{L}_\xi\big(\big(z(t)-\psi\big(t,z(h_1(t))\big)\big)\big)(\lambda)=z(0)-\psi\big(0,z(h_1(0))\big)+\mathcal{L}_\xi(\varphi(t,z(h_2(t))))(\lambda).$$

Hence,

$$\begin{aligned}\mathcal{L}_\xi\big(\big(z(t)-\psi\big(t,z(h_1(t))\big)\big)\big)(\lambda) &= \big(\lambda+B\big)^{-1}\big(z(0)-\psi\big(0,z(h_1(0))\big)\big)\\ &+ \big(\lambda+B\big)^{-1}\mathcal{L}_\xi(\varphi(t,z(h_2(t))))(\lambda).\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}_\xi\big(\big(z(t)-\psi\big(t,z(h_1(t))\big)\big)\big)(\lambda) &= \big(\lambda+B\big)^{-1}\big(z_0-\phi(z)-\psi\big(0,z(h_1(0))\big)\big)\\ &+ \big(\lambda+B\big)^{-1}\mathcal{L}_\xi(\varphi(t,z(h_2(t))))(\lambda).\end{aligned}$$

So, we have

$$\big(\lambda+B\big)^{-1}\big(z(0)-\psi\big(0,z(h_1(0))\big)\big) = \mathcal{L}_\xi\left(S(\frac{t^\xi}{\xi})\Big(z_0-\phi(z)-\psi\big(0,z(h_1(0)\big)\Big)(\lambda)\right).$$

Hence,

$$\begin{aligned}\big(\lambda+B\big)^{-1}\mathcal{L}_\xi(\varphi(t,z(h_2(t))))(\lambda) &= \mathcal{L}_\xi\left(\int_0^{\frac{t^\xi}{\xi}} S(\frac{t^\xi}{\xi}-s)\varphi((\xi s)^{\frac{1}{\xi}},z(h_2((\xi s)^{\frac{1}{\xi}}))))ds\right)(\lambda)\\ &= \mathcal{L}_\xi\left(\int_0^t s^{\xi-1}S(\frac{t^\xi}{\xi}-\frac{s^\xi}{\xi})\varphi(s,z(h_2(s)))ds\right)(\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}_\xi\big(\big(z(t)-\psi\big(t,z(h_1(t))\big)\big)\big)(\lambda) &= \mathcal{L}_\xi\left(S(\frac{t^\xi}{\xi})\Big(z_0-\phi(z)-\psi\big(0,z(h_1(0)\big)\Big)(\lambda)\right)\\ &+ \mathcal{L}_\xi\left(\int_0^t s^{\xi-1}S(\frac{t^\xi}{\xi}-\frac{s^\xi}{\xi})\varphi(s,z(h_2(s)))ds\right)(\lambda).\end{aligned}$$

By employing the inverse fractional Laplace transform, we obtain the following formula

$$\begin{aligned}z(t)-\psi\big(t,z(h_1(t)\big) &= S(\frac{t^\xi}{\xi})\Big(z_0-\phi(z)-\psi\big(0,z(h_1(0)\big)\Big)\\ &+ \mathcal{L}_\xi\left(\int_0^t s^{\xi-1}S(\frac{t^\xi}{\xi}-\frac{s^\xi}{\xi})\varphi(s,z(h_2(s)))ds\right)(\lambda).\end{aligned}$$

Then, we obtain

$$\begin{aligned}z(t) &= \psi\big(t,z(h_1(t)\big)+S(\frac{t^\xi}{\xi})\Big(z_0-\phi(z)-\psi\big(0,z(h_1(0)\big)\Big)\\ &+ \int_0^t s^{\xi-1}S(\frac{t^\xi}{\xi}-\frac{s^\xi}{\xi})\varphi(s,z(h_2(s)))ds.\end{aligned}$$

□

Theorem 3.1 *Assuming that $(H1)-(H7)$ hold, then the fractional problem (1.2) has at least one integral solution provided that*

$$2\Big(L_1 + \big(L_1 + L_2\frac{\tau^\xi}{\xi}\big)\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big) < 1.$$

Proof: In order to apply the Darbo-Sadovskii fixed-point theorem, we set

$$B_r = \{z \in Y \ / \ \|z\|_\xi \leq r\} \ \text{ with } \ Y = \mathcal{C}\big([0,\tau], D(B^\xi)\big) \ \text{ and } \ \|z\|_\xi = \sup_{t\in[0,\tau]}\|z(t)\|_\xi.$$

We define operator $\Gamma : Y \longrightarrow Y$ by

$$\Gamma(z)(t) \;=\; \psi\big(t, z(h_1(t)) + S(\frac{t^\xi}{\xi})\Big(z_0 - \phi(z) - \psi\big(0, z(h_1(0))\big)\Big) + \int_0^t s^{\xi-1}S(\frac{t^\xi - s^\xi}{\xi})\varphi(s, z(h_2(s)))ds.$$

The proof will be given in four steps.

Step 1. Prove that there exists a radius $\gamma > 0$ such that $\Gamma : B_\gamma \longrightarrow B_\gamma$.
Let $z \in Y$, we have

$$\begin{aligned}
\|\Gamma(z)(t)\|_\xi &\leq \|\psi\big(t, z(h_1(t))\|_\xi + \|S(\frac{t^\xi}{\xi})\Big(z_0 - \phi(z) - \psi\big(0, z(h_1(0))\big)\Big)\|_\xi \\
&+ \int_0^t s^{\xi-1}\|S(\frac{t^\xi - s^\xi}{\xi})\varphi(s, z(h_2(s)))\|_\xi ds \\
&\leq \|\psi\big(t, z(h_1(t))\|_\xi + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + \|\phi(z)\|_\xi + \|\psi\big(0, z(h_1(0))\|_\xi\Big) \\
&+ \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\int_0^t s^{\xi-1}\|\varphi(s, z(h_2(s)))\|_\xi ds \\
&\leq \|\psi\big(t, z(h_1(t))\|_\xi + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + ar + b + \|\psi\big(0, z(h_1(0))\|_\xi \\
&+ \frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big) \\
&\leq M + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + ar + b + M + \frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big).
\end{aligned}$$

With $M = \sup_{t\in[0,\tau]}\|\psi\big(t, z(h_1(t))\|_\xi$.
Hence,

$$M + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + ar + b + M + \frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big) \leq r.$$

Then,

$$M + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + b + M + \frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big) \leq r\Big(1 - a\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big).$$

Precisely, we can choose that γ such that

$$\gamma \geq \frac{M + \sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi + b + M + \frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big)}{1 - a\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|}.$$

Step 2. Prove that $\Gamma : B_\gamma \longrightarrow B_\gamma$ is continuous.
Let $z_n \in B_\gamma$ such that $\lim_{n\to\infty} z_n = z$ in B_γ, we have

$$\begin{aligned}\|\Gamma(z_n)(t) - \Gamma(z)(t)\|_\xi &\leq \|\psi\big(t, z_n(h_1(t)) - \psi\big(t, z(h_1(t))\|_\xi \\ &+ \sup_{t\in[0,\tau]} \|S(\frac{t^\xi}{\xi})\| \Big(\|\phi(z) - \phi(z_n)\|_\xi + \|\psi\big(0, z(h_1(0)\big) - \psi\big(0, z_n(h_1(0)\big)\|_\xi \Big) \\ &+ \sup_{t\in[0,\tau]} \|S(\frac{t^\xi}{\xi})\| \int_0^t s^{\xi-1} \|\varphi(s, z_n(h_2(s))) - \varphi(s, z(h_2(s)))\|_\xi ds.\end{aligned}$$

By using $(H2)$, we get

$$\|s^{\xi-1}\varphi(s, z_n(h_2(s))) - \varphi(s, z(h_2(s)))\|_\xi \leq 2s^{\xi-1}|\mu_r|_{\mathcal{L}^\infty\big([0,\tau],\mathbb{R}^+\big)},$$

and

$$\lim_{n\to\infty} \varphi(s, z_n(h_2(s))) = \varphi(s, z(h_2(s))).$$

The Lebesgue dominated convergence theorem proves that

$$\lim_{n\to\infty} \int_0^t s^{\xi-1} \|\varphi(s, z_n(h_2(s))) - \varphi(s, z(h_2(s)))\|_\xi ds = 0.$$

According to the continuity of the functions ψ and ϕ, we deduce that Γ is continuous.

Step 3. Prove that $\Gamma(B_\gamma)$ is equicontinuous.
For $z \in B_\gamma$ and $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$, we have

$$\begin{aligned}\Gamma(z)(t_2) - \Gamma(z)(t_1) &= \psi\big(t_2, z(h_1(t_2))\big) - \psi\big(t_1, z(h_1(t_1))\big) \\ &+ \Big(S(\frac{t_2^\xi - t_1^\xi}{\xi}) - I\Big)\Big(S(\frac{t_1^\xi}{\xi})\Big(z_0 - \phi(z)) - \psi\big(0, z(h_1(0)\big)\Big)\Big) \\ &+ \int_0^{t_1} s^{\xi-1}\Big(S(\frac{t_2^\xi - s^\xi}{\xi}) - S(\frac{t_1^\xi - s^\xi}{\xi})\Big)\varphi(s, z(h_2(s)))ds \\ &+ \int_{t_1}^{t_2} s^{\xi-1} S(\frac{t_2^\xi - s^\xi}{\xi})\varphi(s, z(h_2(s)))ds \\ &= \psi\big(t_2, z(h_1(t_2))\big) - \psi\big(t_1, z(h_1(t_1))\big) \\ &+ \Big[S(\frac{t_2^\xi - t_1^\xi}{\xi}) - I\Big]\Big[S(\frac{t_1^\xi}{\xi})\Big(z_0 - \phi(z)) - \psi\big(0, z(h_1(0)\big)\Big) \\ &\quad + \int_0^{t_1} s^{\xi-1} S(\frac{t_1^\xi - s^\xi}{\xi})\varphi(s, z(h_2(s)))ds\Big] \\ &+ \int_{t_1}^{t_2} s^{\xi-1} S(\frac{t_2^\xi - s^\xi}{\xi})\varphi(s, z(h_2(s)))ds.\end{aligned}$$

Hence,

$$\begin{aligned}\|\Gamma(z)(t_2) - \Gamma(z)(t_1)\|_\xi &\leq \|\psi\big(t_2, z(h_1(t_2))\big) - \psi\big(t_1, z(h_1(t_1))\big)\|_\xi \\ &+ \|S(\frac{t_2^\xi - t_1^\xi}{\xi}) - I\| \Big[\|S(\frac{t_1^\xi}{\xi})\|\Big(\|z_0\|_\xi + \|\phi(z))\|_\xi + \|\psi\big(0, z(h_1(0)\big)\|_\xi\Big)\Big] \\ &+ \|S(\frac{t_2^\xi - t_1^\xi}{\xi}) - I\| \int_0^{t_1} s^{\xi-1}\|S(\frac{t_1^\xi - s^\xi}{\xi})\|\|\varphi(s, z(h_2(s)))\|_\xi ds \\ &+ \int_{t_1}^{t_2} s^{\xi-1}\|S(\frac{t_2^\xi - s^\xi}{\xi})\|\|\varphi(s, z(h_2(s)))\|_\xi ds.\end{aligned}$$

By using assumptions $(H1),(H2)$ and $(H5)$, we obtain

$$\begin{aligned}\|\Gamma(z)(t_2)-\Gamma(z)(t_1)\|_\xi &\leq L\|t_2-t_1\|+\|S(\frac{t_2^\xi-t_1^\xi}{\xi})-I\|\\ &\quad\times\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big(\|z_0\|_\xi+a\gamma+b+M+\frac{\tau^\xi}{\xi}|\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big)\\ &+\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\||\mu_r|_{\mathcal{L}^\infty\left([0,\tau],\mathbb{R}^+\right)}\Big(\frac{t_2^\xi-t_1^\xi}{\xi}\Big).\end{aligned}$$

The above inequality combined with the uniform continuity of the family $(S(t))_{t\geq 0}$ proves that $\Gamma(B_\gamma)$ is equicontinuous on $[0,\tau]$.

Step 4. We prove that $\Gamma: B_\gamma \longrightarrow B_\gamma$ is σ_c-contraction operator.
Let $D\subseteq B_\gamma$, then by lemma 2.1 there exists a countable set D_0 such that $D_0=\{z_n\}\subseteq D$, Hence, $\Gamma(D_0)$ becomes a countable subset of $\Gamma(D)$. Then, lemma 2.1 prove that $\sigma_c\big(\Gamma(D)\big)\leq 2\sigma_c\big(\Gamma(D_0)\big)$. Since, $\Gamma(D_0)$ is bounded and equicontinuous. Then by using Lemma 2.4, we have

$$\sigma_c\big(\Gamma(D_0)\big)=\max_{t\in[0,\tau]}\Big(\sigma\big(\Gamma(D_0(t))\big)\Big).$$

Then, one has

$$\begin{aligned}\sigma_c\big(\Gamma(D)\big) &\leq 2\sigma_c\big(\Gamma(D_0)\big)\\ &\leq 2\max_{t\in[0,\tau]}\Big(\sigma\big(\Gamma(D_0(t))\big)\Big)\\ &\leq 2\max_{t\in[0,\tau]}\Big(\sigma\Big(\psi\big(t,D_0(h_1(t))\big)+S\big(\frac{t^\xi}{\xi}\big)\Big(z_0-\phi(D_0)-\psi\big(0,D_0(h_1(0))\big)\Big)\\ &+\int_0^t s^{\xi-1}S(\frac{t^\xi-s^\xi}{\xi})\varphi(s,D_0(h_2(s)))ds\Big)\Big).\end{aligned}$$

By using point (4) of Lemma 2.2, we deduce that

$$\begin{aligned}\sigma_c\big(\Gamma(D)\big) &\leq 2\max_{t\in[0,\tau]}\Big(\sigma\Big(\psi\big(t,D_0(h_1(t))\big)\Big)+\sigma\Big(S\big(\frac{t^\xi}{\xi}\big)\Big(z_0-\phi(D_0)-\psi\big(0,D_0(h_1(0))\big)\Big)\Big)\\ &+\sigma\Big(\int_0^t s^{\xi-1}S(\frac{t^\xi-s^\xi}{\xi})\varphi(s,D_0(h_2(s)))ds\Big)\Big)\\ &\leq 2\max_{t\in[0,\tau]}\Big(\sigma\Big(\psi\big(t,D_0(h_1(t))\big)\Big)+2\max_{t\in[0,\tau]}\Big(\sigma\Big(S\big(\frac{t^\xi}{\xi}\big)\Big(z_0-\phi(D_0)-\psi\big(0,D_0(h_1(0))\big)\Big)\Big)\Big)\\ &+2\max_{t\in[0,\tau]}\Big(\int_0^t s^{\xi-1}S(\frac{t^\xi-s^\xi}{\xi})\varphi(s,D_0(h_2(s)))ds\Big).\end{aligned}$$

Since ϕ is compact, then $S\big(\frac{t^\xi}{\xi}\big)\big(z_0-\phi(D_0)\big)$ is relatively compact. Hence, using point 1 of Lemma 2.2 in the above inequality, we obtain

$$\begin{aligned}\sigma_c\big(\Gamma(D)\big) &\leq 2L_1\max_{t\in[0,\tau]}\Big(\sigma\Big(D_0(t)\Big)\Big)+2L_1\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\max_{t\in[0,\tau]}\Big(\sigma\Big(D_0(t)\Big)\Big)\\ &+2L_2\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\max_{t\in[0,\tau]}\Big(\sigma\Big(D_0(t)\Big)\Big)\frac{\tau^\xi}{\xi}\\ &\leq 2\Big(L_1+\big(L_1+L_2\frac{\tau^\xi}{\xi}\big)\sup_{t\in[0,\tau]}\|S(\frac{t^\xi}{\xi})\|\Big)\sigma_c(D).\end{aligned}$$

Since $2\left(L_1 + \left(L_1 + L_2\frac{\tau^\xi}{\xi}\right)\sup_{t\in[0,\tau]}\|S\left(\frac{t^\xi}{\xi}\right)\|\right) < 1$, then Γ is a σ_c-contraction operator.

Then, by employing Lemma 2.5, we establish that Γ has at least one fixed point in B_γ, which is a mild of solution of the Cauchy problem (1.2).

□

4. Conclusion

We have demonstrated the existence of mild solutions for a class of neutral fractional differential equations with nonlocal conditions in a Banach space without imposing the compactness condition on the semigroup family and the Lipschitz condition on the nonlocal condition. The main result is obtained through the utilization of semigroup theory combined with the Darbo-Sadovskii fixed point theorem.

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