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Some Common Fixed Point Theorems in $\mathcal{F}$-Bipolar Metric Spaces and Applications

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ABSTRACT: In this paper, we prove some common fixed point theorems for generalized rational type contraction in $\mathcal{F}$-bipolar metric spaces. These theorems also generalize and extend several interesting results of metric fixed point theory to the $\mathcal{F}$-bipolar metric context. In addition, there are some examples and applications for the obtained results.

Key Words: Common fixed point, rational type contraction, $\mathcal{F}$-bipolar metric spaces.

Contents

1. Introduction and Preliminaries

Frechet [3], in 1906, firstly initiated the metric space theory. Thereafter, many researchers have generalized the notion of metric by either weakening the metric condition or modifying the domain and range of the function (see for instance, [2], [5], [12], [13], [14], [16], [17], [20] [21]). Basically, we consider the distance between points of a single set in the process to generalize the metric. So, the question of distance between elements of two different sets can arise naturally, and such problems of measuring distance can be encountered in various fields of the mathematics and other sciences. Although, in 2016, Mutlu and Gurdal [13] introduced the concept of bipolar metric space to encounter such cases. Also, this new notion of generalization and improvement of a metric space leads to the existence and the development of fixed point theorems. However, in bipolar metric spaces, a lot of significant work has been done to the existence for a fixed point of various mappings (see [8], [9], [10], [11], [15], [18], [19]) and references therein. Very recently, in 2022, Rawat et al. [20] introduced a new generalized notion named as, $\mathcal{F}$-bipolar metric space by using the notions of $\mathcal{F}$-metric and bipolar metric. They also showed that every bipolar metric space and $\mathcal{F}$-metric space is an $\mathcal{F}$-bipolar metric space but the converse is not true in general. Next, they define a topology τ_{F_b} on $\mathcal{F}$-bipolar metric spaces using the concept of balls. Further, they proved some fixed point theorems which are extensions and generalizations of the Banach contraction principle in the setting of $\mathcal{F}$-bipolar metric spaces, along with an application to integral equation and homotopy theory. Hence, $\mathcal{F}$-bipolar metric fixed point theory is an active research area and it is capturing a lot of attention for further work.

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Moreover, the bipolar metric space is defined as the following.

Definition 1.1 ([13]) *Let X, $Y \neq \phi$ and $\varrho : X \times Y \to [0, \infty)$ be a mapping which satisfying the following properties*

(B1) $\mu = \nu$, *if* $\varrho(\mu, \nu) = 0$;

(B2) $\varrho(\mu, \nu) = 0$, *if* $\mu = \nu$;

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(B3) $\varrho(\mu,\nu) = \varrho(\nu,\mu)$, *if* $\mu,\nu \in X \cap Y$;

(B4) $\varrho(\mu_1,\nu_2) \le \varrho(\mu_1,\nu_1) + \varrho(\mu_2,\nu_1) + \varrho(\mu_2,\nu_2)$ *for all* $\mu_1,\mu_2 \in X$ *and* $\nu_1,\nu_2 \in Y$.

Then the mapping ϱ *is called a bipolar metric on the pair* (X,Y) *and the triple* (X,Y,ϱ) *is called a bipolar metric space.*

In 2018 Jleli and Samet [5] introduced the notion of $\mathcal{F}$-metric space in the following way.
Let $\mathcal{F}$ be the set of function f : $(0,\infty) \longrightarrow$ R satisfying the following conditions:

$(\mathcal{F}_1)$ f is non-decreasing, i.e., $0 < s < t \Longrightarrow f(s) \le f(t)$.

$(\mathcal{F}_2)$ for every sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} f(t_n) = -\infty$

Definition 1.2 ([5]) *Suppose that* X *is a nonempyt set, and* $\varrho : X \times X \longrightarrow [0,\infty)$ *is a given mapping. Suppose that there exists* $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ *such that for all* $(\mu,\nu) \in X \times X$,

(ϱ_1) $\varrho(\mu,\nu) = 0 \Longleftrightarrow \mu = \nu$;

(ϱ_2) $\varrho(\mu,\nu) = \varrho(\nu,\mu)$;

(ϱ_3) *for every* $N \in \mathbb{N}, N \ge 2$, *and for every* $(x_i)_{i=1}^N \subset X$ *with* $(x_1,x_N) = (\mu,\nu)$, *we have*

$$\varrho(\mu,\nu) > 0 \implies f(\varrho(\mu,\nu)) \le f\left(\sum_{i=1}^{N-1} \varrho(x_i,x_{i+1})\right) + \alpha.$$

Then ϱ *is said to be an* $\mathcal{F}$*-metric on X, and the pair* (X,ϱ) *is said to be an* $\mathcal{F}$*-metric space.*

Definition 1.3 ([20]) *Let* X, $Y \neq \phi$ *and* $\varrho : X \times Y \to [0,\infty)$ *be a mapping. Suppose that there exist* $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ *such that for all* $(\mu,\nu) \in X \times Y$:

(D_1) $\varrho(\mu,\nu) = 0, \Longleftrightarrow \mu = \nu$;

(D_2) $\varrho(\mu,\nu) = \varrho(\nu,\mu)$, *if* $\mu,\nu \in X \cap Y$;

(D_3) *for every* $N \in \mathbb{N}, N \ge 2$, *and for every* $(x_i)_{i=1}^N \subset X$ *and* $(y_i)_{i=1}^N \subset Y$ *with* $(x_1,y_N) = (\mu,\nu)$, *we have*

$$\varrho(\mu,\nu) > 0 \implies f(\varrho(\mu,\nu)) \le f\left(\sum_{i=1}^{N-1} \varrho(x_{i+1},y_i) + \sum_{i=1}^{N} \varrho(x_i,y_i)\right) + \alpha.$$

Then the mapping ϱ *is called an* $\mathcal{F}$*-bipolar metric on the pair* (X,Y) *and the triple* (X,Y,ϱ) *is called an* $\mathcal{F}$*-bipolar metric space.*

Definition 1.4 ([20]) *Let* (X_1,Y_1) *and* (X_2,Y_2) *be two pair of sets. A map* $S : X_1 \cup Y_1 \to X_2 \cup Y_2$ *is called*

(i) covariant if $S(X_1) \subseteq X_2$ *and* $S(Y_1) \subseteq Y_2$, *and denoted as* $S : (X_1,Y_1) \rightrightarrows (X_2,Y_2)$;

(ii) contravariant if $S(X_1) \subseteq Y_2$ *and* $S(Y_1) \subseteq X_2$, *and denoted as* $S : (X_1,Y_1) \rightleftarrows (X_2,Y_2)$.

Definition 1.5 ([20]) *Let* (X, Y, ϱ) *be an* $\mathcal{F}$*-bipolar metric space. Then,*

(1) X = *set of left points;* Y = *set of right points;* $X \cap Y$ = *set of central points.*

In particular, if $X \cap Y = \phi$ *then the space is called disjoint, and otherwise it is called joint. Unless otherwise stated, we shall work with joint spaces.*

(2) *A sequence* (μ_n) *on the set* X *is called a left sequence and a sequence* (ν_n) *on* Y *is called a right sequence. In a* $\mathcal{F}$*-bipolar metric space, a left or a right sequence is called simply a sequence.*

(3) *A sequence* (μ_n) *is said to be convergent to a point* μ*, if and only if* (μ_n) *is a left sequence,* $\lim_{n\to\infty}(\mu_n,\mu) = 0$ *and* $\mu \in Y$*, or* (μ_n) *is a right sequence,* $\lim_{n\to\infty}(\mu,\mu_n) = 0$ *and* $\mu \in X$.

(4) *A bisequence* (μ_n,ν_n) *on* (X, Y, ϱ) *is a sequence on the set* $X \times Y$*. Furthermore, if the sequences* (μ_n) *and* (ν_n) *are convergent, then the bisequence* (μ_n,ν_n) *is said to be convergent. In addition, if* (μ_n) *and* (ν_n) *converge to a common point* $t \in X \cap Y$*, then* (μ_n,ν_n) *is called biconvergent.*

(5) *A bisequence* (μ_n,ν_n) *is a Cauchy bisequence, if* $\lim_{n\to\infty}(\mu_n,\nu_n) = 0$.

Remark 1.1 In a $\mathcal{F}$-bipolar metric space, every convergent Cauchy bisequence is biconvergent.

Definition 1.6 ([20]) *A* $\mathcal{F}$*-bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent*

Definition 1.7 ([20]) *A covariant or a contravariant map* S *from the* $\mathcal{F}$*-bipolar metric space* (X_1, Y_1, ϱ_1) *to the* $\mathcal{F}$*-bipolar metric space* (X_2, Y_2, ϱ_2) *is continuous, if and only if* $(u_n) \to v$ *on* (X_1, Y_1, ϱ_1) *implies* $S(u_n) \to S(v)$ *on* (X_2, Y_2, ϱ_2).

Now, we have the following fixed point theorems on metric spaces.

Theorem 1.1 (Banach's contraction (1922), [1]) *Let* (X, ϱ) *be a complete metric space and* $T : X \to X$ *satisfying* $\varrho(T\mu, T\nu) \leq \mu_1 \varrho(\mu, \nu)$*, for all* $\mu, \nu \in X$*, with* $0 \leq \mu_1 < 1$*. Then* T *has a unique fixed point.*

Theorem 1.2 (Kannan's contraction (1969), [6]) *Let* (X, ϱ) *be a complete metric space and* $T : X \to X$ *satisfying* $\varrho(T\mu, T\nu) \leq \mu_1[\varrho(\mu, T\mu) + \varrho(\nu, T\nu)]$*, for all* $\mu, \nu \in X$*, with* $0 \leq \mu_1 < \frac{1}{2}$*. Then* T *has a unique fixed point.*

Besides, two other fixed point theorems have been obtained under some new contractive conditions, which are the following.

Theorem 1.3 (Khan's contraction (1975), [7]) *Let* (X, ϱ) *be a complete metric space and* $T : X \to X$ *satisfy*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, T\nu) + \varrho(T\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, T\nu) + \varrho(\nu, T\mu)},$$

for all $\mu, \nu \in X$*, with* $0 \leq \mu_1 < 1$*. Then* T *has a unique fixed point.*

Theorem 1.4 (Jaggi's contraction (1977), [4]) *Let* T *be a continuous self map defined on a complete metric space* (X, ϱ)*. Suppose that* T *satisfies the following contractive condition:*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(T\nu, \nu)}{\varrho(\mu, \nu)} + \mu_2 \varrho(\mu, \nu),$$

for all $\mu, \nu \in X$*,* $\mu \neq \nu$*, and for some* $\mu, \nu \in [0, 1)$ *with* $\mu + \nu < 1$*. Then* T *has a unique fixed point in* X*.*

2. Main Results

Now, we establish two common fixed point results for mappings satisfying the generalized rational type contractive conditions which extend Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 to $\mathcal{F}$-bipolar metric spaces. First we prove the following theorem.

Theorem 2.1 *Let (X,Y,ϱ) be a complete $\mathcal{F}$-bipolar metric space, and let $T,S : (X,Y,\varrho) \rightleftarrows (X,Y,\varrho)$ be a contravariant mapping satisfying*

$$\varrho(S\nu,T\mu) \leq \mu_1 \frac{\varrho(\mu,T\mu)\varrho(S\nu,\nu)}{\varrho(\mu,\nu)} + \mu_2\varrho(\mu,\nu) + \mu_3[\varrho(\mu,T\mu) + \varrho(S\nu,\nu)], \tag{2.1}$$

for all $(\mu,\nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$. Then $T,S : X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point, provided that T and S are continuous in (X,Y).

Proof: Let $\mu_0 \in X$ and $\nu_0 \in Y$ then for each $n \in \mathbb{N} \cup \{0\}$, we define

$$S\mu_{2n} = \nu_{2n}\ ,\ T\mu_{2n+1} = \nu_{2n+1}\ ,\ S\nu_{2n} = \mu_{2n+1}\ ,\ T\nu_{2n+1} = \mu_{2n+2}.$$

Then (μ_n,ν_n) is a bisequence in (X,Y,ϱ). Let $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $\mathcal{F}_2$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \alpha. \tag{2.2}$$

Now by (2.1), we get

$$\begin{aligned}
\varrho(\mu_{2n+1},\nu_{2n+1}) &= \varrho(S\nu_{2n},T\mu_{2n+1}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n+1},T\mu_{2n+1})\varrho(S\nu_{2n},\nu_{2n})}{\varrho(\mu_{2n+1},\nu_{2n})} + \mu_2\varrho(\mu_{2n+1},\nu_{2n}) \\
&\quad +\mu_3[\varrho(\mu_{2n+1},T\mu_{2n+1}) + \varrho(S\nu_{2n},\nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n+1},\nu_{2n+1})\varrho(\mu_{2n+1},\nu_{2n})}{\varrho(\mu_{2n+1},\nu_{2n})} + \mu_2\varrho(\mu_{2n+1},\nu_{2n}) \\
&\quad +\mu_3[\varrho(\mu_{2n+1},\nu_{2n+1}) + \varrho(\mu_{2n+1},\nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n+1},\nu_{2n+1}) + \mu_2\varrho(\mu_{2n+1},\nu_{2n}) + \mu_3\varrho(\mu_{2n+1},\nu_{2n+1}) + \mu_3\varrho(\mu_{2n+1},\nu_{2n})
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_{2n+1},\nu_{2n+1}) \leq \frac{\mu_2+\mu_3}{1-\mu_1-\mu_3}\varrho(\mu_{2n+1},\nu_{2n}). \tag{2.3}$$

Also, we have

$$\begin{aligned}
\varrho(\mu_{2n+1},\nu_{2n}) &= \varrho(S\nu_{2n},S\mu_{2n}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n},S\mu_{2n})\varrho(S\nu_{2n},\nu_{2n})}{\varrho(\mu_{2n},\nu_{2n})} + \mu_2\varrho(\mu_{2n},\nu_{2n}) + \mu_3[\varrho(\mu_{2n},S\mu_{2n}) + \varrho(S\nu_{2n},\nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n},\nu_{2n})\varrho(\mu_{2n+1},\nu_{2n})}{\varrho(\mu_{2n},\nu_{2n})} + \mu_2\varrho(\mu_{2n},\nu_{2n}) + \mu_3[\varrho(\mu_{2n},\nu_{2n}) + \varrho(\mu_{2n+1},\nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n+1},\nu_{2n}) + \mu_2\varrho(\mu_{2n},\nu_{2n}) + \mu_3\varrho(\mu_{2n},\nu_{2n}) + \mu_3\varrho(\mu_{2n+1},\nu_{2n})
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_{2n+1},\nu_{2n}) \leq \frac{\mu_2+\mu_3}{1-\mu_1-\mu_3}\varrho(\mu_{2n},\nu_{2n}). \tag{2.4}$$

Since $\mu_1 + \mu_2 + 2\mu_3 \in [0,1)$ and $\frac{\mu_2+\mu_3}{1-\mu_1-\mu_3} = \lambda(\text{say})$, $\lambda \in [0,1)$. Hence, from (2.3) and (2.4), we get

$$\varrho(\mu_{2n+1},\nu_{2n+1}) \leq \lambda^{4n+2}\varrho(\mu_0,\nu_0) \qquad \text{and} \qquad \varrho(\mu_{2n+1},\nu_{2n}) \leq \lambda^{4n+1}\varrho(\mu_0,\nu_0). \tag{2.5}$$

Now, we can get that for any $n \in \mathbb{N}$,

$$\varrho(\mu_{n+1},\nu_{n+1}) \leq \lambda^{2n+2}\varrho(\mu_0,\nu_0); \qquad \varrho(\mu_{n+1},\nu_n) \leq \lambda^{2n+1}\varrho(\mu_0,\nu_0) \qquad \text{and} \qquad \varrho(\mu_n,\nu_n) \leq \lambda^{2n}\varrho(\mu_0,\nu_0).$$

Now, from all the above equation , we get

$$\begin{aligned}
\sum_{i=n}^{m-1}\varrho(\mu_{i+1},\nu_i) + \sum_{i=n}^{m}\varrho(\mu_i,\nu_i) &= (\lambda^{2n+1} + \lambda^{2n+3} + \ldots + \lambda^{2m-1})\varrho(\mu_0,\nu_0) \\
&\quad +(\lambda^{2n} + \lambda^{2n+2} + \ldots + \lambda^{2m})\varrho(\mu_0,\nu_0) \\
&= \lambda^{2n}[1 + \lambda + \lambda^2 + \ldots + \lambda^{2m-2n-1}]\varrho(\mu_0,\nu_0) \\
&\leq \frac{\lambda^{2n}}{1-\lambda}\varrho(\mu_0,\nu_0), \quad m > n.
\end{aligned}$$

Since, $\lim\limits_{n\to\infty} \dfrac{\lambda^{2n}}{1-\lambda}\varrho(\mu_0,\nu_0) = 0,$ there exist $N \in \mathbb{N}$, such that

$$0 < \frac{\lambda^{2n}}{1-\lambda}\varrho(\mu_0,\nu_0) < \delta, \quad n \geq N.$$

For $m > n \geq N,$ using $(\mathcal{F}_1)$ and equation (2.2), we get

$$f\left(\sum_{i=n}^{m-1} \varrho(\mu_{i+1},\nu_i) + \sum_{i=n}^{m} \varrho(\mu_i,\nu_i)\right) \leq f\left(\frac{\lambda^{2n}}{1-\lambda}\varrho(\mu_0,\nu_0)\right) < f(\epsilon) - \alpha. \tag{2.6}$$

From (D_3) and above equation, we get $\varrho(\mu_n,\nu_m) > 0$

$$\Longrightarrow f(\varrho(\mu_n,\nu_m)) \leq f\left(\sum_{i=n}^{m-1} \varrho(\mu_{i+1},\nu_i) + \sum_{i=n}^{m} \varrho(\mu_i,\nu_i)\right) + \alpha < f(\epsilon).$$

Similarly, for $n > m \geq N,$ $\varrho(\mu_n,\nu_m) > 0$

$$\Longrightarrow f(\varrho(\mu_n,\nu_m)) \leq f\left(\sum_{i=m}^{n-1} \varrho(\mu_{i+1},\nu_i) + \sum_{i=m}^{n} \varrho(\mu_i,\nu_i)\right) + \alpha < f(\epsilon).$$

Then by $(\mathcal{F}_1)$, $\varrho(\mu_n,\nu_m) < \epsilon$, for all $m,n \geq N$. Therefore, (μ_n,ν_n) is a cauchy bisequence in (X,Y). By the completeness of (X,Y,ϱ), the bisequence (μ_n,ν_n) biconverges to some $\mu^* \in X \cap Y$ such that $\lim\limits_{n\to\infty}(\mu_n) = \lim\limits_{n\to\infty}(\nu_n) = \mu^*$. Also, $S(\mu_{2n}) = (\nu_{2n}) \longrightarrow \mu^* \in X \cap Y$ implies that $S(\mu_{2n})$ has a unique limit μ^* and $(\mu_n) \longrightarrow \mu^*$ implies that $(\mu_{2n}) \longrightarrow \mu^*$. Now, the continuity of S implies that $S(\mu_{2n}) \longrightarrow S\mu^*$. Therefore, $S\mu^* = \mu^*$.

Similarly, $T(\nu_{2n+1}) = (\mu_{2n+2}) \longrightarrow \mu^* \in X \cap Y$ implies that $T(\nu_{2n+1})$ has a unique limit μ^*, and $(\nu_n) \longrightarrow \mu^*$ implies that $(\nu_{2n+1}) \longrightarrow \mu^*$. Now, the continuity of T implies that $T(\nu_{2n+1}) \longrightarrow T\mu^*$. Therefore, $T\mu^* = \mu^*$. Thus, $S\mu^* = T\mu^* = \mu^*$, i.e., T and S have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. For, if $\nu^* \in X \cap Y$ is another common fixed point of S and T, that is, $S\nu^* = T\nu^* = \nu^* \in X \cap Y$. Then, we get

$$\begin{aligned}
\varrho(\nu^*,\mu^*) &= \varrho(S\nu^*,T\mu^*) \\
&\leq \mu_1\frac{\varrho(\mu^*,T\mu^*)\varrho(S\nu^*,\nu^*)}{\varrho(\mu^*,\nu^*)} + \mu_2\varrho(\mu^*,\nu^*) + \mu_3[\varrho(\mu^*,T\mu^*) + \varrho(S\nu^*,\nu^*)] \\
&= \mu_1\frac{\varrho(\mu^*,\mu^*)\varrho(\nu^*,\nu^*)}{\varrho(\mu^*,\nu^*)} + \mu_2\varrho(\mu^*,\nu^*) + \mu_3[\varrho(\mu^*,\mu^*) + \varrho(\nu^*,\nu^*)].
\end{aligned}$$

Therefore, $\varrho(\nu^*,\mu^*) \leq \mu_2\varrho(\mu^*,\nu^*)$, which is a contradiction and hence, $\mu^* = \nu^*$. This completes the theorem. □

Now, we have the following example to validate the Theorem 2.1.

Example 2.1 *Let $X = \{7,8,11,17\}$ and $Y = \{2,4,17,18\}$. Define $\varrho : X \times Y \to [0,\infty)$ as the usual metric, $\varrho(\mu,\nu) = |\ \mu - \nu\ |$. Then the triple (X,Y,ϱ) is an $\mathcal{F}$-bipolar metric space. The contravariant mapping $T,S : X \cup Y \rightleftarrows X \cup Y$, defined by*

$$T\mu = \begin{cases} 17, & \mu \in X \cup \{18\} \\ 18, & otherwise \end{cases} \qquad and \qquad S\mu = \begin{cases} 17, & \mu \in \{17,18\} \\ 18, & otherwise, \end{cases}$$

satisfy the inequality of Theorem 2.1 taking $\mu_1 = \dfrac{1}{3}$, $\mu_2 = \dfrac{1}{4}$, $\mu_3 = \dfrac{1}{5}$, and 17 $\in X \cap Y$ is the only common fixed point of T and S.

Moreover, by taking $T = S$ in Theorem 2.1 we get the following result which is a generalization of the Theorem 1.1, Theorem 1.2 and Theorem 1.4 in the context of a $\mathcal{F}$-bipolar metric space.

Corollary 2.1 *Let (X, Y, ϱ) be a complete $\mathcal{F}$-bipolar metric space and $T : (X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be a contravariant mapping satisfying*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(T\nu, \nu)}{\varrho(\mu, \nu)} + \mu_2 \varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(T\nu, \nu)],$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$. Then $T : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point, provided that T is continuous in (X, Y).

Remark 2.1 *The results obatained by Rawat et al. [20] are special cases of the Corollary 2.1.*

Now, we prove another common fixed point result in a $\mathcal{F}$-bipolar metric space as follows.

Theorem 2.2 *Let (X, Y, ϱ) be a complete $\mathcal{F}$-bipolar metric space and $T,S : (X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be a contravariant mapping satisfying*

$$\varrho(S\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, S\nu) + \varrho(S\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, S\nu) + \varrho(\nu, T\mu)} + \mu_2 \varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(S\nu, \nu)], \tag{2.7}$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$. Then $T,S : X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point, provided that T,S are continuous in (X, Y).

Proof: Let $\mu_0 \in X$ and $\nu_0 \in Y$ then for each $n \in \mathbb{N} \cup \{0\}$, we define

$$S\mu_{2n} = \nu_{2n}\ ,\ T\mu_{2n+1} = \nu_{2n+1}\ ,\ S\nu_{2n} = \mu_{2n+1}\ ,\ T\nu_{2n+1} = \mu_{2n+2}$$

Then (μ_n, ν_n) is a bisequence in (X, Y, ϱ). Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $\mathcal{F}_2$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \alpha. \tag{2.8}$$

Now, from (2.7), we get

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n+1}) &= \varrho(S\nu_{2n}, T\mu_{2n+1}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n+1}, T\mu_{2n+1})\varrho(\mu_{2n+1}, S\nu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})\varrho(\nu_{2n}, T\mu_{2n+1})}{\varrho(\mu_{2n+1}, S\nu_{2n}) + \varrho(\nu_{2n}, T\mu_{2n+1})} \\
&\quad + \mu_2 \varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, T\mu_{2n+1}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n+1}, \nu_{2n+1})\varrho(\mu_{2n+1}, \mu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})\varrho(\nu_{2n}, \nu_{2n+1})}{\varrho(\mu_{2n+1}, \mu_{2n+1}) + \varrho(\nu_{2n}, \nu_{2n+1})} \\
&\quad + \mu_2 \varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, \nu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1 \varrho(\mu_{2n+1}, \nu_{2n}) + \mu_2 \varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3 \varrho(\mu_{2n+1}, \nu_{2n+1}) + \mu_3 \varrho(\mu_{2n+1}, \nu_{2n})
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_{2n+1}, \nu_{2n+1}) \leq \frac{\mu_1 + \mu_2 + \mu_3}{1 - \mu_3} \varrho(\mu_{2n+1}, \nu_{2n}). \tag{2.9}$$

Also, we have

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n}) &= \varrho(S\nu_{2n}, S\mu_{2n}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n}, S\mu_{2n})\varrho(\mu_{2n}, S\nu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})\varrho(\nu_{2n}, S\mu_{2n})}{\varrho(\mu_{2n}, S\nu_{2n}) + \varrho(\nu_{2n}, S\mu_{2n})} \\
&\quad + \mu_2 \varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, S\mu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n}, \nu_{2n})\varrho(\mu_{2n}, \mu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})\varrho(\nu_{2n}, \nu_{2n})}{\varrho(\mu_{2n}, \mu_{2n+1}) + \varrho(\nu_{2n}, \nu_{2n})} \\
&\quad + \mu_2 \varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, \nu_{2n}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1 \varrho(\mu_{2n}, \nu_{2n}) + \mu_2 \varrho(\mu_{2n}, \nu_{2n}) + \mu_3 \varrho(\mu_{2n}, \nu_{2n}) + \mu_3 \varrho(\mu_{2n+1}, \nu_{2n})
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_{2n+1},\nu_{2n}) \leq \frac{\mu_1+\mu_2+\mu_3}{1-\mu_3}\varrho(\mu_{2n},\nu_{2n}). \tag{2.10}$$

Since, $\mu_1+\mu_2+2\mu_3 \in [0,1)$ and $\frac{\mu_1+\mu_2+\mu_3}{1-\mu_3} = \lambda'(\text{say}),\ \lambda' \in [0,1)$. Hence, from the previous two inequalities (2.9) and (2.10), we get

$$\varrho(\mu_{2n+1},\nu_{2n+1}) \leq \lambda'^{4n+2}\varrho(\mu_0,\nu_0) \qquad \text{and} \qquad \varrho(\mu_{2n+1},\nu_{2n}) \leq \lambda'^{4n+1}\varrho(\mu_0,\nu_0). \tag{2.11}$$

Now, we can get that for any $n \in \mathbb{N}$,

$$\varrho(\mu_{n+1},\nu_{n+1}) \leq \lambda'^{2n+2}\varrho(\mu_0,\nu_0); \qquad \varrho(\mu_{n+1},\nu_n) \leq \lambda'^{2n+1}\varrho(\mu_0,\nu_0) \qquad \text{and} \qquad \varrho(\mu_n,\nu_n) \leq \lambda'^{2n}\varrho(\mu_0,\nu_0).$$

Now, from all the above equation , we get

$$\begin{aligned}
\sum_{i=n}^{m-1}\varrho(\mu_{i+1},\nu_i)+\sum_{i=n}^{m-1}\varrho(\mu_i,\nu_i) &= (\lambda'^{2n+1}+\lambda'^{2n+3}+\ldots+\lambda'^{2m-1})\varrho(\mu_0,\nu_0)\\
&\quad +(\lambda'^{2n}+\lambda'^{2n+2}+\ldots+\lambda'^{2m})\varrho(\mu_0,\nu_0)\\
&= \lambda'^{2n}[1+\lambda'+\lambda'^2+\ldots+\lambda'^{2m-2n-1}]\varrho(\mu_0,\nu_0)\\
&\leq \frac{\lambda'^{2n}}{1-\lambda'}\varrho(\mu_0,\nu_0), \quad m>n.
\end{aligned}$$

Since, $\lim_{n\to\infty}\frac{\lambda'^{2n}}{1-\lambda'}\varrho(\mu_0,\nu_0)=0,$ there exist $N \in \mathbb{N}$, such that

$$0 < \frac{\lambda'^{2n}}{1-\lambda'}\varrho(\mu_0,\nu_0) < \delta, \quad n \geq N.$$

For $m > n \geq N,$ using $(\mathcal{F}_1)$ and equation (2.8), we get

$$f\left(\sum_{i=n}^{m-1}\varrho(\mu_{i+1},\nu_i)+\sum_{i=n}^{m}\varrho(\mu_i,\nu_i)\right) \leq f\left(\frac{\lambda^{2n}}{1-\lambda}\varrho(\mu_0,\nu_0)\right) < f(\epsilon)-\alpha. \tag{2.12}$$

From (D_3) and above equation, we get $\varrho(\mu_n,\nu_m)>0$

$$\Longrightarrow f(\varrho(\mu_n,\nu_m)) \leq f\left(\sum_{i=n}^{m-1}\varrho(\mu_{i+1},\nu_i)+\sum_{i=n}^{m}\varrho(\mu_i,\nu_i)\right)+\alpha < f(\epsilon).$$

Similarly, for $n > m \geq N,\ \varrho(\mu_n,\nu_m)>0$

$$\Longrightarrow f(\varrho(\mu_n,\nu_m)) \leq f\left(\sum_{i=m}^{n-1}\varrho(\mu_{i+1},\nu_i)+\sum_{i=m}^{n}\varrho(\mu_i,\nu_i)\right)+\alpha < f(\epsilon).$$

Then by $(\mathcal{F}_1), \varrho(\mu_n,\nu_m) < \epsilon$, for all $m,n \geq N$. Therefore, (μ_n,ν_n) is a cauchy bisequence in (X,Y). By the completeness of (X,Y,ϱ), the bisequence (μ_n,ν_n) biconverges to some $\mu^* \in X\cap Y$ such that $\lim_{n\to\infty}(\mu_n) = \lim_{n\to\infty}(\nu_n)=\mu^*$. Also, $S(\mu_{2n})=(\nu_{2n}) \longrightarrow \mu^* \in X\cap Y$ implies that $S(\mu_{2n})$ has a unique limit μ^*, and $(\mu_n) \longrightarrow \mu^*$ implies that $(\mu_{2n}) \longrightarrow \mu^*$. Now, the continuity of S implies that $S(\mu_{2n}) \longrightarrow S\mu^*$. Therefore, $S\mu^*=\mu^*$.

Similarly, $T(\nu_{2n+1})=(\mu_{2n+2}) \longrightarrow \mu^* \in X\cap Y$ implies that $T(\nu_{2n+1})$ has a unique limit μ^*, and $(\nu_n) \longrightarrow \mu^*$ implies that $(\nu_{2n+1}) \longrightarrow \mu^*$. Now, the continuity of T implies that $T(\nu_{2n+1}) \longrightarrow T\mu^*$. Therefore, $T\mu^*=\mu^*$. Thus, $S\mu^*=T\mu^*=\mu^*$, i.e., T and S have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. For, if $\nu^* \in X\cap Y$ is another common fixed point of S and T, that is, $S\nu^*=T\nu^*=\nu^* \in X\cap Y$. Then, we get

$$\begin{aligned}
\varrho(\nu^*,\mu^*) &= \varrho(S\nu^*,T\mu^*)\\
&\leq \mu_1\frac{\varrho(\mu^*,T\mu^*)\varrho(\mu^*,S\nu^*)+\varrho(S\nu^*,\nu^*)\varrho(\nu^*,T\mu^*)}{\varrho(\mu^*,S\nu^*)+\varrho(\nu^*,T\mu^*)}+\mu_2\varrho(\mu^*,\nu^*)+\mu_3[\varrho(\mu^*,T\mu^*)+\varrho(S\nu^*,\nu^*)]\\
&= \mu_1\frac{\varrho(\mu^*,\mu^*)\varrho(\mu^*,\nu^*)+\varrho(\nu^*,\nu^*)\varrho(\nu^*,\mu^*)}{\varrho(\mu^*,\nu^*)+\varrho(\nu^*,\mu^*)}+\mu_2\varrho(\mu^*,\nu^*)+\mu_3[\varrho(\mu^*,\mu^*)+\varrho(\nu^*,\nu^*)].
\end{aligned}$$

Therefore, $\varrho(\nu^*,\mu^*) \leq \mu_2\varrho(\mu^*,\nu^*)$, which is a contradiction and hence, $\mu^* = \nu^*$. This completes the theorem. □

The following example shows the validity our Theorem 2.2.

Example 2.2 *Let $X = \{7, 8, 17, 19\}$ and $Y = \{2, 4, 9, 17\}$. Define $\varrho : X \times Y \to [0,\infty)$ as the usual metric, $\varrho(\mu,\nu) = |\ \mu - \nu\ |$. Then the triple (X, Y, ϱ) is an $\mathcal{F}$-bipolar metric space. The contravariant mapping $T,S : X \cup Y \to X \cup Y$, defined as in Example 2.1, satisfy also the inequality of Theorem 2.2 with $\mu_1 = \frac{1}{3}$, $\mu_2 = \frac{1}{4}$, $\mu_3 = \frac{1}{5}$, and 17 $\in X \cap Y$ is the only common fixed point of T and S.*

Moreover, by taking $T = S$ in Theorem 2.2 we get the following results which is a generalization of the Theorem 1.1 , Theorem 1.2 and Theorem 1.3 in the context of a $\mathcal{F}$-bipolar metric space.

Corollary 2.2 *Let (X, Y, ϱ) be a complete $\mathcal{F}$-bipolar metric space and $T : (X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be a mapping satisfying*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, T\nu) + \varrho(T\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, T\nu) + \varrho(\nu, T\mu)} + \mu_2\varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(T\nu, \nu)],$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and for $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$. Then $T : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point, provided that T is continuous in (X, Y).

Remark 2.2 *The results obatained by Rawat et al. [20] are special cases of the Corollary 2.2.*

3. Applications

Now, we study the following application of proved results to the existence of a solution in the homotopy theory.

Theorem 3.1 *Let (S,T,ϱ) be a complete $\mathcal{F}$-bipolar metric space, and let (A,B) be an open subset of (S, T) so that $(\overline{A}, \overline{B})$ is a closed subset of (S, T) and $(A, B) \subseteq (\overline{A}, \overline{B})$. Suppose $L : (\overline{A} \cup \overline{B}) \times [0, 1] \to S \cup T$ is an operator satisfying the following conditions:*

(i) $\mu \neq L(\mu, k)$ for each $\mu \in \partial A \cup \partial B$ and $k \in [0, 1]$, where $(\partial A \cup \partial B$ stands for the boundary of $A \cup B$ in $S \cup T)$.

(ii) $\varrho(L(\nu, k), L(\mu, k)) \leq \mu_1 \dfrac{\varrho(\mu, L(\mu, k)\varrho(L(\nu, k), \nu))}{\varrho(\mu, \nu)} + \mu_2\varrho(\mu, \nu) + \mu_3[\varrho(\mu, L(\mu, k)) + \varrho(L(\nu, k), \nu)]$, for all $\mu \in \overline{A}, \nu \in \overline{B}, k \in [0, 1]$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$.

(ii) There exists an $M > 1$ such that $\varrho(L(\mu, \rho), L(\nu, \sigma)) \leq M\ |\ \rho - \sigma\ |$, for all $\mu \in \overline{A}, \nu \in \overline{B}$ and $\rho, \sigma \in [0, 1]$.

Then $L(., 0)$ has a fixed point if and only if $L(., 1)$ has a fixed point.

Proof: Let $C = \{\rho \in [0, 1] : \mu = L(\mu, \rho), \mu \in A\}$, $D = \{\sigma \in [0, 1] : \nu = L(\nu, \sigma), \nu \in B\}$. Since $L(., 0)$ has a fixed point in $A \cup B$, we have $0 \in C \cap D$. Thus $C \cap D$ is non-empty set. Now, we shall show that $C \cap D$ is both closed and open in $[0, 1]$ and so, by connetedness, $C = D = [0, 1]$. Let $(\{\rho_n\}_{n=1}^{\infty})$, $(\{\sigma_n\}_{n=1}^{\infty}) \subseteq (C, D)$ with $(\rho_n, \sigma_n) \to (\lambda, \lambda) \in [0, 1]$ *as* $n \to \infty$. We also claim that $\lambda \in C \cap D$. Since $(\rho_n, \sigma_n) \in (C, D)$ for $n = 0, 1, 2, 3, \ldots$, there exists a bisequence $(\mu_n, \nu_n) \in (A, B)$ such that $\nu_n = L(\mu_n, \rho_n)$ and $\mu_{n+1} = L(\nu_n, \sigma_n)$. Also, we get

$$\begin{aligned}
\varrho(\mu_{n+1}, \nu_n) &= \varrho(L(\nu_n, \sigma_n), L(\mu_n, \rho_n)) \\
&\leq \mu_1 \frac{\varrho(\mu_n, L(\mu_n, \rho_n))\varrho(L(\nu_n, \sigma_n), \nu_n)}{\varrho(\mu_n, \nu_n)} + \mu_2\varrho(\mu_n, \nu_n) \\
&\quad + \mu_3[\varrho(\mu_n, L(\mu_n, \rho_n) + \varrho(L(\nu_n, \sigma_n), \nu_n)] \\
&= \mu_1 \frac{\varrho(\mu_n, \nu_n)\varrho(\mu_{n+1}, \nu_n)}{\varrho(\mu_n, \nu_n)} + \mu_2\varrho(\mu_n, \nu_n) + \mu_3[\varrho(\mu_n, \nu_n) + \varrho(\mu_{n+1}, \nu_n)],
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_{n+1},\nu_n) \leq \frac{\mu_2+\mu_3}{1-\mu_1-\mu_3}\varrho(\mu_n,\nu_n).$$

And,
$$\begin{aligned}
\varrho(\mu_n,\nu_n) &= \varrho(L(\nu_{n-1},\sigma_{n-1}),L(\mu_n,\rho_n))\\
&\leq \mu_1\frac{\varrho(\mu_n,L(\mu_n,\rho_n))\varrho(L(\nu_{n-1},\sigma_{n-1}),\nu_{n-1})}{\varrho(\mu_n,\nu_{n-1})}+\mu_2\varrho(\mu_n,\nu_{n-1})\\
&+ \mu_3[\varrho(\mu_n,L(\mu_n,\rho_n)+\varrho(L(\nu_{n-1},\sigma_{n-1}),\nu_{n-1})]\\
&= \mu_1\frac{\varrho(\mu_n,\nu_n)\varrho(\mu_n,\nu_{n-1})}{\varrho(\mu_n,\nu_{n-1})}+\mu_2\varrho(\mu_n,\nu_{n-1})+\mu_3[\varrho(\mu_n,\nu_n)+\varrho(\mu_n,\nu_{n-1})],
\end{aligned}$$

$$\Rightarrow \qquad \varrho(\mu_n,\nu_n) \leq \frac{\mu_2+\mu_3}{1-\mu_1-\mu_3}\varrho(\mu_n,\nu_{n-1}).$$

By similar process as used in Theorem 2.1, we can easily prove that (μ_n,ν_n) is a Cauchy bisequence in (A,B). By completeness, there exists $\lambda_1 \in A\cap B$ such that $\lim_{n\to\infty}(\mu_n) = \lim_{n\to\infty}(\nu_n) = \lambda_1$. Now, we have

$$\begin{aligned}
\varrho(L(\lambda_1,\sigma),\nu_n) &= \varrho(L(\lambda_1,\sigma),L(\mu_n,\rho_n))\\
&\leq \mu_1\frac{\varrho(\mu_n,L(\mu_n,\rho_n)\varrho(L(\lambda_1,\sigma),\lambda_1)}{\varrho(\mu_n,\lambda_1)}+\mu_2\varrho(\mu_n,\lambda_1)\\
&\quad +\mu_3[\varrho(\mu_n,L(\mu_n,\rho_n))+\varrho(L(\lambda_1,\sigma),\lambda_1)]\\
&= \mu_1\frac{\varrho(\mu_n,\nu_n)\varrho(L(\lambda_1,\sigma),\lambda_1)}{\varrho(\mu_n,\lambda_1)}+\mu_2\varrho(\mu_n,\lambda_1)+\mu_3[\varrho(\mu_n,\nu_n)+\varrho(L(\lambda_1,\sigma),\lambda_1)].
\end{aligned}$$

Applying limit as $n\to\infty$, we get $\varrho(L(\lambda_1,\sigma),\lambda_1) \leq \mu_3\varrho(L(\lambda_1,\sigma),\lambda_1)$, which is a contradiction. Hence, $\varrho(L(\lambda_1,\sigma),\lambda_1) = 0$, which implies $L(\lambda_1,\sigma) = \lambda_1$. Similarly, $L(\lambda_1,\rho) = \lambda_1$. Therefore $\rho = \sigma \in C\cap D$, and clearly $C\cap D$ is closed in $[0,1]$.

Next, we have to prove that $C\cap D$ is open in $[0,1]$. Suppose $(\rho_0,\sigma_0) \in (C,D)$, then there is a bisequence (μ_0,ν_0) so that $\mu_0 = L(\mu_0,\rho_0)$, $\nu_0 = L(\nu_0,\sigma_0)$. Since $A\cup B$ is open, there is $r>0$ so that $B\varrho(\mu_0,r) \subseteq A\cup B$ and $B\varrho(r,\nu_0) \subseteq A\cup B$. Choose $\varrho \in (\sigma_0-\epsilon,\sigma_0+\epsilon)$ and $\sigma \in (\rho_0-\epsilon,\rho_0+\epsilon)$ such that $\mid \rho-\sigma_0 \mid\leq \frac{1}{M^n} < \frac{\epsilon}{2}$, $\mid \sigma-\rho_0 \mid\leq \frac{1}{M^n} < \frac{\epsilon}{2}$ and $\mid \rho_0-\sigma_0 \mid\leq \frac{1}{M^n} < \frac{\epsilon}{2}$. Thus, we have $\nu \in \overline{B_{C\cup D}(\mu_0,r)}$ $= \{\nu,\nu_0 \in B|\varrho(\mu_0,\nu) \leq r+\varrho(\mu_0,\nu_0)\}$ and $\mu \in \overline{B_{C\cup D}(r,\nu_0)} = \{\mu,\mu_0 \in A|\varrho(\mu,\nu_0) \leq r+\varrho(\mu_0,\nu_0)\}$. Additionally, we have

$$\begin{aligned}
\varrho(L(\mu,\rho),\nu_0) &= \varrho(L(\mu,\rho),L(\nu_0,\sigma_0))\\
&\leq \varrho(L(\mu,\rho),L(\nu,\sigma_0))+\varrho(L(\mu_0,\rho),L(\nu,\sigma_0))+\varrho(L(\mu_0,\rho),L(\nu_0,\sigma_0))\\
&\leq 2M\mid \rho-\sigma_0\mid +\varrho(L(\mu_0,\rho),L(\nu,\sigma_0))\\
&\leq \frac{2}{M^n-1}+\varrho(L(\mu_0,\rho),L(\nu,\sigma_0)).
\end{aligned}$$

Letting n $\to\infty$, we get $\varrho(L(\mu,\rho),\nu_0) \leq \varrho(L(\mu_0,\rho),L(\nu,\sigma_0))$. By (ii), we have

$$\begin{aligned}
\varrho(L(\mu,\rho),\nu_0) &\leq \varrho(L(\mu_0,\rho),L(\nu,\sigma_0))\\
&\leq \mu_1\frac{\varrho(\mu_0,L(\mu_0,\rho)\varrho(L(\nu,\sigma_0),\nu))}{\varrho(\mu_0,\nu)}+\mu_2\varrho(\mu_0,\nu)+\mu_3[\varrho(\mu_0,L(\mu_0,\rho)+\varrho(L(\nu,\sigma_0),\nu)]\\
&\leq \mu_1\frac{\varrho(\mu_0,\mu_0)(\varrho((\nu,\nu))}{\varrho(\mu_0,\nu)}+\mu_2\varrho(\mu_0,\nu)+\mu_3[\varrho(\mu_0,\mu_0)+\varrho(\nu,\nu)]\\
&\leq \mu_2\varrho(\mu_0,\nu).
\end{aligned}$$

$$\Rightarrow \qquad \varrho(L(\mu,\rho),\nu_0) \leq \varrho(\mu_0,\nu) \leq r+\varrho(\mu_0,\nu_0).$$

By analogous manner, we get $\varrho(\mu_0,L(\nu,\sigma)) \leq \varrho(\mu,\nu_0) \leq r+\varrho(\mu_0,\nu_0)$. But, as $n\to\infty$, we have

$$\varrho(\mu_0,\nu_0) = \varrho(L(\mu_0,\rho_0),L(\nu_0,\sigma_0)) \leq M\mid \rho_0-\sigma_0\mid\leq \frac{1}{M^{n-1}} \to 0, \text{ which implies } \mu_0 = \nu_0.$$

Thus, for each fixed σ, $\sigma = \rho \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$ and $L(.,\rho) : \overline{B_{C\cup D}(\mu_0, r)} \to \overline{B_{C\cup D}(\mu_0, r)}$. Since all the hypothesis of Corollary 2.1 hold, $L(.,\rho)$ has a fixed point in $\overline{A} \cap \overline{B}$, which must be in $A \cap B$. Then, $\rho = \sigma \in C \cap D$ for each $\sigma \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$. Hence, $(\sigma_0 - \epsilon, \sigma_0 + \epsilon) \in C \cap D$ which gives $C \cap D$ is open in [0,1]. We can use a similar process for the converse. □

Next, we discuss the existence and uniqueness of the solution of an integral equation as application of Corollary 2.1.

Theorem 3.2 *We consider the integral equation*

$$\gamma(\mu) = f(\mu) + \int_{X\cup Y} p(\mu, \nu, \gamma(\nu))\varrho\nu \quad \text{for } \mu \in X \cup Y, \quad \text{where } X\cup Y \text{ is a Lebesgue measureable set.}$$

Now, suppose the following:

(i) $P : (X^2 \cup Y^2) \times [0, \infty) \to [0, \infty)$ *and* $f \in L^\infty(X) \cup L^\infty(Y)$.

(ii) there is a continuous function $\tau : (X^2 \cup Y^2) \to [0, \infty)$ *such that*

$$\begin{aligned} | P(\mu, \nu, \gamma(\nu)) - P(\mu, \nu, \beta(\nu)) | \quad &\le \quad \tau(\mu, \nu)\Big\{\mu_1 \frac{| \beta(\nu) - T\beta(\nu) || T\gamma(\nu) - \gamma(\nu) |}{| \beta(\nu) - \gamma(\nu) |} \\ &+\mu_2 | \beta(\nu) - \gamma(\nu) | +\mu_3[| \beta(\nu) - T\beta(\nu) | + | T\gamma(\nu) - \gamma(\nu) |]\Big\}, \end{aligned}$$

for $\mu, \nu \in (X^2 \cup Y^2)$.

(iii) $\| \int_{X\cup Y} \tau(\mu, \nu)\varrho\nu \|\le 1$, *that is,* $\sup_{\mu\in X\cup Y} \int_{X\cup Y} | \tau(\mu, \nu) | \varrho\nu \le 1$.

Then the integral equation has a unique solution in $L^\infty(X) \cup L^\infty(Y)$.

Proof: Let $A = L^\infty(X)$ and $B = L^\infty(Y)$ be two normed linear spaces, where X, Y are Lebesgue measureable sets and $m(\text{X}\cup\text{Y}) < \infty$. Consider $\varrho : A \times B \to [0, \infty)$ to be defined by $\varrho(g, h) =\| g - h \|_\infty$, for all $g, h \in A \times B$. Then (A, B, ϱ) is a complete $\mathcal{F}$-bipolar metric space. Define the contravariant mapping $I : L^\infty(X) \cup L^\infty(Y) \to L^\infty(X) \cup L^\infty(Y)$ by

$$I(\gamma(\mu)) = \int_{X\cup Y} p(\mu, \nu, \gamma(\nu))\varrho\nu + f(\mu), \quad \text{where } \mu \in X \cup Y.$$

Now, we have

$$\begin{aligned}
\varrho(I(\gamma(\mu)), I(\beta(\mu))) &= \parallel I(\gamma(\mu)) - I(\beta(\mu)) \parallel \\
&= \Big| \int_{X\cup Y} p(\mu,\nu,\gamma(\nu))\varrho\nu - \int_{X\cup Y} p(\mu,\nu,\beta(\nu))\varrho\nu \Big| \\
&\leq \int_{X\cup Y} \mid p(\mu,\nu,\gamma(\nu)) - p(\mu,\nu,\beta(\nu)) \mid \varrho\nu \\
&\leq \int_{X\cup Y} \tau(\mu,\nu)\Big\{\mu_1 \frac{\mid \beta(\nu) - T\beta(\nu) \mid\mid T\gamma(\nu) - \gamma(\nu) \mid}{\mid \beta(\nu) - \gamma(\nu) \mid} + \mu_2 \mid \beta(\nu) - \gamma(\nu) \mid \\
&\quad + \mu_3\big(\mid \beta(\nu) - T\beta(\nu) \mid + \mid T\gamma(\nu) - \gamma(\nu) \mid \big)\Big\}\varrho\nu \\
&\leq \Big\{\mu_1 \frac{\parallel \beta(\nu) - T\beta(\nu) \parallel\parallel T\gamma(\nu) - \gamma(\nu) \parallel}{\parallel \beta(\nu) - \gamma(\nu) \parallel} + \mu_2 \parallel \beta(\nu) - \gamma(\nu) \parallel \\
&\quad + \mu_3\big(\parallel \beta(\nu) - T\beta(\nu) \parallel + \parallel T\gamma(\nu) - \gamma(\nu) \parallel \big)\Big\} \int_{X\cup Y} |\tau(\mu,\nu)|\varrho\nu \\
&\leq \Big\{\mu_1 \frac{\parallel \beta - T\beta \parallel\parallel T\gamma - \gamma \parallel}{\parallel \beta - \gamma \parallel} + \mu_2 \parallel \beta - \gamma \parallel \\
&\quad + \mu_3\big(\parallel \beta - T\beta \parallel + \parallel T\gamma - \gamma \parallel \big)\Big\} \sup_{\mu\in X\cup Y} \int_{X\cup Y} \mid \tau(\mu,\nu) \mid \varrho\nu \\
&\leq \mu_1 \frac{\parallel \beta - T\beta \parallel\parallel T\gamma - \gamma \parallel}{\parallel \beta - \gamma \parallel} + \mu_2 \parallel \beta - \gamma \parallel + \mu_3(\parallel \beta - T\beta \parallel + \parallel T\gamma - \gamma \parallel) \\
&= \mu_1 \frac{\varrho(\beta, I(\beta))\varrho(I(\gamma),\gamma)}{\varrho(\beta,\gamma)} + \mu_2\varrho(\beta,\gamma) + \mu_3(\varrho(\beta, I(\beta) + \varrho(I(\gamma),\gamma)).
\end{aligned}$$

It follows from Corollary 2.1 that I has a unique fixed point in $A \cup B$. □

4. Conclusion

In this article, we have proved some common fixed point results for generalized rational type contraction in $\mathcal{F}$-bipolar metric spaces. Our results are the extensions of Banach's contraction, Kannan's contraction, Jaggi's contraction and Khan's contraction of the metric space to a $\mathcal{F}$-bipolar metric space. Moreover, we have given some examples for justification of our results as well as provided applications for our obtained results. Henceforth, our results open a direction to new fixed point results and applications in a $\mathcal{F}$-bipolar metric space.

References

1. S. Banach, *Sur les op´erations dans les ensembles abstraits et leur application aux ´equations int´egrales,* Fund. Math. 3, 133-181, (1922).
2. S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform. Univ. Ostraviensis 1(1), 5-11, (1993).
3. M. Frechet, *Sur quelques points du calcul fonctionnel,* Rendiconti del Circolo Matematico di Palermo 22(1), 1-72, (1906).
4. D. S. Jaggi, *Some unique fixed point theorems,* Indian J. Pure Appl. Math. 8(2), 223-230, (1977).
5. M. Jleli, B. Samet, *On a new generalization of Metric Spaces,* J. Fixed Point Theory Appl. 20, 20 pages, (2018).
6. R. Kannan, *Some results on fixed points,* Bull. Cal. Math. Soc. 60, 71-76, (1968).
7. M. S. Khan, *A fixed point theorem for metric spaces,* Rend. Istit. Mat. Univ. Trieste. 8, 69-72, (1976).
8. G. N. V. Kishore, D. R. Prasad, B. S. Rao, V. S. Baghavan, *Some applications via common coupled fixed point theorems in bipolar metric spaces,* J. Crit. Rev. 7(2), 601-607, (2019).
9. G. N. V. Kishore, R. P. Agarwal, B. S. Rao, R. V. N. S. Rao, *Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications,* Fixed Point Theory Appl. 2018, 1-13, (2018).
10. G. N. V. Kishore, K. P. R. Rao, A. Sombabu, R. V. N. S. Rao, *Related results to hybrid pair of mappings and applications in bipolar metric spaces,* J. Math. 2019, 1-7, (2019).

11. G. N. V. Kishore, H. Isik, H. Aydi, B. S. Rao, D. R. Prasad, *On new types of contraction mappings in bipolar metric spaces and applications,* J. linear topol. algeb. 9(4), 253-266, (2020).

12. S. G. Matthews, *Partial metric topology,* The New York Academy of Sciences. Ann. N. Y. Acad. Sci. 1994, 183-197, (1994).

13. A. Mutlu, U. Gurdal, *Bipolar metric spaces and some fixed point theorems,* J. Nonlinear Sci. Appl. 9(9), 5362-5373, (2016).

14. A. Mutlu, K. Ozkan, U. Gurdal, *Locally and weakly contractive principle in bipolar metric spaces,* TWMS J. Appl.Eng. Math. 10(2), 379-388, (2020).

15. A. Mutlu, K. Ozkan, U. Gurdal, *Coupled fixed point theorems on bipolar metric spaces,* Eur. J. Pure Appl. Math. 10(4), 655-667, (2017).

16. J. Paul, U. C. Gairola, *Fixed point for generalized rational type contraction in partially ordered metric spaces,* Jñānābha 52(1), 162-166, (2022).

17. J. Paul, U. C. Gairola, *Existence of fixed point for rational type contraction in F-metric space,* Ganita 72(1), 369-374, (2022).

18. J. Paul, M. Sajid, N. Chandra, U. C. Gairola, *Some common fixed point theorems in bipolar metric spaces and applications,* AIMS Mathematics, 8(8), 19004-19017, (2023).

19. B. S. Rao, G. N. V. Kishore, G. K. Kumar, *Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy,* International Journal of Mathematics Trends and Technology (IJMTT) 63, 25-34, (2018).

20. S. Rawat, R. C. Dimri, A. Bartwal, *F-Bipolar metric spaces and fixed point theorems with applications,* J. Math. Comput SCI-JM. 26(2), 184-195, (2022).

21. S. Shukla, *Partial rectangular metric spaces and fixed point theorems,* Sci. World J. 2014, 1-7, (2014).

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