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Semilocal convergence analysis of Convex acceleration Newton's method under majorant condition in Banach space

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ABSTRACT:
This paper is devoted to give the semilocal convergence analysis of convex acceleration Newton's method under a new type of majorant condition for solving nonlinear operator equation in Banach space. This iterative method is used for finding roots of nonlinear equations. We have used the new type of majorant condition which does not assume existence of it's second derivative. We proposed convergence theorem which established Q-cubic convergence of the method and it's error estimation. We have illustrated two numerical examples based on our analysis to show the efficacy of our result. Computational order of convergence for both the examples have also been given.

Key Words: Majorant Conditions, Smale-type Convergence Criterion, Kantorovich-type Convergence Criterion, Computational Order of Convergence.

Contents

1. Introduction

Let us consider the equation

$$\Lambda(u) = 0 \tag{1.1}$$

where $\Lambda : \mathfrak{D} \subseteq \mathfrak{B}_1 \to \mathfrak{B}_2$ is a nonlinear operator defined on some nonempty convex subset $\mathfrak{D}$ of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two Banach spaces. Then the problem of obtaining an approximate solution ς of (1.1) is done by using iterative processes that is starting from one or several initial approximations u_0 of a solution ς, it generates a sequence of values $\{u_n\}_{n\in\mathbf{N}}$ so that each value of the sequence is a better approximation to the previous approximation of ς, or we can say that the sequence $\{\|u_n - \varsigma\|\}_{n\in\mathbf{N}}$ is convergent to zero.

The well known quadratically convergent method for solving such type of problem is $u_{n+1} = u_n - \Lambda'(u_n)^{-1}\Lambda(u_n), n = 0, 1, 2, \ldots$ where $u_0 \in \mathfrak{D}$ is an initial point. In general, one can be interested to analyze the convergence analysis issue of Newton's method, specially local and semilocal convergence analysis. It should be noted that the local convergence analysis deals with the acquaintance around the solution ς, while the semilocal convergence is based on the acquaintance around the initial guess u_0 of the solution ς. The famous Kantorovich theorem [14] guarantees convergence of Newton's method to a solution using semilocal conditions. It does not require a priori existence of a solution instead proving the existence of the solution and its uniqueness on some region. Also Smale's point theory [21] assumes that

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the nonlinear operator is analytic at the initial point which is an important result concerning Newton's method. Many researcher have studied the convergence issues of Kantorovich-type methods (for details see [3,6,8,20,25,26,28]). Wan and Han in [22,24] had studied the generalization and the particular cases of Smale's point estimate theory.

Let $\mathfrak{B}(u,\mathfrak{a})$ stands for the open ball and $\bar{\mathfrak{B}}(u,\mathfrak{a})$ is the corresponding close ball with radius $\mathfrak{a}$ and center u where $\mathfrak{a}$ is a positive number and $u \in \mathfrak{B}_1$. Following majorant condition is used by Ferreira and Svaiter [4,5] to study the convergence of Newton's method :

$$\|\Lambda'(u_0)^{-1}[\Lambda'(v)-\Lambda'(u)]\| \leq \mathfrak{f}'(\|v-u\|+\|u-u_0\|)-\mathfrak{f}'(\|u-u_0\|)$$

for $u,v \in \mathfrak{B}(u_0,\varrho), \varrho>0$, where$\|v-u\|+\|u-u_0\|<\varrho$ and $\mathfrak{f}:(0,\varrho)\to\mathbb{R}$ is a twice continuously differentiable, convex and strictly increasing function that satisfies $\mathfrak{f}(0)>0$, $\mathfrak{f}'(0)=-1$ and has zero in $(0,\varrho)$. Recently Ling and Xu [15] have discussed convergence analysis of Halley's method which shows a relationship between the majorizing function $\mathfrak{f}$ and the nonlinear operator Λ under the above mentioned conditions.

The convex acceleration of Newton's method to solve nonlinear Eq.(1.1) in Banach space is defined by

$$u_{n+1}=u_n-\big[I+\frac{1}{2}L_\Lambda(u_n)[I-L_\Lambda(u_n)]^{-1}\big]\Lambda'(u_n)^{-1}\Lambda(u_n), \tag{1.2}$$

where for $u \in \mathfrak{B}_1$, $L_\Lambda(u_n)$ is linear operator defined as

$$L_\Lambda(u_n)=\Lambda'(u_n)^{-1}\Lambda''(u_n)\Lambda'(u_n)^{-1}\Lambda(u_n).$$

This method is cubically convergent and the local/semilocal convergence results of this method can be seen in [1,9,18] under several types of continuity conditions. In [16] we presented the local convergence results of this method using majorizing functions.

Now inspired by the ideas of Ferreira and Svaiter [4,5] and Ling and Xu [15], here we present semilocal convergence of the iteration method (1.2) under majorant conditions if the nonlinear operator Λ is analytic at the initial point.

Suppose $\Lambda:\mathfrak{D}\subseteq\mathfrak{B}_1\to\mathfrak{B}_2$ be a twice Fréchet differentiable nonlinear operator on an open convex subset $\mathfrak{D}$ of a Banach space $\mathfrak{B}_1$ with values in a Banach space $\mathfrak{B}_2$. Let $\Lambda'(u_0)^{-1}\in\mathcal{BL}(\mathfrak{B}_2,\mathfrak{B}_1)$ exists at some point $u_0\in\mathfrak{D}$, where $\mathcal{BL}(\mathfrak{B}_2,\mathfrak{B}_1)$ be the set of bounded linear operators from $\mathfrak{B}_2$ into $\mathfrak{B}_1$. The operator Λ'' satisfies the majorant condition, if

$$\|\Lambda'(u_0)^{-1}[\Lambda''(v)-\Lambda''(u)]\| \leq \mathfrak{f}''(\|v-u\|+\|u-u_0\|)-\mathfrak{f}''(\|u-u_0\|), \tag{1.3}$$

for $u,v\in\mathfrak{B}(u_0,\varrho)$, where $\|v-u\|+\|u-u_0\|<\varrho$ and the following assumptions hold:

(A1) $\mathfrak{f}(0)>0$, $\mathfrak{f}''(0)>0$, $\mathfrak{f}'(0)=-1$.

(A2) $\mathfrak{f}''$ is convex and strictly increasing in $(0,\varrho)$.

(A3) $\mathfrak{f}$ has zero(s) in $(0,\varrho)$. Assume that $\bar{m}$ is the smallest zero and $\mathfrak{f}'(\bar{m})<0$.

Under the assumption that the second derivative of Λ satisfies the majorant condition (1.3), we establish a semilocal convergence of the iteration method. The assumption for guarantee of Q-cubic convergence of method are relaxed in our convergence analysis. Also, we obtain a new error estimate based on twice directional derivative of the majorizing function. We drop out the assumption of existence of a second root for the majorizing function, still guarantee Q-cubic convergence. Moreover, the majorizing function even do not need to be defined beyond its first root. Two numerical examples are given to show efficacy of our convergence analysis. In both numerical examples we have also calculated computational order of convergence.

The organization of our paper is as follows. We list some preliminary notions and properties of the majorizing function and the results regarding the majorizing sequence for the method in Section 2. Our main result is stated and proved in Section 3. Two special cases of main result are presented in Section 4. Some remarks and numerical examples are contained in Section 5 and finally section 6 contains conclusions.

2. Preliminaries

We will use the notion of Q-order of convergence (for more details see [13,17]) for a sequence $\{u_n\}$ in $\mathfrak{B}_1$. For the convergence analysis, we consider the following lemmas about elementary convex analysis [4].

Lemma 2.1 *Let $\varrho > 0$ and $x : (0, \varrho) \to \mathbb{R}$ be a continuously differentiable and convex function, then*

(i) $(1-\nu)x'(\nu b) \leq \dfrac{x(b) - x(\nu b)}{b} \leq (1-\nu)x'(b)\ \forall\ b \in (0, \varrho)$ *and* $0 \leq \nu \leq 1$.

(ii) $\dfrac{x(r) - x(\nu r)}{r} \leq \dfrac{x(s) - x(\nu s)}{s}\ \forall\ r, s \in (0, \varrho),\ r < s$ *and* $0 \leq \nu \leq 1$.

Lemma 2.2 *[5] Let $I \subset \mathbb{R}$ be an interval and $x : I \to \mathbb{R}$ be convex. Then*

(i) For any $r_0 \in int(I)$, there exists

$$D^- x(r_0) := \lim_{r \to r_0^-} \frac{x(r_0) - x(r)}{r_0 - r} = \sup_{r < r_0} \frac{x(r_0) - x(r)}{r_0 - r}.$$

(ii) If $r, s, t \in I$ and $r < s < t$, then

$$x(s) - x(r) \leq [x(t) - x(r)]\frac{s - r}{t - r}.$$

For the analysis convenience, we define the majorizing function with respect to the method as follows.

Definition 2.1 *Suppose $\Lambda : \mathfrak{D} \subseteq \mathfrak{B}_1 \to \mathfrak{B}_2$ be a twice continuously differentiable nonlinear operator. For a given guess $u_0 \in \mathfrak{D}$, we assume $\Lambda'(u_0)$ is nonsingular. A twice continuously differentiable function $\mathfrak{f} : (0, \varrho) \to \mathbb{R}$ is said to be majorizing function of Λ at u_0, if Λ'' satisfies the majorant conditions in $\mathfrak{B}(u_0, \varrho) \subset \mathfrak{D}$ and the following initial conditions:*

$$\|\Lambda'(u_0)^{-1}\Lambda(u_0)\| \leq \mathfrak{f}(0), \qquad \|\Lambda'(u_0)^{-1}\Lambda''(u_0)\| \leq \mathfrak{f}''(0).$$

Let

$$\mathfrak{I}_\Lambda(u) := u - \Big[I + \frac{1}{2}H_\Lambda(u)[I - \Lambda_\Lambda(u)]^{-1}\Big]\Lambda'(u)^{-1}\Lambda(u)$$

be the iterative function of convex acceleration Newton's method, where

$$H_\Lambda(u_n) = \Lambda'(u_n)^{-1}\Lambda''(u_n)\Lambda'(u_n)^{-1}\Lambda(u_n).$$

Suppose $\mathfrak{f}$ is the majorizing function of Λ. Then the convex acceleration Newton's method applied to $\mathfrak{f}$ can be denoted as

$$\mathfrak{I}_\mathfrak{f}(m) := m - \Big[I + \frac{H_\mathfrak{f}(m)}{2(1 - H_\mathfrak{f}(m))}\Big]\frac{\mathfrak{f}(m)}{\mathfrak{f}'(m)}, \quad m \in (0, \varrho)$$

where

$$H_\mathfrak{f}(m) = \frac{\mathfrak{f}(m)\mathfrak{f}''(m)}{\mathfrak{f}'(m)^2}.$$

Now we describe some basic properties of the majorizing function $\mathfrak{f}$ by the following lemmas.

Lemma 2.3 *Let $\varrho > 0$ and $\mathfrak{f} : (0, \varrho) \to \mathbb{R}$ be a twice continuously differentiable function which satisfies assumptions $(A1) - (A3)$. Then*

(i) $\mathfrak{f}'$ is strictly convex and strictly increasing on $(0, \varrho)$.

(ii) $\mathfrak{f}$ *is strictly convex on* $(0, \varrho)$, $\mathfrak{f}(m) > 0$ *for* $m \in (0, \bar{m})$ *and equation* $\mathfrak{f}(m) = 0$ *has at most one root in* $(\bar{m}, \varrho)$.

(iii) $-1 < \mathfrak{f}'(m) < 0$ *for* $m \in (0, \bar{m})$.

Lemma 2.4 *Let* $\mathfrak{f} : (0, \varrho) \to \mathbb{R}$ *be a twice continuously differentiable function which satisfies the assumptions* $(A1) - (A3)$. *Then* $0 \leq H_{\mathfrak{f}}(m) \leq \dfrac{1}{2}$ *for* $m \in [0, \bar{m}]$.

Proof: Let us define a function

$$\psi(z) = \mathfrak{f}(m) + \mathfrak{f}'(m)(z - m) + \frac{1}{2}\mathfrak{f}''(m)(z - m)^2, z \in [m, \bar{m}].$$

Then, by lemma 2.3, $\psi(m) = \mathfrak{f}(m) > 0$ for $m \in (0, \bar{m})$. Also

$$\psi(\bar{m}) = \mathfrak{f}(m) + \mathfrak{f}'(m)(\bar{m} - m) + \frac{1}{2}\mathfrak{f}''(m)(\bar{m} - m)^2.$$

By Taylor's formula, one can get

$$\mathfrak{f}(\bar{m}) = \mathfrak{f}(m) + \mathfrak{f}'(m)(\bar{m} - m) + \frac{1}{2}\mathfrak{f}''(m)(\bar{m} - m)^2 + \int_0^1 (1 - \sigma)[\mathfrak{f}''(m + \sigma(\bar{m} - m)) - \mathfrak{f}''(m)](\bar{m} - m)^2 d\sigma.$$

Since $\mathfrak{f}(\bar{m}) = 0$ and $\mathfrak{f}''$ is increasing, it follows that $\psi(\bar{m}) \leq 0$. Thus, there exists a real root of $\psi(z)$ on $[m, \bar{m}]$. So, $\mathfrak{f}'(m)^2 - 2\mathfrak{f}''(m)\mathfrak{f}(m) \geq 0$, which is equivalent to $\dfrac{\mathfrak{f}(m)\mathfrak{f}''(m)}{\mathfrak{f}'(m)^2} \leq \dfrac{1}{2}$. Therefore, $0 \leq H_{\mathfrak{f}}(m) \leq \frac{1}{2}$ for $m \in [0, \bar{m}]$. The proof is complete. □

Lemma 2.5 *Let* $\mathfrak{f} : (0, \varrho) \to \mathbb{R}$ *be a twice continuously differentiable function which satisfies assumptions* $(A1) - (A3)$. *Then for all* $m \in (0, \bar{m})$, $m < \mathfrak{J}_{\mathfrak{f}}(m) < \bar{m}$. *Moreover,* $\mathfrak{f}'(\bar{m}) < 0$ *if and only if there exist* $m \in (\bar{m}, \varrho)$ *such that* $\mathfrak{f}(m) < 0$.

Proof: For $m \in (0, \bar{m})$ since $\mathfrak{f}(m) > 0$, $-1 < \mathfrak{f}'(\bar{m}) < 0$ from lemma 2.3 and $0 \leq H_{\mathfrak{f}}(m) \leq \frac{1}{2}$ from lemma 2.4, we have $m < \mathfrak{J}_{\mathfrak{f}}(m)$. Also, for any $m \in (0, \bar{m}]$, by using the definition of directional derivative and assumption $(A2)$ it follows that $D^-\mathfrak{f}''(m) > 0$.

Thus $\mathfrak{J}_{\mathfrak{f}}(m) < \mathfrak{J}_{\mathfrak{f}}(\bar{m}) = \bar{m}$ for any $m \in (0, \bar{m})$. Now, if $\mathfrak{f}'(\bar{m}) < 0$, then obviously there exists $m \in (\bar{m}, \varrho)$ such that $\mathfrak{f}(m) < 0$. Conversely, noting that $\mathfrak{f}'(\bar{m}) = 0$, by lemma 2.2(ii), we have $\mathfrak{f}(m) > \mathfrak{f}(\bar{m}) + \mathfrak{f}'(m)(\bar{m} - m)$ for $m \in (\bar{m}, \varrho)$, which implies that $\mathfrak{f}'(\bar{m}) < 0$. This completes the proof. □

One can easily note that the condition $\mathfrak{f}'(\bar{m}) < 0$ in $(A3)$ implies:

- $\mathfrak{f}(\bar{m}^*) = 0$ for some $\bar{m}^* \in (\bar{m}, \varrho)$.
- $\mathfrak{f}(m) < 0$ for some $m \in (\bar{m}, \varrho)$.

Lemma 2.6 *Let* $\mathfrak{f} : (0, \varrho) \to \mathbb{R}$ *be a twice continuously differentiable function which satisfies assumptions* $(A1) - (A3)$. *Then*

$$\bar{m} - \mathfrak{J}_{\mathfrak{f}}(m) \leq \Big[\frac{1}{2}\frac{\mathfrak{f}''(\bar{m})^2}{\mathfrak{f}'(\bar{m})^2} + \frac{1}{3}\frac{D^-\mathfrak{f}''(\bar{m})}{-\mathfrak{f}'(\bar{m})}\Big](\bar{m} - m)^3, \quad m \in (0, \bar{m}). \tag{2.1}$$

Proof: We omit the proof as it can be seen in our paper [16]. □

Since $\mathfrak{f}$ is the majorizing sequence to Λ at u_0, let us consider $\{m_k\}$ denotes the majorizing sequence generated by

$$m_0 = 0, m_{k+1} = \mathfrak{J}_{\mathfrak{f}}(m_k) = m_k - \Big[I + \frac{H_{\mathfrak{f}}(m_k)}{2(1 - H_{\mathfrak{f}}(m_k))}\Big]\frac{\mathfrak{f}(m_k)}{\mathfrak{f}'(m_k)}, k = 0, 1, 2, \ldots \tag{2.2}$$

Then by using lemmas 2.4, 2.5 and 2.6 we can conclude that

Theorem 2.1 *Let the sequence* $\{m_k\}$ *be defined by* (2.2). *Then* $\{m_k\}$ *is well defined, strictly increasing and is contained in* $(0, \bar{m})$. *Moreover,* $\{m_k\}$ *satisfies* (2.1) *and converges to* $\bar{m}$ *with* $Q-$*cubic.*

3. Semilocal convergence results

We have studied the semilocal convergence analysis of the method in this section. To establish the semilocal convergence results following lemmas are important.

Lemma 3.1 *Assume $\|u-u_0\| \le m < \bar{m}$. If $\mathfrak{f} : (0,\bar{m}) \to \mathbb{R}$ be a twice continuously differentiable and is the majorizing function to Λ at the initial point u_0. Then $\Lambda'(u)$ is nonsingular and*

$$\|\Lambda'(u)^{-1}\Lambda'(u_0)\| \le -\frac{1}{\mathfrak{f}'(\|u-u_0\|)} \le -\frac{1}{\mathfrak{f}'(m)}.$$

In particular, Λ' is nonsingular in $\mathfrak{B}(u_0,\bar{m})$.

Proof: Take $m \in \bar{\mathfrak{B}}(u_0,\bar{m})$, $0 \le m < \bar{m}$, then

$$\Lambda'(u) = \Lambda'(u_0) + \int_0^1 [\Lambda''(u_0+\sigma(u-u_0)) - \Lambda''(u_0)](u-u_0)d\sigma + \Lambda''(u_0)(u-u_0)$$

and

$$\begin{aligned}\|I-\Lambda'(u_0)^{-1}\Lambda'(u)\| &\le \int_0^1 \|\Lambda'(u_0)^{-1}[\Lambda''(u_0+\sigma(u-u_0)) - \Lambda''(u_0)]\|\|u-u_0\|d\sigma \\ &\quad +\|\Lambda'(u_0)^{-1}\Lambda''(u_0)\|\|u-u_0\| \\ &\le \int_0^1 [\mathfrak{f}''(\sigma(\|u-u_0\|) - \mathfrak{f}''(0)]\|u-u_0\|d\sigma + \|\mathfrak{f}''(0)\|u-u_0\| \\ &= \mathfrak{f}'(\|u-u_0\|) - \mathfrak{f}'(0).\end{aligned}$$

So, we get

$$\|I-\Lambda'(u_0)^{-1}\Lambda'(u)\| \le \mathfrak{f}'(m) - \mathfrak{f}'(0) < 1$$

as $\mathfrak{f}'(0) = -1$ and $-1 < \mathfrak{f}'(m) < 0$ for $(0,\bar{m})$ from Lemma 2.3. Therefore, it follows from Banach lemma [14] that $\Lambda'(u_0)^{-1}\Lambda'(u)$ is nonsingular and the lemma holds. This completes the proof. □

Lemma 3.2 *Suppose $\|u-u_0\| \le m < \bar{m}$. If $\mathfrak{f} : (0,\bar{m}) \to \mathbb{R}$ be a twice continuously differentiable and is the majorizing function to Λ at u_0. Then $\|\Lambda'(u_0)^{-1}\Lambda''(u)\| \le \mathfrak{f}''(\|u-u_0\|) \le \mathfrak{f}''(m)$.*

Proof: By using majorant conditions, we have

$$\begin{aligned}\|\Lambda'(u_0)^{-1}\Lambda''(u)\| &\le \|\Lambda'(u_0)^{-1}[\Lambda''(u) - \Lambda''(u_0)]\| + \|\Lambda'(u_0)^{-1}\Lambda''(u)\| \\ &\le \mathfrak{f}''(\|u-u_0\|) - \mathfrak{f}''(0) + \mathfrak{f}''(0) = \mathfrak{f}''(\|u-u_0\|).\end{aligned}$$

Since $\mathfrak{f}''$ is strictly increasing, we get $\mathfrak{f}''(\|u-u_0\|) \le \mathfrak{f}''(m)$. The proof is complete. □

Lemma 3.3 *Suppose that $\mathfrak{f} : (0,\bar{m}) \to \mathbb{R}$ be twice continuously differentiable and let $\{u_k\}$ be generated by the convex acceleration of Newton's method and $\{m_k\}$ be generated by (2.2). If $\mathfrak{f}$ is the majorizing function to Λ at u_0, then, for all $k = 0,1,2,\dots$ we have*

(i) $\Lambda'(u_k)^{-1}$ *exists and* $\|\Lambda'(u_k)^{-1}\Lambda'(u_0)\| \le -\dfrac{1}{\mathfrak{f}'(\|u_k-u_0\|)} \le -\dfrac{1}{\mathfrak{f}'(m_k)}$.

(ii) $\|\Lambda'(u_0)^{-1}\Lambda''(m_k)\| \le \mathfrak{f}''(m_k)$.

(iii) $\|\Lambda'(u_0)^{-1}\Lambda(u_{k+1})\| \le \mathfrak{f}(m_{k+1})\Big(\dfrac{\|u_{k+1}-u_k\|}{m_{k+1}-m_k}\Big)^3$.

(iv) $[I-H_\Lambda(u_k)]^{-1}$ *exists and* $\|[I-H_\Lambda(u_k)]^{-1}\| \le \dfrac{1}{1-H_{\mathfrak{f}}(m_k)}$.

(v) $\|u_{k+2}-u_{k+1}\| \le (m_{k+2}-m_{k+1})\Big(\frac{\|u_{k+1}-u_k\|}{m_{k+1}-m_k}\Big)^3.$

Proof: For the case $k=0$, $(i)-(ii)$ are obvious. Also, we have

$$\begin{aligned}\|u_1-u_0\| &= \|\Big(I+\frac{1}{2}H_\Lambda(u_0)[I-H_\Lambda(u_0)]^{-1}\Big)\Lambda'(u_0)^{-1}\Lambda(u_0)\| \\ &\le \Big(1+\frac{1}{2}\frac{H_\mathfrak{f}(m_0)}{1-H_\mathfrak{f}(m_0)}\Big)\frac{\mathfrak{f}(m_0)}{\mathfrak{f}'(m_0)} = m_1-m_0\end{aligned}$$

and

$$\Lambda(u_1)=\frac{1}{2}\Lambda''(u_0)H_\Lambda(u_0)(u_1-u_0)^2+\int_0^1(1-\sigma)[\Lambda''(u_0+\sigma(u_1-u_0))-\Lambda''(u_0)](u-u_0)^2d\sigma.$$

Therefore

$$\begin{aligned}\|\Lambda'(u_0)^{-1}\Lambda(u_1)\| &\le \frac{1}{2}\|\Lambda'(u_0)^{-1}\Lambda''(u_0)\|\|H_\Lambda(u_0)\|\|u_1-u_0\|^2+\int_0^1\Big[\mathfrak{f}''(\sigma\|u_1-u_0\|+\|u_0-u_0\|) \\ &\quad -\mathfrak{f}''(\|u_0-u_0\|)\Big]\|u_1-u_0\|^2(1-\sigma)d\sigma \\ &\le \left(\frac{\|u_1-u_0\|}{m_1-m_0}\right)^3\Bigg[\frac{1}{2}\mathfrak{f}''(m_0)\frac{\mathfrak{f}(m_0)\mathfrak{f}''(m_0)}{\mathfrak{f}'(m_0)^2}(m_1-m_0)^2 \\ &\quad +\int_0^1[\mathfrak{f}''(\sigma(m_1-m_0)+m_0)-\mathfrak{f}''(m_0)](m_1-m_0)^2(1-\sigma)d\sigma\Bigg] \\ &= \mathfrak{f}(m_1)\Big(\frac{\|u_1-u_0\|}{m_1-m_0}\Big)^3.\end{aligned}$$

So (iii) hold for $k=0$. And for $k=0$, we have

$$\|H_\Lambda(u_0)\| \le \mathfrak{f}''(0)\mathfrak{f}(0)=\frac{\mathfrak{f}''(0)\mathfrak{f}(0)}{\mathfrak{f}'(0)^2}=H_\mathfrak{f}(0)=H_\mathfrak{f}(m_0)<1.$$

Therefore, by Banach lemma [14], $[I-H_\Lambda(u_0)]^{-1}$ exists and $\|[I-H_\Lambda(u_0)]^{-1}\| \le \dfrac{1}{1-H_\mathfrak{f}(m_0)}$.

This shows that (iv) holds for $k=0$. Now we have

$$\begin{aligned}\|u_2-u_1\| &\le \|I+\frac{1}{2}H_\Lambda(u_1)[I-H_\Lambda(u_1)]^{-1}\|\|\Lambda'(u_1)^{-1}\Lambda(u_1)\| \\ &\le \|I+\frac{1}{2}H_\Lambda(u_1)[I-H_\Lambda(u_1)]^{-1}\|\|\Lambda'(u_1)^{-1}\Lambda'(u_0)\|\|\Lambda'(u_0)^{-1}\Lambda(u_1)\| \\ &\le \Big(1+\frac{1}{2}\frac{H_\mathfrak{f}(m_1)}{1-H_\mathfrak{f}(m_1)}\Big)\frac{\mathfrak{f}(m_1)}{\mathfrak{f}'(m_1)}\Big(\frac{\|u_1-u_0\|}{m_1-m_0}\Big)^3 \\ &= (m_2-m_1)\Big(\frac{\|u_1-u_0\|}{m_1-m_0}\Big)^3.\end{aligned}$$

Therefore, (v) holds for $k=0$. Now assuming that it holds for some $n\in\mathbb{N}$. It follows from Lemmas 3.1 and 3.2 that (i) and (ii) holds for $k=n+1$ respectively. Also by the Theorem 2.1 and the induction hypothesis (v), we have $\|u_{k+1}-u_0\| \le m_{k+1} < \bar{m}$. Now, we have

$$\Lambda(u_{n+1})=\frac{1}{2}\Lambda''(u_n)H_\Lambda(u_n)(u_{n+1}-u_n)^2+\int_0^1(1-\sigma)[\Lambda''(u_n+\sigma(u_{n+1}-u_n))-\Lambda''(u_n)](u-u_n)^2d\sigma.$$

Applying the majorant condition, Lemma 2.2 and the inductive hypotheses $(i)-(ii)$ and $(iv)-(v)$, we have

$$\begin{aligned}
\|\Lambda'(u_0)^{-1}\Lambda(u_{n+2})\| &\leq \frac{1}{2}\|\Lambda'(u_0)^{-1}\Lambda''(u_{n+1})\|\|H_\Lambda(u_{n+1})\|\|u_{n+2}-u_{n+1}\|^2 \\
&\quad + \int_0^1 [\mathfrak{f}''(\sigma\|u_{n+2}-u_{n+1}\|+\|u_{n+1}-u_0\|) \\
&\quad -\mathfrak{f}''(\|u_{n+1}-u_0\|)]\|u_{n+2}-u_{n+1}\|^2(1-\sigma)d\sigma \\
&\leq \Big(\frac{\|u_{k+1}-u_k\|}{m_{k+1}-m_k}\Big)^3\Big[\frac{1}{2}\mathfrak{f}''(m_{n+1})\frac{\mathfrak{f}(m_{n+1})\mathfrak{f}''(m_{n+1})}{\mathfrak{f}'(m_{n+1})^2}(m_{k+2}-m_{k+1})^2 \\
&\quad + \int_0^1 [\mathfrak{f}''(\sigma(m_{n+2}-m_{n+1})+m_{n+1})-\mathfrak{f}''(m_{n+1})](m_{n+2}-m_{n+1})^2(1-\sigma)d\sigma\Big] \\
&= \mathfrak{f}(m_{n+2})\Big(\frac{\|u_{k+1}-u_k\|}{m_{k+1}-m_k}\Big)^3.
\end{aligned}$$

This shows that (iii) holds for $k=n+1$. By inductive hypotheses $(i)-(ii)$ and Lemma 2.4, we see that (iv) holds for $k=n+1$. Also,

$$\begin{aligned}
\|u_{n+2}-u_{n+1}\| &\leq \|I+\frac{1}{2}H_\Lambda(u_{n+1})[I-H_\Lambda(u_{n+1})]^{-1}\|\|\Lambda'(u_{n+1})^{-1}\Lambda(u_{n+1})\| \\
&\leq \Big(1+\frac{1}{2}\frac{H_\mathfrak{f}(m_{n+1})}{1-H_\mathfrak{f}(m_{n+1})}\Big)\frac{\mathfrak{f}(m_{n+1})}{\mathfrak{f}'(m_{n+1})}\Big(\frac{\|u_{n+1}-u_n\|}{m_{n+1}-m_n}\Big)^3 \\
&= (m_{n+2}-m_{n+1})\Big(\frac{\|u_{n+1}-u_n\|}{m_{n+1}-m_n}\Big)^3 \qquad (3.1)
\end{aligned}$$

i.e. (v) holds for $k=n+1$. Therefore, it holds for all $k=0,1,2,\ldots$. This completes the proof. □

Now the semilocal convergence theorem for the convex acceleration Newton's method is given as follows.

Theorem 3.1 *Suppose $\Lambda:\mathfrak{D}\subseteq\mathfrak{B}_1\to\mathfrak{B}_2$ be a twice continuously differentiable nonlinear operator and $\mathfrak{D}$ is open and convex. Consider that for a given initial guess $u_0\in\mathfrak{D}$, $\Lambda'(u_0)$ is nonsingular that is $\Lambda'(u_0)^{-1}$ exists and that $\mathfrak{f}$ is the majorizing function to Λ at u_0 and $\mathfrak{f}$ satisfies the assumptions $(A1)-(A3)$. Then the sequence $\{u_k\}$ generated by the method (1.2) for solving the Eq. (1.1) with a starting point u_0 is well defined, is contained in $\mathfrak{B}(u_0,\bar{m})$ and converges to a solution $\mathfrak{a}^*\in\bar{\mathfrak{B}}(u_0,\bar{m})$ of the Eq.(1.1).*

Proof: Lemma 3.3 conclude that the sequence $\{u_k\}$ is well defined. It follows from Lemma 3.3 (v) and Theorem 2.1 that, $\|u_k-u_0\|\leq m_{k+1}<\bar{m}$ for any $k\in\mathbb{N}$, i.e. $\{u_k\}$ is contained in $\mathfrak{B}(u_0,\bar{m})$. By Eq. (3.1) and Theorem 2.1, we have

$$\sum_{k=N}^{\infty}\|u_{k+1}-u_k\|\leq\sum_{k=N}^{\infty}(m_{k+1}-m_k)=\bar{m}-m_N<+\infty,$$

for any $k\in\mathbb{N}$. Hence $\{u_k\}$ is a Cauchy sequence in $\mathfrak{B}(u_0,\bar{m})$ and so it converges to some $\mathfrak{a}^*\in\bar{\mathfrak{B}}(u_0,\bar{m})$. Also, this implies that $\|\mathfrak{a}^*-u_k\|\leq\bar{m}-m_k$ for any $k\in\mathbb{N}$. It remains to show that $\Lambda(\mathfrak{a}^*)=0$. From Lemma 3.1 it follows that $(\|\Lambda'(u_k)\|)$ is bounded. By Lemma 3.3, we have

$$\|\Lambda(u_k)\|\leq\|\Lambda'(u_k)\|\|\Lambda'(u_k)^{-1}\Lambda(u_k)\|\leq\|\Lambda'(u_k)\|(1-H_\mathfrak{f}(m_k))(1-H_\mathfrak{f}(m_k)/2)^{-1}(m_{k+1}-m_k).$$

By noting that $H_\mathfrak{f}(m_k)$ is bounded by Lemma 2.4 and $\{m_k\}$ is convergent and letting $k\to\infty$, we have $\lim_{k\to\infty}\Lambda(u_k)=0$. Since Λ is continuous in $\bar{\mathfrak{B}}(u_0,\bar{m})$, $\{u_k\}\subset\mathfrak{B}(u_0,\bar{m})$ and $\{u_k\}$ converges to $\mathfrak{a}^*$, also we have $\lim_{k\to\infty}\Lambda(u_k)=\Lambda(\mathfrak{a}^*)$. The proof is complete. □

Theorem 3.2 *Under the assumptions of Theorem 3.1, we have the following error bounds:*

(i) A priori estimate: for all $k \geq 0$, we have the following a priori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \leq (\bar{m} - m_{k+1})\Big(\frac{\|\mathfrak{a}^* - u_k\|}{\bar{m} - m_k}\Big)^3.$$

(ii) Thus the sequence generated by the method converges Q-cubic as follows

$$\|\mathfrak{a}^* - u_{k+1}\| \leq \Big[\frac{1}{2}\frac{\mathfrak{f}''(\bar{m})^2}{\mathfrak{f}'(\bar{m})^2} + \frac{1}{3}\frac{D^-\mathfrak{f}''(\bar{m})}{-\mathfrak{f}'(\bar{m})}\Big](\|\mathfrak{a}^* - u_k\|)^3.$$

(iii) A posteriori estimate: for all $k \geq 0$, we have the following a posteriori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \leq (\bar{m} - u_{k+1})\Big(\frac{\|u_{k+1} - u_k\|}{m_{k+1} - m_k}\Big)^3.$$

Proof: By using standard analytic techniques, one can have

$$\begin{aligned}\mathfrak{a}^* - u_{k+1} &= -\Gamma_\Lambda(u_k)\Lambda'(u_k)^{-1}\int_0^1(1-\sigma)[\Lambda''(u_k + \sigma(\mathfrak{a}^* - u_k)) - \Lambda''(u_k)](\mathfrak{a}^* - u_k)^2 d\sigma \\ &\quad + \frac{1}{2}\Lambda''(u_k)\Gamma_\Lambda(u_k)\Lambda'(u_k)^{-1}\Big[\Lambda'(u_k)^{-1}\int_0^1(1-\sigma)\Lambda''(u_k + \sigma(\mathfrak{a}^* - u_k)) \\ &\quad (\mathfrak{a}^* - u_k)^2 d\sigma\Big](\mathfrak{a}^* - u_k),\end{aligned}$$

where $\Gamma_\Lambda(u) = (I - H_\Lambda(u))^{-1}$. Using condition (1.3), we obtain

$$\begin{aligned}\int_0^1 \|\Lambda'(u_k)^{-1}[\Lambda''(u_k + \sigma(\mathfrak{a}^* - u_k)) - \Lambda''(u_k)]\|(1-\sigma)d\sigma &\leq \int_0^1 [\Lambda''(\sigma\|\mathfrak{a}^* - u_k\| + \|u_k - u_0\|) \\ &\quad -\Lambda''(\|u_k - u_0\|)](1-\sigma)d\sigma.\end{aligned}$$

Also, by Lemma 2.2, we have

$$\begin{aligned}\mathfrak{f}''(\sigma\|\mathfrak{a}^* - u_k\| + \|u_k - u_0\|) - \mathfrak{f}''(\|u_k - u_0\|) &\leq \mathfrak{f}''(\sigma\|\mathfrak{a}^* - u_k\| + m_k) - \mathfrak{f}''(m_k) \\ &\leq [\mathfrak{f}''(\sigma(\bar{m} - m_k) + m_k) - \mathfrak{f}''(m_k)]\frac{\|\mathfrak{a}^* - u_k\|}{\bar{m} - m_k}.\end{aligned}$$

Thus, Lemmas 3.1, 3.2 and the above inequality implies that

$$\begin{aligned}\|\mathfrak{a}^* - u_{k+1}\| &\leq -\frac{1}{(1 - H_\mathfrak{f}(m_k))\mathfrak{f}'(m_k)}\int_0^1[\mathfrak{f}''(\sigma(\bar{m} - m_k) + m_k) - \mathfrak{f}''(m_k)](1-\sigma)d\sigma\frac{(\|\mathfrak{a}^* - u_k\|)^3}{\bar{m} - m_k} \\ &\quad + \frac{1}{2}\frac{\mathfrak{f}''(m_k)}{(1 - H_\mathfrak{f}(m_k))\mathfrak{f}'(m_k)^2}\int_0^1 \mathfrak{f}''(\sigma(\bar{m} - m_k) + m_k)(1-\sigma)d\sigma(\|\mathfrak{a}^* - u_k\|)^3 \\ &= (\bar{m} - m_{k+1})\Big(\frac{\|\mathfrak{a}^* - u_k\|}{\bar{m} - m_k}\Big)^3.\end{aligned}$$

Therefore, Theorem 3.2(i) holds for all $k \in \mathbb{N}$. And it follows from Lemma 2.6 that Theorem 3.2(ii) is true. Now, by using Lemma 3.3 (v), for all $i \geq 0$ and $k \geq k_0 \geq 0$, we have

$$\|u_{k+i+1} - u_{k+i}\| \leq (m_{k+i+1} - m_{k+i})\Big(\frac{\|u_{k_0+1} - u_{k_0}\|}{m_{k_0+1} - m_{k_0}}\Big)^{3^{k-k_0+i}}.$$

So, by Theorem 3.2(i) one can have

$$\begin{aligned}\|\mathfrak{a}^* - u_k\| &\leq \sum_{i=0}^{\infty}(m_{k+i+1} - m_{k+i})\Big(\frac{\|u_{k_0+1} - u_{k_0}\|}{m_{k_0+1} - m_{k_0}}\Big)^{3^{k-k_0+i}}\\ &\leq \sum_{i=0}^{\infty}(m_{k+i+1} - m_{k+i})\Big(\frac{\|u_{k_0+1} - u_{k_0}\|}{m_{k_0+1} - m_{k_0}}\Big)^{3^{k-k_0}}\\ &\leq (\bar{m} - m_k)\Big(\frac{\|u_{k_0+1} - u_{k_0}\|}{m_{k_0+1} - m_{k_0}}\Big)^{3^{k-k_0}}.\end{aligned}$$

And hence by taking $k = k_0$, the proof is complete. □

Theorem 3.3 *Under the assumptions of Theorem 3.1, the limit $\mathfrak{a}^*$ of the sequence $\{u_k\}$ is the unique solution of Eq. (1.1) in $\mathfrak{B}(u_0, \mu)$, where μ is defined as $\mu := \sup\{m \in (\bar{m}, \varrho) : \mathfrak{f}(m) \leq 0\}$.*

Proof: Firstly we have to show that the solution $\mathfrak{a}^*$ of (1.1) is unique in $\bar{\mathfrak{B}}(u_0, \bar{m})$. Assume that there exists another solution $\mathfrak{a}^{**}$ in $\bar{\mathfrak{B}}(u_0, \bar{m})$. So, $\|\mathfrak{a}^{**} - u_0\| \leq \bar{m}$. Now, by using induction we prove that

$$\|\mathfrak{a}^{**} - u_k\| \leq \bar{m} - m_k, \qquad k = 0, 1, 2\ldots$$

For $k = 0$, it holds because $m_0 = 0$. Let us consider it holds for some $n \in \mathbb{N}$. By Theorem 3.2(i), we have

$$\|\mathfrak{a}^{**} - u_{k+1}\| \leq (\bar{m} - m_{k+1})\Big(\frac{\|\mathfrak{a}^{**} - u_k\|}{\bar{m} - m_k}\Big)^3.$$

Then by applying induction hypotheses to the above inequality, we see that it also holds for $k = n + 1$. Since $\{u_k\}$ converges to $\mathfrak{a}^*$ and $\{m_k\}$ converges to $\bar{m}$, we conclude that $\mathfrak{a}^{**} = \mathfrak{a}^*$. Therefore, $\mathfrak{a}^*$ is the unique root of Eq. (1.1) in $\bar{\mathfrak{B}}(u_0, \bar{m})$.

We prove by contradiction that Λ does not have zeros in $\mathfrak{B}(u_0, \mu)/\bar{\mathfrak{B}}(u_0, \bar{m})$. Assume that there exists $\mathfrak{a}^{**} \in \mathfrak{D} \subset \mathfrak{B}_1$ such that $\bar{m} < \|\mathfrak{a}^{**} - u_k\| < \mu$ and $\Lambda(\mathfrak{a}^{**}) = 0$. By Taylor series expansion

$$\begin{aligned}\Lambda(\mathfrak{a}^{**}) &= \Lambda(u_0) + \Lambda'(u_0)(\mathfrak{a}^{**} - u_0) + \frac{1}{2}\Lambda''(u_0)(\mathfrak{a}^{**} - u_0)^2\\ &\quad +(1-\sigma)\int_0^1[\Lambda''(u_0 + \sigma(\mathfrak{a}^{**} - u_0)) - \Lambda''(u_0)](\mathfrak{a}^{**} - u_0)^2 d\sigma \qquad (3.2)\end{aligned}$$

Also, by Eq. (1.3) we obtain

$$\begin{aligned}&\|(1-\sigma)\int_0^1 \Lambda'(u_0)^{-1}[\Lambda''(u_0 + \sigma(\mathfrak{a}^{**} - u_0)) - \Lambda''(u_0)](\mathfrak{a}^{**} - u_0)^2 d\sigma\|\\ \leq\ &\int_0^1 [\mathfrak{f}''(\sigma(\|\mathfrak{a}^{**} - u_0\|)) - \mathfrak{f}''(0)](\|\mathfrak{a}^{**} - u_0\|)^2(1-\sigma)d\sigma\\ =\ &\mathfrak{f}(\|\mathfrak{a}^{**} - u_0\|) - \mathfrak{f}(0) - \mathfrak{f}'(0)\|\mathfrak{a}^{**} - u_0\| - \frac{1}{2}\mathfrak{f}''(0)\|\mathfrak{a}^{**} - u_0\|^2 \qquad (3.3)\end{aligned}$$

and, by applying the initial conditions from Def. 2.1, we have

$$\|\Lambda'(u_0)^{-1}[\Lambda(u_0) + \Lambda'(u_0)(\mathfrak{a}^{**} - u_0) + \frac{1}{2}\Lambda''(u_0)(\mathfrak{a}^{**} - u_0)^2]\| \geq \|\mathfrak{a}^{**} - u_0\| - \mathfrak{f}(0) - \frac{1}{2}\mathfrak{f}''(0)\|\mathfrak{a}^{**} - u_0\|^2. \quad (3.4)$$

Since $\Lambda(\mathfrak{a}^{**}) = 0$ and $\mathfrak{f}'(0) = -1$, combining (3.3) and (3.4), we get from (3.2) that

$$\mathfrak{f}(\|\mathfrak{a}^{**} - u_0\|) - \mathfrak{f}(0) + \|\mathfrak{a}^{**} - u_0\| - \frac{1}{2}\mathfrak{f}''(0)\|\mathfrak{a}^{**} - u_0\|^2 \geq \|\mathfrak{a}^{**} - u_0\| - \mathfrak{f}(0) - \frac{1}{2}\mathfrak{f}''(0)\|\mathfrak{a}^{**} - u_0\|^2$$

which is equivalent to $\mathfrak{f}(\|\mathfrak{a}^{**} - u_0\|) \geq 0$. Since $\mathfrak{f}$ is strictly convex by Lemma 2.3, hence $\mathfrak{f}$ is strictly positive in the interval $(\|\mathfrak{a}^{**} - u_0\|, \varrho)$. So, we get $\mu \leq \|\mathfrak{a}^{**} - u_0\|$, which is a contradiction to the above assumptions. Therefore, Λ does not have zeros in $\mathfrak{B}(u_0, \mu)/\bar{\mathfrak{B}}(u_0, \bar{m})$ and $\mathfrak{a}^*$ is the unique root of Eq. (1.1) in $\mathfrak{B}(u_0, \mu)$. This completes the proof. □

4. Particular Cases

This section consists of two particular cases of the semilocal convergence results obtained in Section 3, namely, convergence results under affine covariant Lipschitz condition and the γ-Condition.

4.1. Semilocal Convergence analysis under Lipschitz condition

The affine covariant Lipschitz condition studied in [10] is given as follows:

$$\|\Lambda'(u_0)^{-1}[\Lambda''(v)-\Lambda''(u)]\| \le \lambda_1\|v-u\|, \quad \forall\, u,v\in\mathfrak{D}. \tag{4.1}$$

The majorizing function for Super-Halley method in [27] is

$$\mathfrak{p}(m) = \frac{b_1}{6}m^3 + \frac{b_2}{2}m^2 - m + b. \tag{4.2}$$

If we consider the cubic polynomial (4.2) as the majorizing function $\mathfrak{f}$ in (1.3), then the majorant condition (1.3) reduced to the Lipschitz condition (4.1) and the assumptions $(A1)$ and $(A2)$ are satisfied for Λ. If the convergence criterion

$$b \le \frac{2(b_2 + 2(b_2^2+2b_1)^{1/2})}{3(b_2+(b_2^2+2b_1)^{1/2})^2} \tag{4.3}$$

holds then the assumption $(A3)$ is also satisfied for Λ. Therefore, the Theorems 3.1, 3.2 and 3.3 can be combined and given as follows:

Theorem 4.1 *Suppose $\Lambda : \mathfrak{D} \subseteq \mathfrak{B}_1 \to \mathfrak{B}_2$ be a twice continuously differentiable nonlinear operator, and $\mathfrak{D}$ is open and convex. Consider that for a given initial guess $u_0 \in \mathfrak{D}$, $\Lambda'(u_0)$ is nonsingular that is $\Lambda'(u_0)^{-1}$ exists and satisfies the affine covariant Lipschitz condition (4.1) and $\|\Lambda'(u_0)^{-1}\Lambda(u_0)\| \le b$, $\|\Lambda'(u_0)^{-1}\Lambda''(u_0)\| \le b_2$. If (4.3) holds, then the sequence $\{u_k\}$ generated by the method (1.2) for solving the Eq. (1.1) with a starting point u_0 is well defined, is contained in $\mathfrak{B}(u_0,\bar{m})$ and converges to a solution $\mathfrak{a}^* \in \bar{\mathfrak{B}}(u_0,\bar{m})$, where $\bar{m}$ is the smallest positive zero of $\mathfrak{p}$ (defined by (4.2)) in $[0,\eta]$ where $\eta = (-b_2+(b_2^2+2b_1)^{1/2})/b_1$ is the positive root of Λ'. The limit $\mathfrak{a}^*$ of the sequence $\{u_k\}$ is the unique zero in $\mathfrak{B}(u_0,\bar{m}^*)$, where $\bar{m}^*$ is the root of $\mathfrak{p}$ in the interval $(\eta,+\infty)$. Moreover, the following error bound holds for all $k \ge 0$:*

(i) A priori estimate: we have the following a priori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \le (\bar{m}-m_{k+1})\Big(\frac{\|\mathfrak{a}^*-u_k\|}{\bar{m}-m_k}\Big)^3,$$

(ii) The sequence generated by the method converges Q-cubic as follows:

$$\|\mathfrak{a}^* - u_{k+1}\| \le \Big[\frac{3(b_1+b_2\bar{m})^2 + 2b_2(1-b_1\bar{m}-b_2\bar{m}^2/2)}{6(1-b_1\bar{m}-b_2\bar{m}^2/2)^2}\Big](\|\mathfrak{a}^*-u_k\|)^3.$$

(iii) A posteriori estimate: for all $k \ge 0$, we have the following a posteriori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \le (\bar{m}-m_{k+1})\Big(\frac{\|u_{k+1}-u_k\|}{m_{k+1}-m_k}\Big)^3.$$

4.2. Semilocal convergence analysis under the γ-Condition

This subsection contains the semilocal convergence results of the method under the γ-Condition. To study the Smale's point estimate theory, Wang and Han [22] introduced the notion of the γ-condition for operators in Banach spaces.

The γ-condition is studied by Smale in [21] for Newton's method under the assumptions that that f is analytic and satisfies

$$\|\Lambda'(u_0)^{-1}\Lambda^{(n)}(u_0)\| \le n!\gamma^{n-1}, \qquad n > 2$$

where u_0 is a given point in $\mathfrak{D}$ and γ is defined by

$$\gamma := \sup_{k>1} \| \frac{\Lambda'(u_0)^{-1}\Lambda^{(n)}(u_0)}{k!} \|^{\frac{1}{k-1}}.$$

Wang and Han in [23] improved Smale's result by introducing a majorizing function

$$\Lambda(m) = b - m + \frac{\gamma m^2}{1-\gamma m}, \qquad m \in \Big[0, \frac{1}{\gamma}\Big).$$

If we consider this function $\Lambda(m)$ as the majorizing function $\mathfrak{f}$, then the majorant condition (1.3) reduced to the following condition:

$$\|\Lambda'(u_0)^{-1}[\Lambda''(v) - \Lambda''(u)]\| \leq \frac{2\gamma}{(1-\gamma\|v-u\| - \gamma\|u-u_0\|)^3} - \frac{2\gamma}{(1-\gamma\|u-u_0\|)^3}, \quad \gamma > 0 \tag{4.4}$$

where $\|v-u\| + \|u-u_0\| < \dfrac{1}{\gamma}$, and the assumptions $(A1)$ and $(A2)$ are satisfied for Λ. Also, if $\mathfrak{a} := b\gamma < 3 - 2\sqrt{2}$, then assumption $(A3)$ is satisfied for Λ. Therefore, the combined form of theorems 3.1, 3.2, 3.3 is given as follows.

Theorem 4.2 *Suppose $\Lambda : \mathfrak{D} \subseteq \mathfrak{B}_1 \to \mathfrak{B}_2$ be a twice continuously differentiable nonlinear operator, and $\mathfrak{D}$ is open and convex. Assume that there is an initial guess $u_0 \in \mathfrak{D}$, such that $\Lambda'(u_0)^{-1}$ exists and satisfies the conditions $\|\Lambda'(u_0)^{-1}\Lambda(u_0)\| \leq b$, $\|\Lambda'(u_0)^{-1}\Lambda''(u_0)\| \leq 2\gamma$ and (4.4). If $\mathfrak{a} := b\gamma < 3 - 2\sqrt{2}$, then the sequence $\{u_k\}$ generated by the the method (1.2) for solving the equation (1.1) with a starting point u_0 is well defined, is contained in $\mathfrak{B}(u_0, \bar{m})$ and converges to a solution $\mathfrak{a}^* \in \bar{\mathfrak{B}}(u_0, \bar{m})$. The limit $\mathfrak{a}^*$ of the sequence $\{u_k\}$ is the unique zero in $\mathfrak{B}(u_0, \bar{m}^*)$, where $\bar{m}$ and $\bar{m}^*$ are given as*

$$\bar{m} = \frac{\mathfrak{a}+1-\sqrt{(\mathfrak{a}+1)^2 - 8\mathfrak{a}}}{4\gamma} \quad \text{and} \quad \bar{m}^* = \frac{\mathfrak{a}+1+\sqrt{(\mathfrak{a}+1)^2 - 8\mathfrak{a}}}{4\gamma}$$

respectively. Moreover, the following error bound holds for all $k \geq 0$:

(i) A priori estimate: we have the following a priori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \leq (\bar{m} - u_{k+1})\Big(\frac{\|\mathfrak{a}^* - u_k\|}{\bar{m} - m_k}\Big)^3.$$

(ii) The sequence $\{u_k\}$ converges Q-cubic as follows

$$\|\mathfrak{a}^* - u_{k+1}\| \leq \frac{2\gamma^2}{[2(1-\gamma\bar{m})^2 - 1]^2}(\|\mathfrak{a}^* - u_k\|)^3.$$

(iii) A posteriori estimate: we have the following a posteriori estimate

$$\|\mathfrak{a}^* - u_{k+1}\| \leq (\bar{m} - m_{k+1})\Big(\frac{\|u_{k+1} - u_k\|}{m_{k+1} - m_k}\Big)^3.$$

5. Observations and Numerical Examples

The unified form of iterative methods with third order of convergence is given as follows (for more details see [11,12])

$$\left.\begin{array}{l} u_{n+1} = u_n - \Omega(H_\Lambda(u_n))\Lambda'(u_n)^{-1}\Lambda(u_n) \\ \Omega(H_\Lambda(u_n)) = I + \frac{1}{2}H_\Lambda(u_n) + \sum_{k\geq 2} S_k H_\Lambda(u_n)^k \\ H_\Lambda(u_n) = \Lambda'(u_n)^{-1}\Lambda''(u_n)\Lambda'(u_n)\bar{\Lambda}(u_n) \end{array}\right\} \tag{5.1}$$

where $\{S_k\}$ is a positive real sequence with $\sum_{k\geq 2} S_k q^k < +\infty$, $q \in [-\frac{1}{2}, \frac{1}{2}]$ and $S_0 = 1$, $S_1 = \frac{1}{2}$. Thus, if $H_\Lambda(u_n)$ exists and $\|H_\Lambda(u_n)\| \leq \frac{1}{2}$, then Eq.(5.1) is well defined. In particular, when $S_k = \frac{1}{2}$ for any $k \geq 0$, this equation reduces to the convex acceleration Newton's method (1.2).

In [12], Hernandez and Romero studied the semilocal convergence of (5.1) under the following condition:

$$\|\Lambda''(u) - \Lambda''(v)\| \leq |\mathfrak{p}''(u_1) - \mathfrak{p}''(v_1)|, u, v \in \mathfrak{D}, u_1, v_1 \in [a, s] \tag{5.2}$$

such that $\|u - v\| \leq |u_1 - v_1|$ and $\mathfrak{p}$ is a sufficiently differentiable nondecreasing real valued function on an interval $[a, b]$ such that $\mathfrak{p}(a) > 0 > \mathfrak{p}(b)$ and $\mathfrak{p}'''(m) \geq 0$ on $[a, s]$, and s is the unique simple solution of $\mathfrak{p}(m) = 0$ on $[a, b]$.

It should be noted that the condition (1.3) used in our convergence analysis is affine invariant but not condition (5.2) (see [2,3]) and the majorizing sequences used in our analysis are weaker than those in [12] as their convergence analysis involve existence of third order derivative of the majorant function where as our convergence analysis only involves second derivative of the majorant function. So in this case our convergence analysis for the mathod is better than the method used in [12].

In many cases, a posterior error estimates are better than apriori error estimates and so it is a good choice as efficient stopping criteria in numerical applications. Now, we present two numerical examples to illustrate the theoretical results.

Example 5.1 *Let us consider a nonlinear Hammerstein integral equation of the second kind:*

$$x(s) = g(s) + \zeta \int_0^1 G(s,t)x(t)^3 dt,$$

where g is a given continuous function satisfying $g(s) > 0$ for $s \in [0, 1]$, ζ is a real constant and the kernel function $G(s, t)$ is the Green's function on $[0, 1] \times [0, 1]$ defined by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

Let $\mathfrak{B}_1 = \mathfrak{B}_2 = C[0, 1]$ and $\mathfrak{D} = \{x \in C[0, 1] : x(s) \geq 0, s \in [0, 1]\}$. Then finding the solution of the above equation is equivalent to find a solution of $\Lambda(u) = 0$, where $\Lambda : \mathfrak{D} \to C[0, 1]$ is defined by

$$\Lambda(x)(s) = x(s) - g(s) - \zeta \int_0^1 G(s,t)x(t)^3 dt. \tag{5.3}$$

We consider the max-norm and we have

$$\Lambda'(x)y(s) = y(s) - 3\zeta \int_0^1 G(s,t)x(t)^2 y(t)dt, \qquad y \in \mathfrak{D}$$

and

$$\Lambda''(x)[yz](s) = -6\zeta \int_0^1 G(s,t)x(t)(yz)(t)dt, \qquad y, z \in \mathfrak{D}.$$

Now, let $M = \max_{s\in[0,1]} \int_0^1 |G(s,t)|dt$. Then $M = \frac{1}{8}$ and if we choose $u_0(t) = g(t) = 1$ then it follows that

$$\|\Lambda'(u_0)^{-1}\| \leq \frac{8}{8 - 3|\zeta|}, \quad \|\Lambda'(u_0)^{-1}\Lambda(u_0)\| \leq \frac{|\zeta|}{8 - 3|\zeta|}, \quad \|\Lambda'(u_0)^{-1}\Lambda''(u_0)\| \leq \frac{6|\zeta|}{8 - 3|\zeta|}.$$

Also, for any $u, v \in \mathfrak{D}$, we have

$$\|\Lambda'(u_0)^{-1}[\Lambda''(u) - \Lambda''(v)]\| \leq \frac{6|\zeta|}{8 - 3|\zeta|}\|u - v\|.$$

So, we obtain the values of b, b_2 and b_1 as follows

$$b \leq \frac{|\zeta|}{8 - 3|\zeta|}, \qquad b_2 \leq \frac{6|\zeta|}{8 - 3|\zeta|}, \qquad b_1 \leq \frac{6|\zeta|}{8 - 3|\zeta|}.$$

Table 1: Domains of existence and uniqueness of solution for Super-Halley's method

ζ	convergence ball in our work		convergence ball in work of [19]	
	existence	uniqueness	existence	uniqueness
0.25	$\mathfrak{B}(1, 0.03460809002)$	$\mathfrak{B}(1, 4.068140394)$	$\mathfrak{B}(1, 0.03460664877)$	$\mathfrak{B}(1, 9.632060017)$
0.5	$\bar{\mathfrak{B}}(1, 0.07837774562)$	$\mathfrak{B}(1, 2.350261741)$	$\bar{\mathfrak{B}}(1, 0.07833883907)$	$\mathfrak{B}(1, 2.219211821)$
0.75	$\bar{\mathfrak{B}}(1, 0.1382595728)$	$\mathfrak{B}(1, 1.544540158)$	$\bar{B}(1, 0.1375065477)$	$\mathfrak{B}(1, 0.265341129)$
1	$\bar{\mathfrak{B}}(1, 0.236068)$	$\mathfrak{B}(1, 1)$	$\bar{\mathfrak{B}}(1, 0.231578)$	$\mathfrak{B}(1, 1.4350877)$

Table 2: Computational Order of Convergence(COC)

Number of iterations	Error	COC
0	1.0×10^{-1}	.....
1	2.7×10^{-3}	2.85
2	5.3×10^{-8}	2.94
3	4.1×10^{-22}	2.98
4	1.9×10^{-64}	2.99

Therefore, the convergence criterion (4.3) *holds and the Theorem 4.1 is applicable to conclude that the sequence generated by the method* (1.2) *with initial point* u_0 *converges to a zero of* Λ *defined by Eq.* (5.3).

For the special cases of Eq. (5.3) *when* $g(s) = 1$ *and* $\lambda = 0.25, 0.5, 0.75, 1$ *the corresponding domain of existence and uniqueness of solution, together with those obtained in [19] are given in Table 1. One can easily conclude by seeing Table 1 that the existence and uniqueness regions of solutions are improved by our approach.*

Many authors have considered Computational Order of Convergence (COC) in their research work to test numerically the order of convergence of iterative methods. Grau-Sanchez et.al. have discussed some variant of COC in their paper [7].

We have constructed Table 2 for computing the order of convergence of our method. From Table 2 we can see that the order of convergence is three. Here we have taken 10^{-6} *as error tolerance and the initial approximation as* $u_0 = 1$.

Example 5.2 *Let* $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathbb{R}^2$, $\mathfrak{D} = \bar{\mathfrak{B}}(0, 1)$. *Here, we define the analytic function* $\Lambda : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ *on* $\mathfrak{D}$ *for* $u = (u_1, u_2)^T \in \mathfrak{D}$ *by*

$$\Lambda(u) = (w(u), z(u))^T = \Big(10e^{u_1} + 5u_1u_2 - 10,\ 5u_1^2 + \sin u_1 + 10u_2\Big)^T \tag{5.4}$$

endowed with max norm $\|.\| = \|.\|_\infty$. *The first and second Fréchet derivatives of* Λ *are:*

$$\Lambda'(u) = \begin{bmatrix} 10e^{u_1} + 5u_2 & 5u_1 \\ 10u_1 + \cos u_1 & 10 \end{bmatrix}$$

and

$$\Lambda''(u) = \left[\begin{array}{cc} 10e^{u_1} & 5 \\ 5 & 0 \\ \hline 10 - \sin u_1 & 05 \\ 0 & 0 \end{array}\right].$$

Now, we find the inverse of $\Lambda'(u)$ *at the initial point* $u_0 = (a_0, b_0)^T$

$$\Lambda'(u_0)^{-1} = \frac{1}{d}\begin{bmatrix} 10 & -5a_0 \\ -(10a_0 + \cos a_0) & 10e^{a_0} + 5b_0 \end{bmatrix}$$

where,

$$d = det(\Lambda'(u_0)) = 100e^{a_0} + 50b_0 - 5a_0(10a_0 + \cos a_0).$$

Thus we have

$$\Lambda'(u_0)^{-1}\Lambda(u_0) = \frac{1}{d}\big(10w(u_0) - 5a_0z(u_0), (10e^{a_0} + 5b_0)z(u_0) - (10a_0 + \cos a_0)w(u_0)\big)^T.$$

and

$$\Lambda'(u_0)^{-1}\Lambda''(u_0) = \frac{1}{d}\left[\begin{array}{cc} 100e^{a_0} - 5a_0(10 - \sin a_0) & 50 \\ 50 & 0 \\ \hline -10e^{a_0}(10a_0 + \cos a_0) + (10e^{a_0} + 5b_0)(10 - \sin a_0) & -5(10a_0 + \cos a_0) \\ -5(10a_0 + \cos a_0) & 0 \end{array}\right].$$

Note that

$$\|\Lambda'(u_0)^{-1}\Lambda(u_0)\| \leq \frac{1}{|d|}\max\big(|10w(u_0) - 5a_0z(u_0)|, |(10e^{a_0} + 5b_0)z(u_0) - (10a_0 + \cos a_0)w(u_0)|\big) \quad (5.5)$$

and

$$\begin{aligned}\|\Lambda'(u_0)^{-1}\Lambda''(u_0)\| \quad \leq \quad & \frac{1}{|d|}\max\big(|100e^{a_0} - 5a_0(10 - \sin a_0)| + |50| + |5(10a_0 + \cos a_0)| \\ & +|(-10)e^{a_0}(10a_0 + \cos a_0) + (10e^{a_0} + 5b_0)(10 - \sin a_0)|, \\ & |50| + |5(10a_0 + \cos a_0)|\big). \end{aligned} \quad (5.6)$$

Theorem 4.2 together with Eq. (5.5) *and* (5.6) *gives*

$$b = \frac{1}{|d|}\max\big(|10w(x_0) - 5a_0z(x_0)|, |(10e^{a_0} + 5b_0)z(x_0) - (10a_0 + \cos a_0)w(x_0)|\big) \quad (5.7)$$

and

$$\begin{aligned}\gamma \quad \leq \quad & \frac{1}{2|d|}\max\big(|100e^{a_0} - 5a_0(10 - \sin a_0)| + |50| + |5(10a_0 + \cos a_0)| \\ & +|(-10)e^{a_0}(10a_0 + \cos a_0) + (10e^{a_0} + 5b_0)(10 - \sin a_0)|, \\ & |50| + |5(10a_0 + \cos a_0)|\big). \end{aligned} \quad (5.8)$$

Thus we find the value of b, γ, $\bar{m}$ *and* $\bar{m}^*$ *using Eq.*(5.7), *Eq.*(5.8) *and Theorem 4.2 respectively for the initial values* $u_0 = (0.005, 0.005)^T, (0.025, 0.025)^T, (0.05, 0.05)^T, (0.075, 0.075)^T$, *that are included in Table 3, 4. Also one can note from Table 3 that the convergence criterion* $\mathfrak{a} = b\gamma < 3 - 2\sqrt{2}$ *holds for the corresponding initial values. Therefore the sequence* $\{u_k\}$ *generated by the convex acceleration Newton's method* (1.2) *with initial value* u_0, *converges to a zero of* Λ *defined by* (5.4). *We have concluded in Table 4, the domain of existence and uniqueness of solution for the initial values* $u_0 = (0.005, 0.005)^T, (0.025, 0.025)^T, (0.05, 0.05)^T, (0.075, 0.075)^T$.

Table 5 shows that our method converges rapidly to third order of convergence(error tolerance taken is 10^{-6}*). The initial approximation taken here is* $u_0 = (0.005, 0.005)^T$.

6. Conclusions

A semilocal convergence analysis of the convex acceleration of Newton method under the assumption of majorant conditions for solving nonlinear operator equations in Banach space has been studied in this article. Our convergence analysis approach allow us to obtain two important particular cases about the convergence results based on Kantorovich type and Smale type. Some observations has been given about a family of third order methods. The article has been concluded with two numerical examples and a comparison is done to show that our convergence analysis gives better existence and uniqueness ball. The stated convergence analysis can be extended to study semilocal and local convergence analysis of this iterative methods under Riemannian settings.

Table 3: Estimated value of b, γ, $\mathfrak{a}$, $\bar{m}$

u_0	b	γ	$\mathfrak{a} = b\gamma$	$\bar{m}$
$(0.005, 0.005)^T$	0.0004951601...	0.7706521002...	0.0003815962...	0.0004953493...
$(0.025, 0.025)^T$	0.0023823824...	0.753743575...	0.0017957054...	0.00238668362...
$(0.05, 0.05)^T$	0.0045453651...	0.7336059989...	0.0033345071...	0.00456067514...
$(0.075, 0.075)^T$	0.00065102895...	0..7144529459...	0.0046512955...	0.00654100065...

Table 4: Estimated value of $\bar{m}^*$ and domain of existence and uniqueness for solution of (5.4)

u_0	$\bar{m}^*$	Existence	Uniqueness
$(0.005, 0.005)^T$	0.6485534211..	$\bar{\mathfrak{B}}(u_0, 0.0004953493)$	$\mathfrak{B}(u_0, 0.6485534211)$
$(0.025, 0.025)^T$	0.66216007912...	$\bar{\mathfrak{B}}(u_0, 0.00238668362)$	$\mathfrak{B}(u_0, 0.66216007912)$
$(0.05, 0.05)^T$	0.67927677207...	$\bar{\mathfrak{B}}(u_0, 0.00456067514)$	$\mathfrak{B}(u_0, 0.67927677207)$
$(0.075, 0.075)^T$	0.69655029547...	$\bar{\mathfrak{B}}(u_0, 0.00654100065)$	$\mathfrak{B}(u_0, 0.69655029547)$

Table 5: Computational Order of Convergence(COC)

Number of iterations	Error	COC
0	5.0×10^{-3}	.....
1	8.5×10^{-4}	2.89
2	3.3×10^{-6}	2.95
3	1.9×10^{-17}	2.98
4	3.7×10^{-51}	2.99

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