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Sign-Changing Radial Solutions for a Semilinear Problem on Exterior Domains With Nonlinear Boundary Conditions

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ABSTRACT: In this paper we are interested to the existence and multiplicity of radial solutions of problem of elliptic equations $\Delta U(x) + \varphi(|x|)f(U) = 0$ with a nonlinear boundary conditions on exterior of the unite ball centered at the origin in $\mathbb{R}^N$ such that $u(x) \to 0$ as $|x| \to \infty$, with any given number of zeros where the nonlinearity $f(u)$ is odd, superlinear for u larger enough and $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞). The function $\varphi > 0$ is C^1 on $[R, \infty)$ where $0 < \varphi(|x|) \le c_0\, |x|^{-\alpha}$ with $\alpha > 2(N-1)$ and $N > 2$ for large $|x|$.

Key Words: Radial solution, elliptic equations, nonlinear mixed boundary conditions.

Contents

1. Introduction

This paper is concerned with the existence of radial solutions for nonlinear boundary-value problem

$$\Delta U(x) + \varphi(|x|)f(U) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial U}{\partial n} + U\sigma(U) = 0 \quad \text{in } \partial\Omega\,, \tag{1.2}$$

$$\text{and} \quad \lim_{|x|\to\infty} U(x) = 0. \tag{1.3}$$

Where $U : \mathbb{R} \to \mathbb{R}$ and Ω is the complement of the ball of the radius $R > 0$ centered at the origin with $|x|^2 = x_1^2 + \cdots + x_N^2$ is the standard norm of $\mathbb{R}^N$ and $\frac{\partial}{\partial n}$ is the outward normal derivate. And we assuming that $\sigma : [0, \infty) \to (0, \infty)$ is a positive and continuous function.
We furthermore impose that the following assumptions:
(H1) $f : \mathbb{R} \to \mathbb{R}$ is odd and locally Lipschitzian. Moreover, f has one positive zero β s.t

$$\begin{cases} f < 0 \quad \text{on}\,(0,\beta) \quad , \quad f > 0 \quad \text{on}\,(\beta,\infty)\,, \\ \qquad\qquad \text{and} \qquad \limsup_{s\to 0} \dfrac{f(s)}{s} < 0\,. \end{cases}$$

(H2)

$$f(x) = |x|^{q-1}x + g(x) \text{ and } \lim_{|x|\to\infty} \frac{|g(x)|}{|x|^q} = 0 \quad \text{where } q > 1 \ (f \text{ is superlinear at infinity})$$

(H3) The function $\varphi(r)$ is the C^1 on $[R, \infty)$ s.t

$$0 < \varphi(r) \le c_0\, r^{-\alpha} \quad \text{for any } r \ge R\,, \tag{1.4}$$

$$2(N-1) + \frac{r\,\varphi'}{\varphi} < 0 \quad \text{for any } r \ge R\,, \tag{1.5}$$

where $\alpha > 2(N-1)$, $N > 2$ and $c_0 > 0$.

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Remark 1.1.

(i) From **(H2)** *we see that f is superlinear at infinity, i.e* $\lim\limits_{|x|\to\infty} \frac{f(x)}{x} = \infty$.

(ii) By **(H1)-(H2)** *it follows that* $F(u) = \int_0^u f(t)dt$ *is even and has a unique positive zero* $\gamma > \beta$ *with* $F < 0$ *on* $(0, \gamma)$.

(iii) Denoting $F_0 = -F(\beta) > 0$ *it then follows that*

$$F(u) \geq -F_0 \quad \textit{for any } u \in \mathbb{R}\,. \tag{1.6}$$

It is well known that the existence of many solutions on this and similar topics has been studied by several papers. Some have used variational approach, degree theory, or sub/super solutions to prove the existence of a positive solution [4,5,12,14]. Others with more assumptions have been able to prove the existence of an infinite number of solutions [7,8,9,10,13]. A common approach in many of these papers has been the shooting method and the scaling argument.
In [11], the authors studied the problem (1.1)-(1.2) in the case that $0 < \alpha < 2(N-1)$ under the assumptions **(H1)-(H2)** and assuming that $r \to \varphi(r)$ is positive and the C^1, $\varphi(r) \sim r^{-\alpha}$ for larger r and $\lim\limits_{r\to\infty} \frac{r\,\varphi'}{\varphi} = -\alpha$ to prove that (1.1)-(1.2) has an infinitely number of solutions. In this paper, we treat the case that $\alpha > 2(N-1)$ and we have a much weaker hypothesis **(H3)**. Notice that a key difference between this case and the one case already treated in [11] that the "energy function" $\dfrac{U'^2}{2\,\varphi} + F(U)$ associate to radial solution U of (1.1)-(1.2) is strictly decreasing but in our case, it is strictly increasing. Our aim here is to prove the existence of an infinite number of solutions of (1.1)-(1.2) which is convenient to count the number of zeros using ordinary differential equation methods.

Theorem 1.1. *If* **(H1)–(H3)** *are satisfied then* (1.1)–(1.3) *has infinitely many radially symmetric solutions. In addition, for each integer n there exist a radially symmetric solutions of problem* (1.1)–(1.3) *which have exactly n zeros.*

2. Preliminaries

The existence of radially symmetric solution $U(x) = U(r)$ with $r = |x|$ of (1.1)-(1.2) is equivalent to the existence of a solution U of the nonlinear ordinary differential equation

$$U''(r) + \frac{N-1}{r}U'(r) + \varphi(r)\,f(U) = 0 \quad \text{if } r > R, \tag{2.1}$$

$$U'(R) = U(R)\,\sigma(U(R)) \quad \text{and} \lim_{r\to\infty} U(r) = 0. \tag{2.2}$$

Let p be positive reel parameter and denoting $U(r,p) = U_p(r)$ the solution to the initial value problem

$$U''(r) + \frac{N-1}{r}U'(r) + \varphi(r)\,f(U) = 0, \tag{2.3}$$

$$u(R) = p > 0 \quad \text{and} \quad u'(R) = p\,\sigma(p), \tag{2.4}$$

As this initial value problem is not singular so, the existence uniqueness and continuous dependence with respect to p of the solution of (2.3)-(2.4) on $[R, R+\epsilon]$ for some $\epsilon > 0$, it follows by the standard existence-uniqueness and dependence theorem for ordinary differential equations [6].

We now, for a solution U_p of (2.3)-(2.4) we define the energy function as follows

$$E_p(r) = \frac{U_p'^2}{2\,\varphi(r)} + F(U_p) \quad \text{for } r \geq R. \tag{2.5}$$

A simple calculation by using (2.3) yields

$$E_p'(r) = -\frac{U_p'^2}{2r\,\varphi(r)}\Big(2(N-1) + \frac{r\,\varphi'}{\varphi}\Big). \tag{2.6}$$

From (1.4)-(1.5) therefore $E_p' > 0$ which means that the energy is nondecreasing.

On other hand we employing the following transformation

$$t = r^{2-N} \quad \text{and } U_p(r) = V_p(t)\,. \tag{2.7}$$

It then follows that the initial value problem (2.3)-(2.4) is converted to

$$V_p''(t) + H(t)\,f(V_p) = 0 \quad \text{if } 0 < t < T, \tag{2.8}$$

$$V_p(T) = p > 0 \quad \text{and} \quad V_p'(T) = -b(p) < 0 \tag{2.9}$$

where $T = R^{2-N}$, $b(p) = \frac{p\,\sigma(p)R^{N-1}}{N-2} > 0$ and

$$H(t) = \big(\frac{1}{N-2}\big)^2\, t^{-\frac{2(N-1)}{N-2}}\, \varphi(t^{-\frac{1}{N-2}})\,. \tag{2.10}$$

Furthermore from (1.4) we get

$$0 < H(t) \le c_1\, t^{\nu} \quad \text{on } (0,T]\,, \tag{2.11}$$

where $\nu = \frac{2(N-1)-\alpha}{N-2}$ and $c_1 = \frac{c_0}{(N-2)^2} > 0$.
Notice that, since $\alpha > 2(N-1)$ then $\nu > 0$ which implies that $\lim_{t\to 0^+} H(t) = 0$ and it follows that H is continuous on $[0,T]$. In addition, from **(H3)** we have that H is C^1 on $(0,T]$) and also

$$H'(t) = -\frac{t^{-\frac{3N-4}{N-2}}\,\varphi(t^{-\frac{1}{N-2}})}{(N-2)^3}\Big[2(N-1) + t^{-\frac{1}{N-2}}\,\frac{\varphi'(t^{-\frac{1}{N-2}})}{\varphi(t^{-\frac{1}{N-2}})}\Big] > 0\,,$$

which means that H is strictly increasing.

A simple calculation by using (2.8) show that

$$\big(\frac{V_p'^2(t)}{2} + H(t)\,F(V_p)\big)' = H'(t)F(V_p). \tag{2.12}$$

From (2.5) and by integrating (2.12) from t to T gives

$$\frac{V_p'^2(t)}{2} + H(t)\,F(V_p) = \frac{b(p)^2}{2} + H(T)\,F(p) - \int_t^T H'(x)F(V_p)\,dx.$$

From (1.6), since H' and H are positives we assert that

$$\frac{V_p'^2(t)}{2} \le \frac{b(p)^2}{2} + H(T)\big(F_0 + F(p)\big)\,.$$

It then follows that

$$|V_p'(t)| \le c_{2,p}\,, \tag{2.13}$$

where $c_{2,p} = \sqrt{b(p)^2 + 2\,H(T)\big(F_0 + F(p)\big)} > 0$. Also we apply the mean value theorem with the initial conditions (2.9) we get

$$|V_p(t)| \le p + T\,c_{2,p} = c_{3,p}\,. \tag{2.14}$$

Thus V_p and V_p' are bounded on wherever they are defined. For $p > 0$ fixed it then follows that there is a unique solution V_p of (2.8)-(2.9) defined on all $[0,T]$. Which assert from the change variables (2.7) that there is a unique solution U_p of (2.3)-(2.4) defined on $[R,\infty)$.

Lemma 2.1. *Let V_p be a solution of (2.8)-(2.9). Then $V_p(t) > 0$ on $(0,T]$ if p is sufficiently small.*

Proof. As $V_p'(T) = -b(p) = -\frac{p\,\sigma(p)R^{N-1}}{N-2} < 0$ because $\sigma(p) > 0$ so either,

$$\begin{cases} \text{case } (A): & V_p'(t) < 0 \quad \text{on all} \quad t \in (0,T]\,, \\ \text{case } (B): & V_p \text{ has a local maximum at some } m_p \in (0,T). \end{cases}$$

For the case(A). Since V_p is nonincreasing we get $V_p(t) > V_p(T) = p$ on $(0,T]$ and so we are done in this case.
We then consider the case (B). So it follows from (2.8) that $V_p''(m_p) = -H(m_p)f(V_p(m_p)) \le 0$. As $H > 0$ therefore $f\big(V_p(m_p)\big) \ge 0$. Which implies from **(H1)** that $V_p(m_p) \ge \beta$.

Next, we will to show the next Claim:

Claim 1. *$0 < V_p < \beta$ on $(0,T]$ for p close to 0^+.*

If not, so we suppose that for any $p > 0$ sufficiently small there is $t_p \in (m_p, T)$ such that $V_p(t_p) = \beta$ and $V_p' < 0$ on (t_p, T).
Let us $t \in [t_p, T]$ and integrating (2.8) from t to T with the initial conditions (2.9) yields

$$V_p'(t) = b(p) + \int_t^T H(x) f(V_p)\, dx\,. \tag{2.15}$$

Integrating this over $[t,T]$ with the initial conditions (2.9) and using the fact that $b(p)$ is positive we see that

$$V_p(t) \le p - \int_t^T \Big(\int_s^T H(x) f(V_p)\, dx \Big) ds\,. \tag{2.16}$$

Notice that by condition (H1) we see that $x \to \frac{f(x)}{x}$ is bonded below by some $-c_4 < 0$ on $[0,\infty)$. And since $V_p > 0$ is nondecreasing on $[t_p, T]$ and from (2.11)-(2.16) it thus follows that

$$V_p(t) \le p + c_4 \int_t^T \widehat{H}(s)\, V_p(s)\, ds\,,$$

where $\widehat{H}(t) = \int_t^T H(x)\, dx$ is a continuous and positive function on $[0,T]$ because H is continuous on$[0,T]$. We can apply the Cornwall inequality [6] it follows that

$$V_p(t) \le p\, e^{c_4 \int_t^T \widehat{H}(x)\, dx}. \tag{2.17}$$

We observe that the function $t \to e^{c_4 \int_t^T \widehat{H}(x)\, dx} > 0$ is positive and bounded above by some $c_5 > 0$ on $[0,T]$. Thus taking $t = t_p$ in (2.17) and letting $p \to 0^+$ we get

$$0 < V_p(t_p) = \beta \le c_5\, p \to 0. \tag{2.18}$$

This is a contradiction and the claim1 is proven. Consequently, we have $V_p > 0$ on $(0,T]$ for p sufficiently small. Finally, the result is established for both cases. Which completes the proof of Lemma 2.1. □

Lemma 2.2. *Let V_p be a solution of (2.8)-(2.9). Then V_p has a local maximum m_p on $(0,T)$ if p is sufficiently large. In addition,*

$$V_p(m_p) \to \infty \quad \text{as } p \to \infty\,, \tag{2.19}$$

$$\text{and } m_p \to T \quad \text{as } p \to \infty\,. \tag{2.20}$$

Proof. From the above discussion at the beginning in the proof of lemma 2.1, we will to assert that the case (A) is not occurs, if $p > 0$ is large enough. To the contrary we suppose that $V_p' < 0$ on $(0,T]$ for any $p > 0$ large enough. Therefore we have that $V_p(t) \ge V_p(T) = p > 0$ on $(0,T]$ for any $p > 0$ sufficiently large. Consequently, $V_p(t) \to \infty$ as $p \to \infty$ for all $t \in (0,T]$. Thus if $p > 0$ is sufficiently large we get

$$V_p(t) > \beta \quad \text{for any } t \in (0,T]. \tag{2.21}$$

Let us fixed $t_0 \in (0,T)$ and $p>0$ we denote

$$\Omega_p = \inf_{t_0 \le t \le T} \left\{ H(t)\, \frac{f(V_p)}{V_p} \right\}.$$

By virtue of (2.21) and since $H' > 0$ and $V_p' < 0$ we deduce that

$$\Omega_p \ge H(t_0) \inf_{p \le x \le V_p(t_0)} \left\{ \frac{f(x)}{x} \right\} \quad \text{for } p \text{ sufficiently large.} \tag{2.22}$$

From (i) of Remark 1.1 (superlinearity of f) with $H>0$ and taking $p \to \infty$ in (2.22) consequently we have that

$$\Omega_p \to \infty \quad \text{as } p \to \infty\,. \tag{2.23}$$

It is well known the eigenvectors of the operator $-\frac{d^2}{dt^2}$ in (t_0,T) with Dirichlet boundary conditions can be chosen as $\psi_k(t) = \sqrt{\frac{2}{T-t_0}}\,\sin\left(\frac{k\pi(t-t_0)}{T-t_0}\right)$ of eigenvalues $\mu_k = \left(\frac{k\,\pi}{T-t_0}\right)^2$ where k is nonnegative integer. Also, $t_1 = t_0 + \frac{T-t_0}{2}$ is a zero of the second eigenfunction ψ_2 on (t_0,T). In addition, from (2.23) therefore for suitable large $p>0$ it follows that $\Omega_p > \mu_2$. This allows us to apply the Sturm comparison theorem [6] and consequently, V_p has at least one zero in (t_0,T) which contradicts to (2.21). Hence, V_p has a local maximum at some $m_p \in (0,T]$ for p sufficiently large.

It remains to be shown (2.20). By integrating (2.10) from m_p to $t<T$ gives

$$-V_p'(t) = \int_{m_p}^{t} H(x) f(V_p)\, dx\,. \tag{2.24}$$

By the condition **(H2)** we see that $f(x) \ge c_6\, x^q$ on $[0,\infty)$ for some positive constant $c_6 > 0$. This and from (2.24) and using the fact that $V_p > 0$ is nonincreasing on (m_p, t) implies that

$$c_6\, V_p^q(t) \int_{m_p}^{t} H(x)\, dx \le -V_p'(t). \tag{2.25}$$

Dividing both sides by $V_p^q(t)$ and integrating both sides of the resultant inequality over (m_p,T) we obtain

$$\frac{1}{(q-1)\, V_p^{q-1}(m_p)} + c_6 \int_{m_p}^{T}\int_{m_p}^{s} H(x)\, dx\, ds \le \frac{1}{(q-1)\, p^{q-1}}\,.$$

Since $q>1$, $V_p(m_p) > 0$ and $H>0$ together leads to

$$0 < \int_{m_p}^{T}\int_{m_p}^{s} H(x)\, dx\, ds \le \frac{1}{c_6\,(q-1)\, p^{q-1}}\,.$$

Finally, by making $p \to \infty$ of this so the limit is necessarily zero and consequently (2.20) is proven. Ends of the proof of Lemma 2.2. □

Lemma 2.3. *Let V_p be a solution of (2.8)-(2.9). Then V_p has an arbitrary large of number of zeros on $(0,T]$ if p is large enough.*

Proof. To prove this lemma, it is sufficient to show that U_p has an arbitrary large of number of zeros on $[R,\infty)$ if p is large enough. Using the results obtained in Lemma 2.2 and the change of variables (2.7) we can assert that U_p has a local maximum at $M_p \in (R,\infty)$ for p large enough and also,

$$M_p \to R \quad \text{as} \quad p \to \infty\,, \tag{2.26}$$

$$\text{and } U_p(M_p) \to \infty \quad \text{as} \quad p \to \infty\,. \tag{2.27}$$

Now, we set

$$\lambda_p^{\frac{2}{q-1}} = U_p(M_p) \quad \text{and } \omega_{\lambda_p}(r) = \lambda_p^{-\frac{2}{q-1}} U_p(M_p + \frac{r}{\lambda_p}) \quad r \geq 0\,.$$

From (2.3) an easy computation shows

$$\omega''_{\lambda_p}(r) + \frac{N-1}{\lambda_p M_p + r}\omega'_{\lambda_p}(r) + \lambda_p^{-\frac{2q}{q-1}} \varphi\big(M_p + \frac{r}{\lambda_p}\big)\, f(\lambda_p^{\frac{2}{q-1}}\omega_{\lambda_p}) = 0 \quad \text{if } r > 0, \tag{2.28}$$

$$\omega_{\lambda_p}(0) = 1 \quad \text{and } \omega'_{\lambda_p}(0) = 0\,. \tag{2.29}$$

It then follows that

$$\Big(\frac{\omega'^2_{\lambda_p}}{2} + \lambda_p^{-\frac{2(q+1)}{q-1}} \varphi\big(M_p + \frac{r}{\lambda_p}\big) F(\lambda_p^{\frac{2}{q-1}}\,\omega_{\lambda_p})\Big)' = -\frac{N-1}{\lambda_p M_p + r}\omega'^2_{\lambda_p} + \lambda_p^{-\frac{3q+1}{q-1}} \varphi'\big(M_p + \frac{r}{\lambda_p}\big)\, F(\lambda_p^{\frac{2}{q-1}}\omega_{\lambda_p})\,. \tag{2.30}$$

From (1.5) we observe that $\varphi' < 0$ and by using (1.6)-(2.30) we get

$$\Big(\frac{\omega'^2_{\lambda_p}}{2} + \lambda_p^{-\frac{2(q+1)}{q-1}} \varphi\big(M_p + \frac{r}{\lambda_p}\big) F(\lambda_p^{\frac{2}{q-1}}\,\omega_{\lambda_p})\Big)' \leq -F_0\, \lambda_p^{-\frac{3q+1}{q-1}} \varphi'\big(M_p + \frac{r}{\lambda_p}\big)\,.$$

Integrating both sides of this inequality over $(0, r)$ gives

$$\frac{\omega'^2_{\lambda_p}}{2} + \lambda_p^{-\frac{2(q+1)}{q-1}} \big(M_p + \frac{r}{\lambda_p}\big) F(\lambda_p^{\frac{2}{q-1}}\,\omega_{\lambda_p}) \leq \lambda_p^{-\frac{2(q+1)}{q-1}} F(\lambda_p^{\frac{2}{q-1}}) + F_0\, \lambda_p^{-\frac{2(q+1)}{q-1}} \Big(\varphi(M_p) - \varphi(M_p + \frac{r}{\lambda})\Big)\,.$$

This implies that

$$\frac{\omega'^2_{\lambda_p}}{2} \leq \lambda_p^{-\frac{2(q+1)}{q-1}} \Big(F(\lambda_p^{\frac{2}{q-1}}) + F_0\, \varphi(M_p)\Big) \quad (\text{since } \varphi > 0). \tag{2.31}$$

On other hand, from **(H2)** it follows that

$$F(s) = \frac{1}{q+1}|s|^{q+1} + G(s) \quad \text{and } \lim_{|s|\to\infty} \frac{G(s)}{s^{q+1}} = 0\,,$$

where $G(s) = \int_0^s g(x)\,\mathrm{d}x$. Which implies that

$$\lim_{|s|\to\infty} \frac{F(s)}{|s|^{q+1}} = \frac{1}{q+1}. \tag{2.32}$$

From the continuity of φ and (2.26) we deduce that $\varphi(M_p) \to \varphi(R)$ as $p \to \infty$. Also, by (2.27) and $q > 1$ we obtain $\lambda_p^{\frac{2(q+1)}{q-1}} \to \infty$ as $p \to \infty$. This implies from (2.32) that

$$\frac{F(\lambda_p^{\frac{2}{q-1}})}{\lambda_p^{\frac{2(q+1)}{q-1}}} \to \frac{1}{q+1} \quad \text{and } \frac{F_0\, \varphi(M_p)}{\lambda_p^{\frac{2(q+1)}{q-1}}} \to 0 \quad \text{as } p \to \infty.$$

Therefore from (2.31), if p is sufficiently large we have that

$$|\omega'_{\lambda_p}| \leq \frac{2}{\sqrt{q+1}} \quad \text{for any } r \geq 0.$$

Consequently, ω_{λ_p} and ω'_{λ_p} are uniformly bounded. By the application of Arzela-Ascoli theorem there is a subsequence (again label ω_{λ_p}) such that $\omega_{\lambda_p} \to \omega$ and $\omega'_{\lambda_p} \to \omega'$ as $p \to \infty$ on compact subset of $[0, \infty)$.

We know from (2.27) and since $q > 1$ that

$$\lambda_p \to \infty \quad \text{as } p \to \infty\,. \tag{2.33}$$

By using (2.26)-(2.33) and the continuity of φ therefore we have that

$$\frac{N-1}{\lambda_p M_p + r} \to 0 \quad \text{and } \varphi\big(M_p + \frac{r}{\lambda_p}\big) \to \varphi(R) \quad \text{as } p \to \infty \quad \text{for any } r \in [0, \infty).$$

Furthermore from **(H2)** and (2.33) we get

$$\lambda_p^{-\frac{2q}{q-1}}\, g\big(\lambda_p^{\frac{2}{q-1}}\, \omega_{\lambda_p}(r)\big) \to 0 \quad \text{as } p \to \infty\,,$$

which implies that

$$\lambda_p^{-\frac{2q}{q-1}}\, f\big(\lambda_p^{\frac{2}{q-1}}\, \omega_{\lambda_p}(r)\big) = |\omega_{\lambda_p}|^{q-1}\, \omega_{\lambda_p} + \lambda_p^{-\frac{2q}{q-1}}\, g\big(\lambda_p^{\frac{2}{q-1}}\, \omega_{\lambda_p}(r)\big) \to |\omega(r)|^{q-1}\omega(r) \quad \text{as } p \to \infty\,,$$

for any $r \in [0, \infty)$. Consequently from (2.28) and (2.29) ω satisfies

$$\omega''(r) + \varphi(R)\, |\omega(r)|^{q-1}\omega(r) = 0 \quad \text{if } r > 0,$$
$$\omega(0) = 1 \quad \text{and} \quad \omega'(0) = 0.$$

It is well known that ω has an infinite number of zeros on $[0, \infty)$ we see [1] (lemma 10, with $p = 2$). Since $\omega_{\lambda_p} \to \omega$ as $p \to \infty$ uniformly on compact subsets of $[0, \infty)$. Then it follows that ω_p has an arbitrary large number of zeros for p large enough. Finally, since $U_p(M_p + \dfrac{r}{\lambda_p}) = \lambda_p^{\frac{2}{q-1}}\, \omega_{\lambda_p}(r)$ therefore U_p has an arbitrary large number of zeros on $[R, \infty)$ for p large enough and which also allows to obtain the same conclusion for V_p on a interval $(0, T]$. Which completes the proof of Lemma 2.3. □

Remark 2.1. *V_p has only simple zeros on $(0, T]$ for any $p > 0$.*

Proof. If not, we suppose there is some point $t_0 \in (0, T]$ such that $V_p(t_0) = V_p'(t_0) = 0$. Then by applying the uniqueness of solutions of initial value problem (2.8)-(2.9) we assert that $V_p = 0$ which contradicts to initial conditions (2.9). Thus V_p has only simple zeros. □

3. Proof of the main result

To prove the main theorem we need to recall the technical lemma which has been proved in [13] (Lemma 4) and it is generalized in [7] (Lemma 2.7) on (R, ∞).

Technical lemma: If U_{p_k} is a solution of (2.3)-(2.4) with $k \in \mathbb{N}$ zeros on (R, ∞) and in addition $U_{p_k}(r) \to 0$ as $r \to \infty$ then U_{p_k} has at most $k+1$ zeros on (R, ∞), if p is sufficiently close to p_k.

In what follows, for any integer $k \geq 1$ we construct the following sets

$$S_k = \{p > 0 : \quad V_p \text{ has at least } k \text{ zero on } (0, T]\}\,.$$

By Lemmas 2.3 and 2.2 we see that $S_1 \neq \emptyset$ and is bounded from below by some positive constant. Thus we can let

$$p_0 = \inf S_1 > 0\,.$$

Now, we want to claim the following result first

Claim 2. $V_{p_0} > 0$ *on* $(0, T]$.

Proof. Otherwise, so we suppose that $V_{p_0}(z) = 0$ for some point $z \in (0, T]$. By continuous dependence of solutions on initial conditions it follows that $V_{p_0} \geq 0$ on $(0, T]$. It then follows that $V_{p_0}(z) = V_{p_0}'(z) = 0$. Which contradicts to Remark 2.1. Thus $V_{p_0} > 0$ on $(0, T]$. By the definition of p_0, if $p > p_0$ therefore V_p must have a zero z_p on $(0, T]$. Ends of the proof of Claim 2. □

Next, we aim to prove the second claim

Claim 3. $z_p \to 0 \quad as\ p \to p_0^+$.

Proof. To the contrary, so a subsequence of (z_p) would converge to a $z \in (0, T]$ (still denoted (z_p)). From (2.13)-(2.14) and as F and σ are continuous it then follows that V_p and V_p' are uniformly bounded on $[0, T]$ for p near to p_0. Moreover, from (2.8) V_p'' is also uniformly bounded on $[0, T]$ for p close to p_0. Thus by using the Arzela-Ascoli theorem a subsequence of V_p and V_p' converges uniformly on $[0, T]$ to V_{p_0} and V_{p_0}'. This implies that $V_{p_0}(z) = 0$ which contradicts to $V_{p_0} > 0$ on $(0, T]$. Which completes the proof of Claim 3. □

From Claim 3 and since $V_p(z_p) = 0$ it follows that $V_{p_0}(0) = 0$ and $V_{p_0} > 0$ on $(0, T]$. To refer of the change variables (2.7) therefore U_{p_0} is a positive solution of (2.3)-(2.4) and also $U_{p_0}(r) \to 0$ as $r \to \infty$.

Now, by Lemmas 2.3 and 2.2 the set S_2 is non empty and is bounded from below by some positive constant. And thus we let $p_1 = \inf S_2$.

On other hand, by the technical lemma, we see that V_p has at most one zero on $(0, T]$ if $p \to p_0$. By definition of p_0 if p is sufficiently close to p_0^+ it then follows that V_p has exactly one zero on $(0, T]$. Thus $p_1 > p_0$ and by the same argument as above, we also show that V_{p_1} has exactly one zero on $(0, T]$ and $V_{p_1}(0) = 0$. Consequently there is a solution of (2.3)-(2.4) which has exactly one zero on (R, ∞) and $U_{p_1}(r) \to 0$ as $r \to \infty$.

Proceeding inductively we can show that for every nonnegative integer n there is a solution of (2.1)-(2.2) which has exactly n zeros on (R, ∞). Finally, the proof of Theorem 1.1 is complete as well.

4. Conclusion

By this work, we managed to establish the existence of infinitely many sign-changing radial solution to superlinear problem (1.1)-(1.3) on exterior domain in $\mathbb{R}^N$, when f grows superlinearity at infinity, the proof presented here seems more natural and more easier.
We make the change of variables $U(r) = V(r^{2-N})$ and investigate the differential equation for V on $[0, R^{2-N}]$ this allows us to obtain some qualitative properties of zeros of solutions. Finally, by approximating solutions of (2.8)-(2.9) with an appropriate linear equation, we deduce that there are localized solutions with any prescribed number of zeros.

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