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New Separation Axiom in Multiset Topology

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ABSTRACT: In this article, we introduce some M-topological operators called multiset kernel and multiset shell operators. Thereafter, we define a new separation axiom termed as multiset T_D- spaces and investigate some of its basic properties. It is observed that this space precisely lies between multiset T_0 and multiset T_1-spaces. Also, we characterize multiset T_0, T_1 and T_D-spaces in the light of the mentioned operators.

Key Words: Multiset, Multiset topology, Multiset kernel operator, Multiset shell operator, Multiset T_D-spaces.

Contents

1. Introduction

Many generalizations have been explored following an appropriate application of classical set theory in order to overcome some of the natural difficulties encountered in modeling real-life problems. Zadeh [26] introduced fuzzy set theory, which was the primary amongst those to offer a general framework based on the membership value of the elements, followed by soft set reported in [17], rough set by [18], neutrosophic set by [24], fuzzy soft set by [16]. The notion of multiset (mset) is one of the generalization of the classical sets. In a multiset, any element may occur more than once and the number of times occurrence of an element is called the multiplicity of the element, which is a natural number. One can say that a Zermelo-Fraenkel set is a particular case of a multiset, if the multiplicity of each element is equal to 1. In real world this is very much essential, as there are identical things like in a statistical survey repeated data, in a water molecule repetitions of hydrogen atoms, repetitions strands of DNA and RNA, repetitions of roots in a polynomial and many more.

Over the years the application of multiset has been observed not only in philosophy, logic, linguistics, and physics, but also in mathematics and computer science and this leads to the development of a comprehensive theory of multisets. Blizard [1,2] provided the excellent overview of the theory of multisets. In 1986, Yager [25] initiated the algebraic properties of multisets. After that many researchers have put their efforts for studying it rigorously. The basic properties of the multisets can be found in [9,11,25]. In 2012, Girish and John [7] procured the notion of multiset topology. Since then many topological properties of these concepts have been studied by researchers, for instances [5,6,8,12,13,14,15,20,21,22]. Very recently El-Sharkasy and Badr [3], El-Sharkasy et al. [4] portrayed beautifully multiset topology in DNA and RNA mutations, and these articles demonstrate how multiset topology can be used to detect diseases and help biologists in disease treatment.

The points play a significant role in the study of topological space. After proposing the definition of points in multiset topological spaces, Shravan and Tripathy [23] have introduced the notion of quasi-coincidence

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which has a vital role in the neighborhood structure of multipoint. In a multiset topological space (M, τ) in general it is not true that $cl(A) = A \cup A'$ for all submultiset A of M, which was shown by Jakaria [10]. Nevertheless, after modifying the definition of accumulation point, Shravan and Tripathy [23] have proved that the above result is true for all submultiset A of M.
The separation axioms are just axioms in the sense that we could add these conditions as extra axioms to the definition of topological space to achieve a more restricted definition of what a topological space is. For developing topology, separation axioms play an important role. In multiset topology, separation axioms were first introduced by [20]. Later on, it was studied by Ray and Dey [19] by extending it in the mixed multiset topology. In a topological space, most of the separation axioms are described in the light of open sets, closed sets, or closure of a set. In this article we study separation axioms in a different way. First we introduce two multiset operators and then we study their basic properties. Thereafter, we give the notion of a new separation axiom called Multiset T_D-spaces. We show that this space precisely lies between multiset T_0 and multiset T_1-spaces.
This article is being written in a fine thread of logic and furnished a well ordered perspective to the development of the new concepts. Lucid examples are properly placed to make the treatise more illustrative and self-contained. To reinforce and solidify the understanding some of the well expected notions, a new approach has been incorporated and discussed in detail.

2. Preliminaries

In this section, we recall some necessary definitions and results which will be useful for this study.

Definition 2.1. *[7] An mset M drawn from the set X is represented by a function count M or $C_M : X \longrightarrow N$, where N represents the set of all non negative integers.*
The mset M drawn from the set $X = \{x_1, x_2, ..., x_k\}$ is denoted by $M = \{m_1/x_1, m_2/x_2, ..., m_k/x_k\}$, where M is an mset with x_1 appearing m_1 times, x_2 appearing m_2 times and so on.
Here, $C_M(x)$ is the number of occurrences of the element x in the mset M. However those elements which are not included in the mset M have zero count. It is clear that a classical set is a special case of an mset.

Definition 2.2. *[7] A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times.*
If $X = \{x_1, x_2, ..., x_k\}$ then
$[X]^m = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : \text{for } i = 1, 2, ..., k; m_i \in \{0, 1, 2, ..., m\}\}$.
Let $M, N \in [X]^m$. Then the following are defined:
(i) $M = N$ if $C_M(x) = C_N(x)$, for all $x \in X$.
(ii) $M \subseteq N$ if $C_M(x) \leq C_N(x)$, for all $x \in X$.
(iii) $P = M \cup N$ if $C_P(x) = max\{C_M(x), C_N(x)\}$, for all $x \in X$.
(iv) $P = M \cap N$ if $C_P(x) = min\{C_M(x), C_N(x)\}$, for all $x \in X$.
(v) $P = M \ominus N$ if $C_P(x) = max\{C_M(x) - C_N(x), 0\}$, for all $x \in X$.

Definition 2.3. *[11] Let $M \in [X]^m$. Then the complement M^c of M is defined by $C_{M^c}(x) = m - C_M(x)$, for all $x \in X$.*

On the basis of multiplicity of elements there can be defined three types of submset of M from $[X]^m$.

Definition 2.4. *[7] A submset N of M is a whole submset of M with each element in N having full multiplicity as in M, i.e., $C_M(x) = C_N(x)$, for all $x \in N$.*

Definition 2.5. *[7] A submset N of M is a partial whole submset of M is a partial whole submset of M with at least one element in N having full multiplicity as in M, i.e., $C_M(x) = C_N(x)$, for some $x \in N$.*

Definition 2.6. *[7] A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in N, i.e., $C_N(x) \leq C_M(x)$, for all $x \in N$.*

Definition 2.7. *[7] Let $M \in [X]^m$. The power set of M is the support set of the power mset $P(M)$ and is denoted by $P^*(M)$.*

Definition 2.8. *[7] Let $M \in [X]^m$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology on M if τ satisfies the following properties*
(i) $\emptyset$ and M are in τ;
(ii) The union of the elements of any subcollection of τ is in τ;
(iii) The intersection of any two elements of τ is in τ.
The ordered pair (M, τ) is called multiset topological space. By an M-topology we shall mean the multiset topology.

Definition 2.9. *[7] Let (M, τ) be an M-topological space. Let A be a submset of M. The closure of an mset A is defined as the intersection of all closed msets containing A and is denoted by $cl(A)$, i.e., $cl(A) = \cap\{K \subseteq M : K$ is a closed mset and $A \subseteq K\}$ and $C_{cl(A)}(x) = Min\{C_K(x) : A \subseteq K\}$.*

Definition 2.10. *[7] Let (M, τ) be an M-topological space. Let A be a submset of M. The union of all open msets contained in A is defined as the interior of an mset A, and is denoted by $int(A)$, i.e., $int(A) = \cup\{K \subseteq M : K$ is an open mset and $K \subseteq A\}$ and $C_{int(A)}(x) = Max\{C_K(x) : K \subseteq A\}$.*

Definition 2.11. *[23] Let $[X]^m$ be a space of multisets. A multipoint is a multiset M in X such that*

$$C_M(x) = \begin{cases} k, & for\ x \in X; \\ 0, & for\ y \neq x, y \in X. \end{cases}$$

Remark 2.12. *[23] A multipoint $\{k/x\}$ is a subset of a multiset M or $\{k/x\} \in M$ if $k \leq C_M(x)$.*

Definition 2.13. *[23] Let M be a multiset in the space $[X]^m$. Let $k/x \in M$, then k/x is said to be quasi-coincident with $j/y \in M$ if $k + j > m$.*

Definition 2.14. *[23] Let M be a multiset in the space $[X]^m$. Let $N \subseteq M$, then k/x is said to be quasi-coincident with N if $k > C_{N^c}(x)$.*

Definition 2.15. *[23] A multiset M is said to be quasi-coincident with N, i.e., MqN at x iff $C_M(x) > C_{N^c}(x)$. By $M \not{q} N$ we shall mean the msets M and N are not quasi-coincident.*

Remark 2.16. *[23] If M and N are quasi-coincident at x then both $C_M(x)$ and $C_N(x)$ are non-zero and so M and N intersect at x.*

Definition 2.17. *[23] A multiset N in an M-topological space (M, τ) is said to be Q-neighbourhood (Q-nbd)of k/x if and only if there exists an open mset P such that $k/xqP \subset N$.*

Proposition 2.18. *[23] Let $[X]^m$ be a space of multisets. Let $M, N \in [X]^m$. Then $M \subseteq N$ iff M and N^c are not quasi-coincident, i.e., $k/x \in M$ iff k/x is not quasi-coincident with M^c.*

3. M-topological operators

In this section, we define M-kernel and M-shell as two M-topological operators. Then the connections between these two operators and M-closure and M-derived set operators are obtained.

Definition 3.1. *Given a submset A of an M-topological space (M, τ), the M-kernel of A is defined by: $ker(A) = \cap\{K \subseteq M : K$ is an open mset and $A \subseteq K\}$ and $C_{ker(A)}(x) = Min\{C_K(x) : A \subseteq K\}$.*

Based on the above definition, we formulate the following result.

Proposition 3.2. *Let $A, B \in M$ where $M \in [X]^m$. Then we have the following properties.*
(i) $A \subseteq ker(A)$.
(ii) $A \subseteq B$ implies $ker(A) \subseteq ker(B)$.
(iii) $ker(A) \subseteq ker(ker(A))$.
(iv) $ker(A \cap B) \subseteq ker(A) \cap ker(B)$.
(v) $ker(A \cup B) = ker(A) \cup ker(B)$.

In an M-topological space (M, τ), it is found that for any submset A of M, $cl(A) = A \cup A'$ is not true in general, which was verified by Jakaria [10]. Shravan and Tripathy [23] have made a remarkable observation on here that whether it is possible to think the definition of accumulation point in another way, so that the above can be true. Nonetheless, they proved that by changing the definition of accumulation point the above mentioned result is always true. We now add those results.

Proposition 3.3. *[23] Let (M, τ) be an M-topological space. Then $k/x \in cl(A)$ iff each Q-nbhd of k/x is q-coincident with A.*

Definition 3.4. *[23] A multipoint k/x is said to be an accumulation point of an mset M iff (i) k/x is the closure point of M.*
(ii) Every Q-nbhd of k/x and M are q-coincident at some point different from $supp(k/x)$.

Theorem 3.5. *[23] Let (M, τ) be an M-topological space. Then $cl(A) = A \cup A'$ for all submset A of M.*

It can be obtained from the above Definitions 2.9 and 3.1 that:

Proposition 3.6. *$cl(\{k/x\}) = \cap\{K \subseteq M : K$ is closed mset and $k/x \in K\}$ and $C_{cl(\{k/x\})}(x) = Min\{C_K(x) : k/x \in K\}$.*

Proposition 3.7. *$ker(\{k/x\}) = \cap\{K \subseteq M : K$ is an open mset and $k/x \in K\}$ and $C_{ker(k/x)}(x) = Min\{C_K(x) : k/x \in K\}$.*

Theorem 3.8. *[7] Let $\mathfrak{c} : P^*(M) \longrightarrow P^*(M)$ be an operator satisfying the following conditions:*
(c1) $\mathfrak{c}(\phi) = \phi$;
(c2) $A \subseteq \mathfrak{c}(A)$;
(c3) $\mathfrak{c}(\mathfrak{c}(A)) = \mathfrak{c}(A)$;
(c4) $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$.
Then we can associate an M-topology in the following way:
$\tau = \{A^c \in P^*(M) : \mathfrak{c}(A) = A\}$.
Moreover, with this M-topology τ, $cl(A) = \mathfrak{c}(A)$ for every submset A of M.

Note 1. *The operator $\mathfrak{c}$ is called M-closure operator.*

Theorem 3.9. *Let $\mathfrak{d} : P^*(M) \longrightarrow P^*(M)$ be an operator satisfying the following conditions:*
(d1) $\mathfrak{d}(\emptyset) = \emptyset$;
(d2) $k/x \in \mathfrak{d}(A)$ iff $x \in \mathfrak{d}(A - \{k/x\})$;
(d3) $\mathfrak{d}(A \cup \mathfrak{d}(A)) \subseteq A \cup \mathfrak{d}(A)$;
(d4) $\mathfrak{d}(A \cup B) = \mathfrak{d}(A) \cup \mathfrak{d}(B)$.
Then we can associate an M-topology in the following way:
$\tau = \{A \in P^*(M) : \mathfrak{d}(A^c) \subseteq A^c\}$.
Moreover, with this M-topology τ, $der(A) = \mathfrak{d}(A)$, for every submset A of M.

Proof. (i) By (d1), $M \in \tau$ and by the definition of $\mathfrak{d}$, $\mathfrak{d}(M) \subseteq M$, thus $\emptyset \in \tau$.
(ii) Let $A, B \in \tau$, then $\mathfrak{d}(A^c) \subseteq A^c$ and $\mathfrak{d}(B^c) \subseteq B^c$. Now, $\mathfrak{d}(A^c \cup B^c) = \mathfrak{d}(A^c) \cup \mathfrak{d}(B^c)$ [by (d4)] and thus $\mathfrak{d}(A^c \cup B^c) \subseteq A^c \subseteq B^c$. Therefore, $\mathfrak{d}(A \cap B)^c \subseteq (A \cap B)^c$ and hence $A \cap B \in \tau$.
(iii) Let $\{A_i : i \in \Lambda\} \in \tau$. Then $\mathfrak{d}(A_i^c) \subseteq A_i^c$ for all $i \in \Lambda$. Since it is true that $\cap_{i\in\Lambda} A_i^c \subseteq A_i^c$ for all $i \in \Lambda$, $\mathfrak{d}(\cap_{i\in\Lambda} A_i^c) \subseteq \mathfrak{d}(A_i^c) \subseteq \mathfrak{d}(A_i^c) \cup A_i^c = A_i^c$ [by monotinicity of $\mathfrak{d}$, one can show it easily, since $\mathfrak{d}(A_i^c) \subseteq A_i^c$, $\forall i \in \Lambda$]. Thus $\mathfrak{d}(\cap_{i\in\Lambda} A_i^c) \subseteq \cap_{i\in\Lambda} A_i^c$ and so $\cap_{i\in\Lambda} A_i^c \in \tau$. Consequently $\cup_{i\in\Lambda} A_i^c \in \tau$.

As, $cl(A) = c(A)$, we have $A \cup der(A) = A \cup \mathfrak{d}(A)$. Therefore, $(A \ominus \{k/x\}) \cup der(A \ominus \{k/x\}) = (A \ominus \{k/x\}) \cup \mathfrak{d}(A \ominus \{k/x\})$ for every $k/x \in M$. This implies that $\{k/x\} \in der(A \ominus \{k/x\}) \Longleftrightarrow \{k/x\} \in \mathfrak{d}(A \ominus \{k/x\})$. By the given hypothesis (d2), $\{k/x\} \in \mathfrak{d}(A) \Longleftrightarrow \{k/x\} \in \mathfrak{d}(A \ominus \{k/x\})$, Therefore $\{k/x\} \in der(A) \Longleftrightarrow \{k/x\} \in \mathfrak{d}(A)$ and thus $der(A) = \mathfrak{d}(A)$. □

Note 2. *The operator $\mathfrak{d}$ is called M-derived set operator.*

Theorem 3.10. *If an operator $\mathfrak{c}^*$ is defined by $\mathfrak{c}^*(A) = A \cup \mathfrak{d}(A)$ for every submset A of M, then $\mathfrak{c}^* = \mathfrak{c}$.*

Proof. (c1) Since $\mathfrak{d}(\emptyset) = \emptyset$, by the definition of $\mathfrak{c}^*$, we have $\mathfrak{c}^*(\emptyset) = \emptyset$.
(c2) It is clear from the definition that $A \subseteq \mathfrak{c}^*(A)$.
(c3) $\mathfrak{c}^*(A \cup B) = [A \cup B] \cup \mathfrak{d}[A \cup B] = [A \cup B] \cup [\mathfrak{d}(A) \cup \mathfrak{d}(B)] = [A \cup \mathfrak{d}(A)] \cup [B \cup \mathfrak{d}(B)] = \mathfrak{c}^*(A) \cup \mathfrak{c}^*(B)$ [by (d4)].
(c4) $\mathfrak{c}^*(\mathfrak{c}^*(A)) = [A \cup \mathfrak{d}(A)] \cup \mathfrak{d}[A \cup \mathfrak{d}(A)] = A \cup \mathfrak{d}(A) = \mathfrak{c}^*(A)$ [by (d3)].
Therefore, we have $\mathfrak{c}^* = \mathfrak{c}$. □

Definition 3.11. *Let (M, τ) be an M-topological space. A multipoint k/x is said to be weakly separated from A if there exists an open mset $U \in \tau$ of k/x such that $U \not\!\!\between A$ and which is denoted by $k/x \vdash A$.*

Based on the above Propositions 3.6, 3.7 and Definition 3.11, we formulate the following:

Remark 3.12. *Let (M, τ) be an M-topological space and $k_1/x, k_2/y$ be two multipoints in M. Then*
(i) $cl(k_1/x) = \{k_2/y : k_2/y \not\vdash k_1/x\}$.
(ii) $ker(k_1/x) = \{k_2/y : k_1/x \not\vdash k_2/y\}$.

Definition 3.13. *In an M-topological space (M, τ) we define:*
(i) The M-derived set of k/x as $der(k/x) = cl(k/x) \ominus \{k/x\}$.
(ii) The M-shell of k/x as $shell(k/x) = ker(k/x) \ominus \{k/x\}$.

Based on the above results we have the following:

Remark 3.14. *Let (M, τ) be an M-topological space. Then for any two multipoints k_1/x and k_2/y,*
(i) $der(k_1/x) = \{k_2/y : k_2/y \neq k_1/x, k_2/y \not\vdash k_1/x\}$.
(ii) $shell(k_1/x) = \{k_2/y : k_2/y \neq k_1/x, k_1/x \not\vdash k_2/y\}$.

Proposition 3.15. *Let k_1/x and k_2/y be two multipoints in M. Then we have the following properties:*
(i) $k_2/y \in ker(k_1/x)$ if and only if $k_1/x \in cl(k_2/y)$.
(ii) $k_2/y \in shell(k_1/x)$ if and only if $k_1/x \in der(k_2/y)$.
(iii) $k_2/y \in cl(k_1/x)$ implies $cl(k_2/y) \subseteq cl(k_1/x)$.
(iv) $k_2/y \in ker(k_1/x)$ implies $ker(k_2/y) \subseteq ker(k_1/x)$.

Proof. The assertions (i) and (ii) are straightforward.
The statement (iii) follows from the definition of closure immediately.
(iv) Let $k_3/z \in ker(k_2/y)$. Then by (i), $k_2/y \in cl(k_3/z)$ and so by (iii), $cl(k_2/y) \subseteq cl(k_3/z)$. By the given condition, $k_2/y \in ker(k_1/x)$ and again by (i), $k_1/x \in cl(k_2/y)$ and hence by (iii), $cl(k_1/x) \in cl(k_2/y)$. Therefore, $cl(k_1/x) \subseteq cl(k_3/z)$ and since $k_1/x \in cl(k_1/x)$, we have $k_1/x \in cl(k_3/z)$. By (i), $k_3/z \in ker(k_1/x)$. Thus $ker(k_2/y) \subseteq ker(k_1/x)$. □

Proposition 3.16. *Let (M, τ) be an M-topological space and let k_1/x and k_2/y be any two multipoints in M, then $ker(k_1/x) \neq ker(k_2/y)$ if and only if $cl(k_1/x) \neq cl(k_2/y)$.*

Proof. Let $ker(k_1/x) \neq ker(k_2/y)$. Then there is a multipoint k_3/z such that $k_3/z \in ker(k_1/x)$ but $k_3/z \notin ker(k_2/y)$. Now, $k_3/z \in ker(k_1/x)$ implies $k_1/x \in cl(k_3/z)$ [by Proposition 3.15] and so $cl(k_1/x) \subseteq cl(k_3/z)$. Also, from $k_3/z \notin ker(k_2/y)$ we get $k_2/y \notin cl(k_3/z)$ [by Proposition 3.15] and this implies $cl(k_3/z) \cap k_2/y = \emptyset$. Since $cl(k_1/x) \subseteq cl(k_3/z)$, we have $cl(k_1/x) \cap k_2/y = \emptyset$ and so $k_2/y \notin cl(k_1/x)$. Therefore, $cl(k_1/x) \neq cl(k_2/y)$.

The converse part can be verified in a similar manner to the 1st part. □

4. Multiset T_D-spaces

In this section, we introduce a new separation axiom, which is called as 'Multiset T_D-spaces'. We show that this space lies precisely in between T_0 and T_1-spaces. Also, we study some of its basic properties.

Definition 4.1. *[20] Let (M,τ) be an M-topological space. Then M is said to be a multiset T_0-space iff for any pair of multipoints k_1/x, k_2/y in M such that $x \neq y$, there exists a τ-open mset G such that $k_1/x \in G$, $k_2/y \notin G$ or there exists a τ-open mset H such that $k_1/x \notin H$, $k_2/y \in H$.*

Definition 4.2. *[20] Let (M,τ) be an M-topological space. Then M is said to be a multiset T_1-space iff for any pair of multipoints k_1/x, k_2/y in M such that $x \neq y$, there exist τ-open msets G, H such that $(k_1/x \in G,\ k_2/y \notin G)$ and $(k_1/x \notin H,\ k_2/y \in H)$.*

Definition 4.3. *Let (M,τ) be an M-topological space. Then M is said to be a multiset T_D-space if $der(k/x)$ is a closed mset for every multipoint k/x in M.*

Theorem 4.4. *If an M-topological space (M,τ) is T_D-space then it is T_0-space.*

Proof. Let k_1/x and k_2/y be any two multipoints of M such that $x \neq y$. If $k_2/y \in der(k_1/x)$, then $[der(k_1/x)]^c$ is an open mset [since $der(k_1/x)$ is closed mset] such that $k_1/x \in [der(k_1/x)]^c$ and $k_2/y \notin [der(k_1/x)]^c$. If $k_2/y \notin der(k_1/x)$, then we have $k_2/y \in [cl(k_1/x)]^c$ in such a way that $k_1/x \notin [cl(k_1/x)]^c$. Thus there exist τ-open msets $[der(k_1/x)]^c$ and $[cl(k_1/x)]^c$ such that $(k_1/x \in [der(k_1/x)]^c, k_2/y \notin [der(k_1/x)]^c)$ or $(k_1/x \notin [ck(k_1/x)]^c, k_2/y \in [cl(k_1/x)]^c)$. Hence M is T_0-space. □

Remark 4.5. *The converse of the above theorem may not be true in general.*

Example 4.6. *Let $X = \{a,b,c\}$ and $M = \{2/a, 3/b, 1/c\}$. Consider $\tau = \{M, \emptyset, \{1/a, 1/b\}, \{1/a, 3/b\}, \{1/c\}, \{1/a, 1/b, 1/c\}, \{2/a, 1/b, 1/c\}, \{1/a, 3/b, 1/c\}\}$. Then τ is an M-topology on M. We show that M is a T_0-space but not a T_D-space.*
Here, we have $cl(2/a) = \{2/a, 3/b\}$, and so $der(2/a) = \{3/b\}$, which is not closed mset and eventually it is showing that M is not a T_D-space but one can easily verified that it is T_0-space.

Theorem 4.7. *If an M-topological space (M,τ) is T_1-space then it is T_D-space.*

Proof. Let k/x be any multipoint of M. Since M is a T_1-space, $cl(k/x) = \{k/x\}$ and therefore by the Definition 3.13, we have $der(k/x) = \emptyset$ and so $der(k/x)$ is closed mset in τ. Thus M is T_D-space. □

Remark 4.8. *The converse of the above theorem may not be true. The following example justifies the claim.*

Example 4.9. *Let $X = \{a,b,c\}$ and $M = \{1/a, 2/b, 1/c\}$. Let $\tau = \{M, \emptyset, \{1/a\}, \{1/b\}, \{2/b\}, \{1/a, 1/b\}, \{1/a, 2/b\}\}$. Then τ forms an M-topology on M. We show that (M,τ) is a T_D-space but not a T_1-space.*
Here, $cl(1/c) = \{1/c\}$, $cl(2/b) = \{2/b, 1/c\}$ and $cl(1/a) = \{1/a, 1/c\}$. Therefore, $der(1/a) = \{1/c\}$, $der(2/b) = \{1/c\}$ and $der(1/c) = \emptyset$.
Since $\emptyset$ and $\{1/c\}$ are closed msets in (M,τ), we have $der(k/x)$ is closed mset for every multipoint $k/x \in M$ and hence M is T_D-space. Also, one can easily verify that M is not a T_1-space.

Theorem 4.10. *Let (M,τ) be an M-topological space. Then M is a T_D-space if and only if for every $k/x \in M$ there exist an open mset G and a closed mset F such that $\{k/x\} = G \cap F$.*

Proof. Let the space (M,τ) be T_D-space. Then $der(k/x)$ is closed mset for every $k/x \in M$. By Definition 3.13, we have $der(k/x) = cl(k/x) \ominus \{k/x\}$. This implies that $\{k/x\} = cl(k/x) \ominus der(k/x)$. If we let $G = [der(k/x)]^c$ and $F = cl(k/x)$, then G and F are the required open and closed msets respectively such that $\{k/x\} = G \cap F$.

Let for every multipoint $k/x \in M$ there exist an open mset G and a closed mset F such that $\{k/x\} = G \cap F$. Then one can replace F by $cl(k/x)$. Hence, $der(k/x) = cl(k/x) \ominus \{k/x\} = cl(k/x) \ominus (G \cap cl(k/x)) = cl(k/x) \cap G^c$. Which is showing that $der(k/x)$ is closed mset, as intersection of two closed msets is closed. □

We now give some important characterizations of T_D-space.

Proposition 4.11. *Let (M, τ) be an M-topological space. Then the following statements are equivalent.*
(i) M is T_D-space.
(ii) $der(der(A)) \subseteq der(A)$ for every mset A.
(iii) $der(A)$ is closed mset for every mset A.

Proof. $(i) \Longrightarrow (ii)$: The assertion immediately follows from the definition of derived set.
$(ii) \Longrightarrow (iii)$: It follows from the fact $cl(der(A)) = der(der(A)) \cup der(A) \subseteq der(A)$.
$(iii) \Longrightarrow (i)$: If for every submset A of M, $der(A)$ is closed mset, then $der(k/x)$ is closed mset for all multipoint $k/x \in M$. □

5. Multiset T_i-spaces, for $i = 0, 1$

In this particular section, we establish several characterizations of multiset T_0 and T_1-spaces in terms of the multiset operators introduced in Section 3.

Proposition 5.1. *Let (M, τ) be an M-topological space. Then the following statements are equivalent.*
(i) M is T_0-space.
(ii) For any multipoints $k_1/x, k_2/y$ in M with $x \neq y$, either $k_1/x \vdash k_2/y$ or $k_2/y \vdash k_1/x$.
(iii) $k_2/y \in cl(k_1/x)$ implies $k_1/x \notin cl(k_2/y)$.
(iv) For any pair of multipoints $k_1/x, k_2/y$ with $x \neq y$, $cl(k_1/x) \neq cl(k_2/y)$.

Proof. $(i) \Longrightarrow (ii)$: It is straightforward.
$(ii) \Longrightarrow (iii)$: Let $k_2/y \in cl(k_1/x)$. Then each Q-nbd of k_2/y is quasi-coincident with k_1/x. Suppose G is a Q-nbd of k_2/y. Then k_1/xqG and this implies $k_2/y \nvdash k_1/x$ and therefore by the hypothesis (ii), $k_1/x \vdash k_2/y$. Hence, there exists a Q-nbd H of k_1/x such that $k_2/y \not q H$ and so $k_1/x \notin cl(k_2/y)$.
$(iii) \Longrightarrow (iv)$: Suppose that $cl(k_1/x) \neq cl(k_2/y)$ is not true. Then $cl(k_1/x) \subseteq cl(k_2/y)$ and $cl(k_2/y) \subseteq cl(k_1/x)$. Since $k_2/y \in cl(k_2/y)$, we have $k_2/y \in cl(k_1/x)$. By the given condition we get $k_1/x \notin cl(k_2/y)$ and this implies $k_1/x \notin cl(k_1/x)$, which cannot be possible.
$(iv) \Longrightarrow (i)$: Let for any pair of multipoints $k_1/x, k_2/y$ with $x \neq y$, $cl(k_1/x) \neq cl(k_2/y)$. This implies that either $k_1/x \notin cl(k_2/y)$ or $k_2/y \notin cl(k_1/x)$. Let us assume that $k_1/x \notin cl(k_2/y)$. Set $U = [cl(k_2/y)]^c$. Then, there exists an open mset U such that $k_1/x \in U$ and $k_2/y \notin U$ and so M is T_0. □

Proposition 5.2. *Let (M, τ) be an M-topological space. Then the following statements are equivalent.*
(i) M is T_0-space.
(ii) For any pair of multipoints $k_1/x, k_2/y$ in M, $k_2/y \in ker(k_1/x)$ implies $k_1/x \notin ker(k_2/y)$.
(iii) For any pair of multipoints $k_1/x, k_2/y$ with $x \neq y$, $ker(k_1/x) \neq ker(k_2/y)$.

Theorem 5.3. *An M-topological space (M, τ) is said to be T_0-space iff $k_2/y \in der(k_1/x)$ implies $cl(k_2/y) \subseteq der(k_1/x)$ for every multipoints $k_1/x, k_2/y$ in M.*

Proof. Let $k_1/x, k_2/y$ in M. If $k_2/y \in der(k_1/x)$, then we have $k_1/x \neq k_2/y$ and $k_1/x \notin cl(k_2/y)$ [since M is T_0], and hence $cl(k_2/y) \subseteq der(k_1/x)$.

Let $k_1/x, k_2/y$ in M with $k_1/x \neq k_2/y$. If $k_2/y \in der(k_1/x)$, then $cl(k_2/y) \subseteq der(k_1/x)$. This shows that $k_2/y \in cl(k_1/x)$ and $k_1/x \notin cl(k_2/y)$. Thus by Proposition 5.1, we have M is T_0-space. □

Theorem 5.4. *An M-topological space (M, τ) is said to be T_0-space iff $k_2/y \in shell(k_1/x)$ implies $ker(k_2/y) \subseteq shell(k_1/x)$ for every multipoints $k_1/x, k_2/y$ in M.*

Proof. This can be established using standard technique. □

Theorem 5.5. *An M-topological space (M, τ) is said to be T_0-space iff $der(k/x) \cap shell(k/x) = \emptyset$ for every multipoints k/x in M.*

Proof. If $der(k/x) \cap shell(k/x) \neq \emptyset$, then there exists k_1/y such that $k_1/y \in der(k/x)$ and $k_1/y \in shell(k/x)$. Thus $k/x \neq k_1/y$ and so $k_1/y \in cl(k/x)$ and $k_1/y \in ker(k/x)$. Therefore, by Remark 3.14, we have $k_1/y \nvdash k/x$ and $k/x \nvdash k_1/y$, This shows that M cannot be T_0, which is a contradiction and whence the result follows.

For the converse part, if $der(k/x) \cap shell(k/x) = \emptyset$, then for each $k_1/y \neq k/x$, we have either $k_1/y \in cl(k/x)$ or $k_1/y \in ker(k/x)$ and by the Proposition 5.1, M is T_0. □

Theorem 5.6. *An M-topological space (M, τ) is said to be T_0-space iff $der(k/x)$ is an mset union of closed msets for every multipoints k/x in M.*

Proof. Let (M, τ) be a T_0-space. Since, $der(k/x)$ is closed mset for every k/x in M, for all $k_1/y \in der(k/x)$ there must exists an open mset $G \in \tau$ such that $k/x \in G$ and $k_1/y \notin G$. Thus, for all $k_1/y \in der(k/x)$, we get $k_1/y \in G^c \cap cl(k/x) \subseteq der(k/x)$. Since $G^c \cap cl(k/x)$ is closed mset, so $der(k/x)$ is an mset union of closed msets.

For the converse part, let $der(k/x) = \cup_{i \in J} A_i$, where A_i is closed mset for all $i \in J$. If $k_1/y \in der(k/x)$, then $k_1/y \in A_i$ for some $i \in J$ and $k/x \notin A_i$. Thus, there exists open mset $A_i^c \in \tau$ such that $k/x \in A_i^c$ and $k_1/y \notin A_i^c$. If $k_1/y \in der(k/x)$ and $k_1/y \neq k/x$, then $k_1/y \in [cl(k/x)]^c$, which is an open mset not containing k/x. This shows that M is T_0-space. □

Proposition 5.7. *Let (M, τ) be an M-topological space. Then the following statements are equivalent.*
(i) M is T_1-space.
(ii) For any multipoints $k_1/x, k_2/y$ in M with $x \neq y$, $k_1/x \vdash k_2/y$.
(iii) $cl(k/x) = \{k/x\}$ for every multipoint $k/x \in M$.
(iv) $der(k/x) = \emptyset$ for every multipoint $k/x \in M$.
(v) $ker(k/x) = \{k/x\}$ for every multipoint $k/x \in M$.
(vi) $shell(k/x) = \emptyset$ for every multipoint $k/x \in M$.
(vii) For any pair of multipoints $k_1/x, k_2/y$ with $x \neq y$, $cl(k_1/x) \cap cl(k_2/y) = \emptyset$.
(viii) For any pair of multipoints $k_1/x, k_2/y$ with $x \neq y$, $ker(k_1/x) \cap ker(k_2/y) = \emptyset$.

Proof. $(i) \Longleftrightarrow (ii)$: Using the definition it can be easily established.
$(i) \Longleftrightarrow (iii)$: It is straightforward.
$(iii) \Longrightarrow (iv)$: It follows directly from the Definition 3.13.
$(iv) \Longrightarrow (v)$: If $ker(k/x) \neq \{k/x\}$, then there is a multipoint k_3/z such that $x \neq z$ and $k_3/z \in ker(k/x)$. Hence, $k/x \in cl(k_3/z)$ and so $k/x \in cl(k_3/z) \ominus \{k_3/z\} = der(k_3/z)$, a contradiction.
$(v) \Longrightarrow (vi)$: It follows immediately from the Definition 3.13.
$(vi) \Longleftrightarrow (vii)$: If $cl(k_1/x) \cap cl(k_2/y) \neq \emptyset$, then there is k_3/z in M such that $k_3/z \in cl(k_1/x)$ and $k_3/z \in cl(k_2/y)$. Then we have $k_1/x \in ker(k_3/z)$ and $k_2/y \in ker(k_3/z)$. Thus, $k_1/x, k_2/y \in ker(k_3/z) \ominus \{k_3/z\} = shell(k_3/z)$, which is a contradiction.
$(vii) \Longrightarrow (viii)$: The given hypothesis implies $k_1/x \notin cl(k_2/y)$ and $k_2/y \notin cl(k_1/x)$. Equivalently, $k_2/y \notin ker(k_1/x)$ and $k_1/x \notin ker(k_2/y)$. Consequently, it follows that $ker(k_1/x) \cap ker(k_2/y) = \emptyset$.
$(viii) \Longrightarrow (i)$: The given condition implies $k_1/x \notin ker(k_2/y)$ and $k_2/y \notin ker(k_1/x)$. Equivalently, $k_2/y \notin cl(k_1/x)$ and $k_1/x \notin cl(k_2/y)$. Then there exist two open msets G and H such that $k_1/x \in G$, $k_2/y \notin G$ and $k_2/y \in H$, $k_1/x \notin G$ and eventually, M is T_1.
□

6. Conclusion

In this article, we have introduced some M-topological operators called them multiset kernel and multiset shell operators. Then we have discussed the interrelations between these two mset operators with the M-derived set or M-closure operators. Further, we have defined a new separation axiom termed as multiset T_D-spaces and investigated some of its basic properties. It has observed that this space precisely lies between multiset T_0 and T_1-spaces. Lastly, we have given several characterizations of multiset T_0, multiset T_1 and multiset T_D-spaces in the light of the above mentioned operators.

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