Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.66924

Characterization of Optional Submartingales of a New Class (Σ^l)

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ABSTRACT: In this paper, we shall define a new class denote by (Σ^l) of optional submartingales of the form $X_t = M_t + A_t$, where $(M_t)_{t\geq 0}$ is a càdlàg (right continuous with left limits) local martingale, $(A_t)_{t\geq 0}$ is a làdlàg increasing process, which can be decomposed as $A_t = A_t^d + A_t^g$ such that A^d is a càdlàg increasing process, A^g is a càglàd increasing process, the measure (dA^d) is carried by the set $\{t : X_{t-} = 0\}$, and the measure (dA_+^g) is carried by the set $\{t : X_t = 0\}$. Our main purpose in this work is to study the positive and negative parts of these processes, and establish some martingale characterizations, then show the formula of relative martingales in terms of last passage times, finally calculate a predictable compensator by using balayage formula.

Key Words: Optional submartingales, relative martingales, last passage times, predictable compensator, balayage formula.

Contents

1. Introduction

The purpose of this paper is to develop a new framework for studying the properties of optional submartingale.

The paths of an optional semimartingale possess limits from the left and from the right, but may have double jumps, this leads to quite interesting phenomena in integration theory. Such processes have been studied extensively by Lenglart [1] and Gal'čuk ([2], [3], [4]). In [3] Gal'čuk has introduced a stochastic integral with respect to an optional martingale with a larger domain, but the integral of [3] is not the unique (continuous and linear) extension of the elementary integral. In [5] Kühn *et al* has introduced a mathematically tractable domain of integrands which is between the small set of predictable integrands and the large domain in [3], then they have characterized the integral as the unique continuous and linear extension of the elementary integral and show completeness of the space of integrals.

In the literature, the theory of semimartingales is a major part of the general theory of stochastic processes (see [6]). This theory has undergone massive growth during the last decade. Much of the impetus for the rapid advances in this branch of pure mathematics comes from efforts to solve applied problems. For example, the theory of stochastic integration relative to semimartingales is the right tool for the analysis of stochastic dynamical systems, so for a large class of studies carried in the fields of theoretical physics [7], theory of controlled systems [8], probability theory [9] and statistics [10].
Notice also that semimartingale processes become increasingly popular for modelling market fluctuations, both for risk management and option pricing purposes, where the (discounted) asset price process must be a semimartingale in order to preclude arbitrage opportunities (see [11, Theorems 1.4, 1.6]) for details, see also [12]). The question whether a given process is a semimartingale is also of importance in stochastic modeling, where long memory processes with possible jumps and high volatility are considered as driving processes for stochastic differential equations. Examples of such processes include various fractional, or more generally, Volterra processes driven by Lévy processes.

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2010 *Mathematics Subject Classification:* 60G07, 60G46, 60H46.
Submitted January 31, 2023. Published December 04, 2025

To develop this theory of semimartingale, Yor [13] introduced the class (Σ) of local submartingale, and further studied by [14], [15], [16], [17] and [18].
Nikeghbali [15] has considered the class (Σ) for local submartingales of the form $X_t = M_t + A_t$, where M_t is a càdlàg local martingale with $M_0 = 0$ and $(A_t)_{t\geq 0}$ is a continuous increasing process with $A_0 = 0$, and the measure (dA) is carried by the set $\{t : X_t = 0\}$, which is referred to as Skorokhod's reflection equation. It plays a capital role in martingale theory as the family of Azema–Yor martingales [19], the resolution of Skorokhod's embedding problem [20], [21], the study of Brownian local times and the study of zeros of continuous martingales (see [22], [23], [15], [24]). It also plays an important role in the study of some diffusion processes (see [22], [24], [25]) and in the study of zeros of continuous martingales [19]. In this direction Nikeghbali has provided in [15] a general framework and methods, based on martingale techniques, such as establishing some martingale characterizations for these processes and compute explicitly some distributions involving the pair (X_t, A_t), and associate with X a solution to the Skorokhod's stopping problem for probability measures on the positive half-line. In [14] Cheridito *et al.* developed a framework for studying various properties of continuous-time stochastic processes such as the behavior of last passage times, running maxima and drawdowns.
The objective of this paper is to extend some characterizations of the classes (Σ) and (Σ^r)(see [16]) for a new class of optional submartingales, we denote by (Σ^l) a class of optional submartingales. The difference between these classes is that in class (Σ), the process A is a continuous increasing process, and it is a càdlàg (right continuous with left limits) increasing process in class (Σ^r), but in this new class (Σ^l), we consider that A is a làdlàg (with right and left limits) increasing process, which can be represented in the form $A = A^d + A^g$ (see [26]), where A^d is a càdlàg increasing process, and A^g is a càglàd (left continuous with right limits) increasing process. The processes of the class (Σ^l) are the optional làdlàg submartingales, which is our motivation by bring attention to the theory of semimartingales in a general framework where the semimartingales are optional, because the development of stochastic calculus of optional processes marks the beginning of a new and more general form of stochastic analysis. Recently such processes have been studied in mathematical finance (see [27], [28], [26], [29], [30] and [31]).
The paper is organized as follows, in section 2, we shall study positive and negative parts of processes of class (Σ^l) and we shall establish some martingale characterizations, in section 3, we will show how we obtain the formula of relative martingales in terms of last passage times for these processes, and finaly in section 4, we will calculate a predictable compensator by using balayage formula.

2. A first characterization of the process of class (Σ^l)

Let consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where the family $(\mathcal{F}_t)_{t\geq 0}$ is not necessarily right-continuous. All stochastic processes will be indexed by $t \in \mathbb{R}^+$. Throughout the paper, the stochastic integral of the predictable process f with respect to the làdlàg stochastic process Y is defined as in [1] by

$$f \bullet Y := \int f dY_+ - f \Delta^+ Y$$

where $\Delta^+ Y = Y_+ - Y$, we denote by $\int f dY$ the value of the process $f \bullet Y$.
Our purpose is the study of these stochastic processes defined as follows :

Definition 2.1 *Let $(X_t)_{t\geq 0}$ be a local submartingale, which decomposes as*

$$X_t = M_t + A_t.$$

We say that $(X_t)_{t\geq 0}$ is of class (Σ^l) if:

(1) $(M_t)_{t\geq 0}$ is a càdlàg local martingale, with $M_0 = 0$.

(2) $(A_t)_{t\geq 0}$ is a làdlàg (with left and right limits) increasing process, with $A_0 = 0$, which can be represented in the form

$$A_t = A^d_t + A^g_t$$

where A^d is a càdlàg increasing process, and A^g is a càglàd increasing process.

(3) the measure (dA^d) is carried by the set $\{t : X_{t-} = 0\}$, and the measure (dA^g_+) is carried by the set $\{t : X_t = 0\}$.

Recall that a stochastic process X is said of class (D) if $\{X_T : T$ is a finite stopping time$\}$ is uniformly integrable. If additionally, (X_t) is of class (D), we shall say that (X_t) is of class $(\Sigma^l D)$.
In this work X is a làdlàg process, and therefore X_+ is a càdlàg process.

Let us recall the change of variables formula for optional semimartingales which are not necessarily càdlàg. The result can be seen as a generalization of the classical Itô formula and can be found in [3].

Lemma 2.1 (Theorem 8.2. [3]) *Let $n \in \mathbb{N}$. Let X be n-dimensional optional semimartingale, i.e, $X = (X^1, \dots, X^n)$ is an n-dimensional optional process with decomposition $X^k = X^k_0 + M^k + A^k + B^k$, for all $k \in \{1, \dots, n\}$, where M^k is a (càdlàg) local martingale, A^k is a right–continuous process of finite variation such that $A_0 = 0$, and B^k is a left–continuous process of finite variation which is purely discontinuous and such that $B_0 = 0$. Let $h(x) = h(x^1, \dots, x^n)$ is twice continuously differentiable function on $\mathbb{R}^n$. Then $h(X)$ is a semimartingale, and for all $t \in \mathbb{R}_+$,*

$$
\begin{aligned}
h(X_t) &= h(X_0) + \sum_{k=1}^{n} \int_{]0,t]} D^k h(X_{s-}) d(A^k + M^k)_s \\
&+ \frac{1}{2} \sum_{1 \le k,l \le n} \int_{]0,t]} D^k D^l h(X_{s-}) d\left\langle M^{kc}, M^{lc} \right\rangle_s \\
&+ \sum_{0<s\le t} \left\{ h(X_s) - h(X_{s-}) - \sum_{k=1}^{n} D^k h(X_{s-}) \Delta X^k_s \right\} \\
&+ \sum_{k=1}^{n} \int_{[0,t[} D^k h(X_s) d(B^k)_{s+} \\
&+ \sum_{0\le s<t} \left\{ h(X_{s+}) - h(X_s) - \sum_{k=1}^{n} D^k h(X_s) \Delta^+ X^k_s \right\}
\end{aligned}
$$

where D^k is the differentiation operator with respect to the k–th coordinate, the process M^{kc} denotes the continuous part of M^k, and $\Delta X_t = X_t - X_{t-}$, $\Delta^+ X_t = X_{t+} - X_t$.

In the following proposition, we will give some characteristics of the processes of class (Σ^l).

Proposition 2.1 *Assume X is of class (Σ^l). Then the following hold:*

(1). X^+ and X^- are local submartingales.

(2). If X has no negative jumps, then X^+ is again of class (Σ^l). If X has no positive jumps, then X^- is of class (Σ^l).

(3). If X is nonnegative and has no positive jumps, then it is a local submartingale with $A_t = \sup_{u \le t}(-M_u) \vee 0$.

(4). If X is of class $(\Sigma^l D)$, then M is a uniformly integrable martingale and A has integrable total variation, in particular, there exist integrable random variables X_∞, M_∞, A_∞ such that $X_t \to X_\infty$, $M_t \to M_\infty$, $A_t \to A_\infty$ almost surely and in L^1.

Proof:

(1). From Lenglart [1], we have the following formula for a làdlàg process

$$
\begin{aligned}
X^+_t &= \int_0^t \mathbf{1}_{\{X_{s-}>0\}} dX_{s+} + U_t \\
&= \int_0^t \mathbf{1}_{\{X_{s-}>0\}} dM_s + \int_0^t \mathbf{1}_{\{X_{s-}>0\}} dA^d_s + \int_0^t \mathbf{1}_{\{X_s>0\}} dA^g_{s+} + U_t,
\end{aligned}
$$

for the increasing finite variation process

$$U_t = \sum_{0\le s<t}\left\{\mathbf{1}_{\{X_{s-}\le 0\}}X^+_{s+}+\mathbf{1}_{\{X_{s-}>0\}}X^-_{s+}\right\}+ \\ +\mathbf{1}_{\{X_{t-}\le 0\}}X^+_t+\mathbf{1}_{\{X_{t-}>0\}}X^-_t+\frac{1}{2}L^0_t, \tag{2.1}$$

where the process L^0_t is a local time of X at the level 0 and time t. Since (dA^d_t) is carried by the set $\{t : X_{t-}=0\}$, and (dA^g_{t+}) is carried by the set $\{t : X_t = 0\}$, then

$$X^+_t = \int_0^t \mathbf{1}_{\{X_{s-}>0\}}dM_s + U_t, \tag{2.2}$$

since the pocess $\int_0^t \mathbf{1}_{\{X_{s-}>0\}}dM_s$ is a local martingale, hence this shows that X^+ is a local submartingale. For X^-, as above we have

$$\begin{aligned} X^-_t &= -\int_0^t \mathbf{1}_{\{X_{s-}<0\}}dX_{s+}+U_t \\ &= -\int_0^t \mathbf{1}_{\{X_{s-}<0\}}dM_s-\int_0^t \mathbf{1}_{\{X_{s-}<0\}}dA^d_s-\int_0^t \mathbf{1}_{\{X_s<0\}}dA^g_{s+}+U_t \\ &= -\int_0^t \mathbf{1}_{\{X_{s-}<0\}}dM_s+U_t, \end{aligned}$$

where U is the increasing finite variation process given in (2.1). Therefore X^- is also a local submartingale.

(2). If X has no negative jumps, (2.2) become

$$X^+_t = \int_0^t \mathbf{1}_{\{X_{s-}>0\}}dM_s+\sum_{0\le s<t}\left\{\mathbf{1}_{\{X_{s-}\le 0\}}X^+_{s+}\right\} \\ +\mathbf{1}_{\{X_{t-}\le 0\}}X^+_t+\frac{1}{2}L^0_t(X). \tag{2.3}$$

$\int_0^t \mathbf{1}_{\{X_{s-}>0\}}dM_s$ is a local martingale, and the local time L is continuous and has the property $\int_0^t \mathbf{1}_{\{X_{s-}\neq 0\}}dL^0_s=0$, $t\in\mathbb{R}_+$. So we should show that the process $Z_t=\sum_{0\le s<t}\left\{\mathbf{1}_{\{X_{s-}\le 0\}}X^+_{s+}\right\}+\mathbf{1}_{\{X_{t-}\le 0\}}X^+_t$ can be decomposed into the sum of a local martingale and an adapted làdlàg increasing process D satisfying $D_0=0$ and $D_t=D^d_t+D^g_t$ where D^d is a càdlàg process, D^g is a càglàd process and $\int_0^t \mathbf{1}_{\{X^+_{s-}\neq 0\}}dD^d_s=0$, $\int_0^t \mathbf{1}_{\{X^+_s\neq 0\}}dD^g_{s+}=0$ for all $t\ge 0$.
One has M and $\int_0^t \mathbf{1}_{\{X_{s-}>0\}}dM_s$ are local martingales, there exists a sequence of stopping times T_n, $n\in\mathbb{N}$, increasing to ∞ such that

$$\mathbb{E}\left[(X_{T_n})^+\right]=\mathbb{E}[(M_{T_n}+A_{T_n})^+]<\infty$$

and

$$\mathbb{E}\left[\int_0^{T_n}\mathbf{1}_{\{X_{s-}>0\}}dM_s\right]=0,\quad n\in\mathbb{N}.$$

Then $\mathbb{E}\left[Z_{T_n}\right]\le\mathbb{E}\left[(X_{T_n})^+\right]<\infty$ for all $n\in\mathbb{N}$. Hence, from Remark p.p. 528 of [1], there exists a làdlàg increasing predictable process D starting at 0 such that $Z-D$ is a local martingale.

Moreover, there exists a sequence of stopping times R_n, $n \in \mathbb{N}$, increasing to ∞ such that

$$\begin{aligned}\mathbb{E}\left[\int_0^{t\wedge R_n} \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dD_{s+}\right] &= \mathbb{E}\left[\int_0^{t\wedge R_n} \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dZ_{s+}\right] \\ = \mathbb{E}\Bigg[\sum_{0\leq s<t\wedge R_n} &\left\{\mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}}\mathbf{1}_{\{X_{s-}\leq 0\}}X^+_{s+}\right\} \\ &+ \mathbf{1}_{\left\{X^+_{(t\wedge R_n)-}\neq 0\right\}}\mathbf{1}_{\left\{X_{(t\wedge R_n)-}\leq 0\right\}}X^+_{t\wedge R_n}\Bigg] \\ = 0\ ,\end{aligned}$$

for all $n \in \mathbb{N}$. By monotone convergence we get

$$\begin{aligned}\mathbb{E}\left[\int_0^{t} \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dD_{s+}\right] &= \mathbb{E}\left[\int_0^{t} \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dZ_{s+}\right] \\ = \mathbb{E}\left[\sum_{0\leq s<t}\left\{\mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}}\mathbf{1}_{\{X_{s-}\leq 0\}}X^+_{s+}\right\} + \mathbf{1}_{\left\{X^+_{t-}\neq 0\right\}}\mathbf{1}_{\{X_{t-}\leq 0\}}X^+_{t}\right] &= 0,\end{aligned}$$

hence

$$\mathbb{E}\left[\int_0^t \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dD^d_s\right] = \mathbb{E}\left[\int_0^t \mathbf{1}_{\left\{X^+_{s}\neq 0\right\}} dD^g_{s+}\right] = 0.$$

Then $\int_0^t \mathbf{1}_{\left\{X^+_{s-}\neq 0\right\}} dD^d_s = \int_0^t \mathbf{1}_{\left\{X^+_{s}\neq 0\right\}} dD^g_{s+} = 0$, hence X^+ is of class (Σ^l). Also X^- is of class (Σ^l) if X has no positive jumps follows from the same arguments applied to $-X$.

(3). Since X is nonnegative, hence $A_t \geq A_s \geq -M_s \vee 0$ for all $t \geq s$, and therefore, $A_t \geq \sup_{s\leq t}(-M_s)\vee 0$. We assume

$$\mathbb{P}\left[A_t > \sup_{s\leq t}(-M_s)\vee 0\right] > 0. \tag{2.4}$$

Consider the random time $T = \sup\left\{s \leq t : A_s = \sup_{u\leq s}(-M_u)\vee 0\right\}$, we have $A_T = \sup_{s\leq T}(-M_s)\vee 0$. Moreover, since $X_s > 0$ and has no positive jumps on $\{T < s \leq t\}$, it follows from $\int_0^t \mathbf{1}_{\{X_{s-}\neq 0\}} dA^d_s = 0$ and $\int_0^t \mathbf{1}_{\{X_s\neq 0\}} dA^g_{s+} = 0$ that $A_t = A_T$, a contradiction to (2.4). Then, $A_t = \sup_{s\leq t}(-M_s)\vee 0$.

(4). If X is of class $(\Sigma^l D)$, then from the first property (1), the processes X^+ and X^- are submartingales of class (D). Therefore, from [4] both have a decomposition into the sum of a uniformly integrable martingale and a predictable increasing process of integrable total variation:

$$X^+_t = M^1_t + A^1_t\ , \quad X^-_t = M^2_t + A^2_t.$$

Since the predictable finite variation part of a special semimartingale is unique, one must have $M_t = M^1_t - M^2_t$ and $A_t = A^1_t - A^2_t$. So M is a uniformly integrable martingale and A has integrable total variation. It follows that there exist integrable random variables X_∞, M_∞, A_∞ such that $X_t \to X_\infty$, $M_t \to M_\infty$, $A_t \to A_\infty$ almost surely and in L^1.

□

Proposition 2.2 *Let $(X^1_t), \ldots, (X^n_t)$ be nonnegative processes of class (Σ^l) such that $[X^i, X^j]_t = 0$ for $i \neq j$. Then $(\prod_{i=1}^n X^i_t)_{t\geq 0}$ is also of class (Σ^l).*

Proof: The proof is a simple change to the proof of the right-continuous case see [16]. □

Now, we shall give some examples of processes in the class (Σ^l).

Example 2.1 *1- Let M be an optional local martingale, starting from 0, we have (see [1]):*

$$|M_t| = \int_0^t sign(M_s)dM_{s+} + A_t,$$

where

$$A_t = L_t^0 + \sum_{0<s\leq t} (|M_s| - |M_{s-}| - sign(M_{s-})\Delta M_s) + \sum_{0\leq s<t} \left(|M_{s+}| - |M_s| - sign(M_s)\Delta^+ M_s\right),$$

with L_t^0 is the local time of M_+ at the level 0 and time t. We can decompose the process A_t to:

$$A_t^d = \frac{1}{2}L_t^0 + \sum_{0<s\leq t} (|M_s| - |M_{s-}| - sign(M_{s-})\Delta M_s),$$

and

$$A_t^g = \frac{1}{2}L_t^0 + \sum_{0\leq s<t} \left(|M_{s+}| - |M_s| - sign(M_s)\Delta^+ M_s\right),$$

such that the measure (dA^d) is carried by the set $\{t : M_{t-} = 0\}$, and the measure (dA^g_+) is carried by the set $\{t : M_t = 0\}$. Then $|M_t|$ is of class (Σ^l).

2- For any $\alpha > 0$, $\beta > 0$, let M be an optional local martingale, starting from 0, the process:

$$\alpha M_t^+ + \beta M_t^- = \int_0^t \left(\alpha \mathbf{1}_{\{M_s>0\}} - \beta \mathbf{1}_{\{M_s\leq 0\}}\right) dM_{s+} + (\alpha+\beta)A'_t,$$

where

$$A'_t = L_t^0 + 2\sum_{0\leq s<t} \left\{\mathbf{1}_{\{M_{s-}\leq 0\}} M_{s+}^+ + \mathbf{1}_{\{M_{s-}>0\}} M_{s+}^-\right\} + 2\left(\mathbf{1}_{\{M_{t-}\leq 0\}} M_t^+ + \mathbf{1}_{\{M_{t-}>0\}} M_t^-\right).$$

Then the process $\alpha M_t^+ + \beta M_t^-$ is of class (Σ^l).

3- Let M be right continuous local martingale which vanishes at zero with only negative jumps and let S its supremum process. Then according to the property (3) of Proposition 2.1, the process

$$X_t = S_t - M_t \tag{2.5}$$

is of class (Σ^l).

The next theorem gives the martingale characterization for the processes of class (Σ^l):

Theorem 2.1 *The following are equivalent:*

1. *The positive local submartingale (X_t) is of class (Σ^l).*

2. *There exists an increasing, adapted and làdlàg process (D_t) such that for every locally bounded, $\mathcal{C}^1$ Borel function f, and*

$$F(x) = \int_0^x f(z)dz,$$

the process

$$\begin{aligned} N_t &= F(D_t) - f(D_t)X_t + \\ &- \sum_{0<s\le t}\{\Delta(F(D_s)) - f(D_{s-})\Delta D_s\} + \sum_{0\le s<t}\{\Delta^+(F(D_s)) - f(D_s)\Delta^+ D_s\} + \\ &+ \sum_{0<s\le t}\{\Delta(f(D_s)X_s) - f'(D_{s-})X_{s-}\Delta D_s - f(D_{s-})\Delta X_s\} + \\ &+ \sum_{0\le s<t}\{\Delta^+(f(D_s)X_s) - f'(D_s)X_s\Delta^+ D_s - f(D_s)\Delta^+ X_s\} \end{aligned}$$

is a local martingale.

Proof: (1) $\Longrightarrow$ (2)
Let us take $D_t = A_t$. From Theorem 8.2 p.p 463 of [3], we get :

$$\begin{aligned} f(A_t)X_t &= \int_0^t f(A_{s-})dX_{s+} + \int_0^t f'(A_{s-})X_{s-}dA_{s+} + \\ &+ \sum_{s\le t}\{f(A_s)X_s - f(A_{s-})X_{s-} - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} + \\ &+ \sum_{0\le s<t}\{f(A_{s+})X_{s+} - f(A_s)X_s - f'(A_s)X_s\Delta^+ A_s - f(A_s)\Delta^+ X_s\} \\ &= \int_0^t f(A_{s-})dM_s + \int_0^t f(A_{s-})dA^d_s + \int_0^t f(A_s)dA^g_{s+} + \\ &+ \int_0^t f'(A_{s-})X_{s-}dA^d_s + \int_0^t f'(A_s)X_s dA^g_{s+} + \\ &+ \sum_{0<s\le t}\{f(A_s)X_s - f(A_{s-})X_{s-} - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} + \\ &+ \sum_{0\le s<t}\{f(A_{s+})X_{s+} - f(A_s)X_s - f'(A_{s-})X_s\Delta^+ A_s - f(A_{s-})\Delta^+ X_s\} \end{aligned}$$

Moreover

$$\begin{aligned} F(A_t) &= \int_0^t f(A_{s-})dA^d_s + \int_0^t f(A_s)dA^g_{s+} + \sum_{0<s\le t}\{\Delta(F(A_s)) - f(A_{s-})\Delta A_s\} + \\ &+ \sum_{0\le s<t}\{\Delta^+(F(A_{s+})) - f(A_s)\Delta^+ A_s\}. \end{aligned}$$

For simplification, we denote by

$$\begin{aligned} \Gamma_t &= \sum_{0<s\le t}\{\Delta(F(A_s)) - f(A_{s-})\Delta A_s\} + \\ &\quad + \sum_{0\le s<t}\{\Delta^+(F(A_s)) - f(A_s)\Delta^+ A_s\} \qquad (2.6) \\ \Lambda_t &= \sum_{0<s\le t}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} + \\ &\quad + \sum_{0\le s<t}\{\Delta^+(f(A_s)X_s) - f'(A_s)X_s\Delta^+ A_s - f(A_s)\Delta^+ X_s\} \qquad (2.7) \end{aligned}$$

then from (2.6) and (2.7), the expression of N_t is reduced to

$$N_t = F(A_t) - f(A_t)X_t - \Gamma_t + \Lambda_t. \qquad (2.8)$$

Since (dA_t^d) is carried by the set $\{t : X_{t-} = 0\}$, and (dA_{t+}^g) is carried by the set $\{t : X_t = 0\}$, we have

$$\int_0^t f'(A_{s-})X_{s-}dA_s^d = 0, \quad \text{and} \quad \int_0^t f'(A_s)X_s dA_{s+}^g = 0.$$

Therefore, from (2.8), we obtain

$$\begin{aligned} N_t &= F(A_t) - f(A_t)X_t - \Gamma_t + \Lambda_t \\ &= \left(\int_0^t f(A_{s-})dA_s^d + \int_0^t f(A_s)dA_{s+}^g + \Gamma_t\right) + \\ &\quad - \left(\int_0^t f(A_{s-})dM_s + \int_0^t f(A_{s-})dA_s^d + \int_0^t f(A_s)dA_{s+}^g + \Lambda_t\right) - \Gamma_t + \Lambda_t \\ &= -\int_0^t f(A_{s-})dM_s. \end{aligned} \tag{2.9}$$

Consequently, $(N_t)_{t\geq 0}$ is a local martingale.
(2) $\Longrightarrow$ (1)
First take $F(x) = x$, we obtain

$$N_t = D_t - X_t = -M_t,$$

hence N_t is a local martingale. Next, we take $F(x) = x^2$, we get

$$\begin{aligned} N_t &= D_t^2 - 2D_tX_t - \sum_{s\leq t}\left\{\Delta D_s^2 - 2D_{s-}\Delta D_s\right\} - \sum_{s<t}\left\{\Delta^+ D_s^2 - 2D_s\Delta^+ D_s\right\} + \\ &\quad + \sum_{s\leq t}\{2\Delta\left(D_sX_s\right) - 2X_{s-}\Delta D_s - 2D_{s-}\Delta X_s\} + \\ &\quad + \sum_{s<t}\left\{2\Delta^+\left(D_sX_s\right) - 2X_s\Delta^+ D_s - 2D_s\Delta^+ X_s\right\} \end{aligned}$$

we develop D_t^2 and D_tX_t by using Lemma 2.1 and integration by parts, we obtain:

$$\begin{aligned} D_t^2 &= 2\int_0^t D_{s-}dD_s^d + 2\int_0^t D_s dD_{s+}^g + \\ &\quad + \sum_{s\leq t}\left\{\Delta D_s^2 - 2D_{s-}\Delta D_s\right\} + \sum_{s<t}\left\{\Delta^+(D^2)_s - 2D_s\Delta^+ D_s\right\} \end{aligned}$$

and

$$\begin{aligned} D_tX_t &= \int_0^t D_{s-}dX_{s+} + \int_0^t X_{s-}dD_{s+} \\ &\quad + \sum_{s\leq t}\{\Delta\left(D_sX_s\right) - X_{s-}\Delta D_s - D_{s-}\Delta X_s\} \\ &\quad + \sum_{s<t}\left\{\Delta^+\left(D_sX_s\right) - X_s\Delta^+ D_s - D_s\Delta^+ X_s\right\} \\ &= \int_0^t D_{s-}dM_s + \int_0^t X_{s-}dD_s^d + \int_0^t X_s dD_{s+}^g \\ &\quad + \sum_{s\leq t}\{\Delta\left(D_sX_s\right) - X_{s-}\Delta D_s - D_{s-}\Delta X_s\} \\ &\quad + \sum_{s<t}\left\{\Delta^+\left(D_sX_s\right) - X_s\Delta^+ D_s - D_s\Delta^+ X_s\right\} \end{aligned}$$

hence

$$N_t = -2\int_0^t D_{s-}dM_s - 2\int_0^t X_{s-}dD^d_s - 2\int_0^t X_s dD^g_{s+},$$

since N_t is a local martingale, and X is a positive process, therefore

$$\int_0^t X_{s-}dD^d_s = 0 \quad \text{and} \quad \int_0^t X_s dD^g_{s+} = 0$$

then the measure (dD^d) is carried by the set $\{t : X_{t-} = 0\}$, and the measure (dD^g_+) is carried by the set $\{t : X_t = 0\}$. So X is of class (Σ^l). □

Corollary 2.1 *If f is a nonnegative and locally bounded $\mathcal{C}^1$ function, then for all X in (Σ^l), the process*

$$\begin{aligned} f(A_t)X_t &- \sum_{0<s\le t}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} \\ &+ \sum_{0\le s<t}\{\Delta^+(f(A_s)X_s) - f'(A_s)X_s\Delta^+ A_s - f(A_s)\Delta^+ X_s\} \end{aligned}$$

is also of class (Σ^l) and its non-decreasing part is

$$F(A_t) - \sum_{0<s\le t}\{\Delta(F(A_s)) - f(A_{s-})\Delta A_s\} + \sum_{0\le s<t}\{\Delta^+(F(A_s)) - f(A_s)\Delta^+ A_s\}.$$

Proof: In Theorem 2.1, we showed that $(F(A_t) - f(A_t)X_t - \Gamma_t + \Lambda_t$ is a local martingale, where the expressions of Λ and Γ are given in (2.6) and (2.7) . In addition $(F(A_t) - \Gamma_t)_{t\ge 0}$ is a ládlág, non-decreasing process which vanishes at zero since f is a nonnegative function. Then for all X in (Σ^l), $(f(A_t)X_t - \Lambda_t)_{t\ge 0}$ is again of class (Σ^l) . □

Corollary 2.2 *If f is a nonnegative and locally bounded $\mathcal{C}^1$ Borel function, and if X is positive of class (Σ^l) such that the process*

$$\begin{aligned} f(A_t)X_t &- \sum_{0<s\le t}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} \\ &+ \sum_{0\le s<t}\{\Delta^+(f(A_s)X_s) - f'(A_s)X_s\Delta^+ A_s - f(A_s)\Delta^+ X_s\} \end{aligned}$$

is of class (D), then

$$\begin{aligned} N_t &= F(A_t) - f(A_t)X_t \\ &\quad - \sum_{0<s\le t}\left\{\Delta(F(A_s)) - f(A_{s-})\Delta A_s + \sum_{0\le s<t}\Delta^+(F(A_s)) - f(A_s)\Delta^+ A_s\right\} \\ &\quad + \sum_{0<s\le t}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} \\ &\quad + \sum_{0\le s<t}\{\Delta^+(f(A_s)X_s) - f'(A_s)X_s\Delta^+ A_s - f(A_s)\Delta^+ X_s\} \end{aligned}$$

is a uniformly integrable martingale, hence we obtain

$$N_T = \mathbb{E}[N_\infty \mid \mathcal{F}_T] \text{ for every stopping time } T. \tag{2.10}$$

Proof: If X is of class (Σ^l) and $(f(A_t)X_t - \Lambda_t)_{t\ge 0}$ is of class (D), then from Proposition 2.1, we obtain that $N_t = F(A_t) - f(A_t)X_t - \Gamma_t + \Lambda_t$ is a uniformly integrable martingale, where the expressions of Λ and Γ are given in (2.6) and (2.7). From Doob's optional stopping theorem (Theorem 3.2, p.69 [32]), we get a formula (2.10). □

3. Representation formula for relative martingales

In this section we prove the formula of relative martingales for process of class (Σ^l) in terms of last passage times L given by

$$L := \sup\{t \in \mathbb{R}_+ : X_t = 0 \text{ and } X_{t-} = 0\} \quad \text{with the convention} \quad \sup\emptyset = 0.$$

Theorem 3.1 *Let X be a process of class (Σ^l) and $f : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^1$ Borel function, for notational simplification, we write as in (2.7) for all $t \in \mathbb{R}_+$*

$$\begin{aligned} Z_t &:= f(A_t)X_t - \Lambda_t \\ &= f(A_t)X_t - \sum_{s\le t}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} \\ &\quad + \sum_{s<t}\{\Delta(f(A_s)X_s) - f'(A_s)X_{s-}\Delta A_s - f(A_s)\Delta X_s\}. \end{aligned}$$

If X is of class (D), then there exist integrable random variables X_∞, M_∞, A_∞ such that $X_t \to X_\infty$, $M_t \to M_\infty$, $A_t \to A_\infty$ almost surely in L^1 and

$$Z_T = \mathbb{E}\left[Z_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] \quad \textit{for every stopping time } T.$$

In particular

$$X_T = \mathbb{E}\left[X_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] \quad \textit{for all stopping time } T < \infty.$$

Proof: If X is of class $(\Sigma^l D)$, it follows from Proposition 2.1 that M is a uniformly integrable martingale and A of integrable total variation. So there exist integrable random variables X_∞, M_∞, A_∞ such that $X_t \to X_\infty$, $M_t \to M_\infty$, $A_t \to A_\infty$ almost surely and in L^1. For a given stopping time T, denote

$$d_T = \inf\{t > T : X_t = 0 \text{ and } X_{t-} = 0\} \quad \text{with the convention} \quad \inf\emptyset = \infty.$$

Since $X_\infty \mathbf{1}_{\{L\le T\}} = X_{d_T}$ and since dA^d is carried by the set $\{t : X_{t-} = 0\}$ and dA^g_+ is carried by the set $\{t : X_t = 0\}$, then for all $t > T$, $dA^d_t = dA^g_{t+} = 0$ on $\{L \le T\}$ so one has $A_T = A_{d_T}$, it follows from Doob's optional stopping theorem that

$$\mathbb{E}\left[X_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] = \mathbb{E}\left[M_{d_T} + A_{d_T} \mid \mathcal{F}_T\right] = M_T + A_T = X_T.$$

Since $A_\infty = A_T$ almost everywhere on $\{L \le T\}$, hence

$$\sum_{s>T}\{\Delta(f(A_s)X_s) - f'(A_{s-})X_{s-}\Delta A_s - f(A_{s-})\Delta X_s\} = 0 \quad \text{on} \quad \{L \le T\},$$

and from (2.7), one has $Z_t = f(A_t)X_t - \Lambda_t$, then $\Lambda_\infty \mathbf{1}_{\{L\le T\}} = \Lambda_T$, and therefore

$$\begin{aligned} \mathbb{E}\left[Z_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] &= \mathbb{E}\left[(f(A_\infty)X_\infty - \Lambda_\infty)\mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] \\ &= \mathbb{E}\left[f(A_T)X_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] - \mathbb{E}\left[\Lambda_\infty \mathbf{1}_{\{L\le T\}} \mid \mathcal{F}_T\right] \\ &= f(A_T)X_T - \Lambda_T = Z_T. \end{aligned}$$

□

Corollary 3.1 *Let X is of class (Σ^l), with no negative jumps such that X^+ is of class (D). Denote $g = \sup\{t \in \mathbb{R}_+ : X_t \le 0\}$. Then*

$$X^+_T = \mathbb{E}\left[X^+_\infty \mathbf{1}_{\{g\le T\}} \mid \mathcal{F}_T\right] \tag{3.1}$$

for every stopping time T.

Proof: X is of class (Σ^l) with no negative jumps, so it follows from Proposition 2.1 that X^+ is a local submartingale of class (Σ^l). One has X^+ is of class (D) and $L = g$, where $g = \sup\left\{t \in \mathbb{R}_+ : X_t^+ = X_{t-}^+ = 0\right\}$, therefore by using Theorem 3.1, we come to the desired equation (3.1). □

Corollary 3.2 *Let $(X_t^1), \ldots, (X_t^n)$ be processes of class (Σ^l) that are bounded from below and have a no negative jumps. Assume $[X^i, X^j]_t = 0$ for $i \neq j$. Denote $g^i = \sup\left\{t \in \mathbb{R}_+ : X_t^i \leq 0\right\}$. Then*

$$\prod_{i=1}^{n} X_T^{i+} = \mathbb{E}\left[\prod_{i=1}^{n} X_\infty^{i+} \mathbf{1}_{\{g^i \leq T\}} \mid \mathcal{F}_T\right] \tag{3.2}$$

for every stopping time T.

Proof: From Proposition 2.1, the X^{i+} are a local submartingales of class (Σ^l) and since $[X^i, X^j]_t = 0$ for $i \neq j$, So we obtain using Proposition 2.2 that $\prod_{i=1}^{n} X^{i+}$ is again of class (Σ^l) and this, since all X^i are bounded, is bounded. From Theorem 3.1 we get (3.2). □

4. Balayage Formula and Predictable Compensator

The first theorem in this subsection is inspired from balayage formula of Azéma and Yor Theorem 6.1. [33].
Let X a local submartingale of class (Σ^l), and $(k_t; t \geq 0)$ be a locally bounded, optional process. Denote by α_t and β_t respectively the last zero of X before t and the first zero of X after t, namely:

$$\begin{aligned} \alpha_t &= \sup\{s < t;\ X_s = 0\}, \\ \beta_t &= \inf\{s > t;\ X_s = 0\}. \end{aligned}$$

Next theorem presents the balayage formula for optional submartingale.

Theorem 4.1 *Let $\gamma_t = \alpha_{t+}$, the process $(k_{\gamma_t} X_t; t \geq 0)$ is a local submartingale , more precisely:*

$$k_{\gamma_t} X_t = \int_0^t k_{\gamma_s} dX_s$$

where the stochastic integral with respect to the process X is defined as by Lenglart in [1].

Proof: We assume that k is a simple optional process, i.e. there exists a stopping time T such that $k_u = \mathbf{1}_{[0,T[}(u)$ for any $u \geq 0$.
Let $\delta_t = \beta_{t-}$, then, there is the following sequence of easy identities:

$$\begin{aligned} k_{\gamma_t} X_t &= \mathbf{1}_{\gamma_t < T} X_t = \mathbf{1}_{t < \delta_T} X_t = X_{t \wedge \delta_T} = X_{(t \wedge \delta_T)+} - \Delta^+ X_{t \wedge \delta_T} \\ &= \int_0^t \mathbf{1}_{s < \delta_T} dX_s = \int_0^t k_{\gamma_s} dX_s \end{aligned}$$

□

Definition 4.1 *Let H be an integrable, increasing, right-continuous process. There exists a unique predictable, increasing, right-continuous process $H^{(p)}$ such that, for any positive optional process k:*

$$\mathbb{E}\left[\int_0^\infty k_s dH_s\right] = \mathbb{E}\left[\int_0^\infty k_s dH_s^{(p)}\right].$$

We shall say that $H^{(p)}$ is the predictable compensator of H.

Theorem 4.2 *If X is a process of class (Σ^l), with no positive jumps, and $(k_t)_{t\geq 0}$ be a locally bounded, optional process. Then*

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty k_s d(\mathbf{1}_{\{\gamma\geq s\}})\right] &= \mathbb{E}\left[\int_0^\infty k_s d(\mathbf{1}_{\{\gamma\geq s\}})_s^{(p)}\right] \\ &= \frac{1}{2}\mathbb{E}\left[\int_0^\infty k_s dL_s^0\right]. \end{aligned} \tag{4.1}$$

where $\gamma=\gamma_1$ and L_t^0 is the local time of X at the level 0 and time t. Therefore, the predictable compensator of $(\mathbf{1}_{\{\gamma\geq t\}})$ is $\frac{1}{2}L_t^0$.

Proof: X is of class (Σ^l) and M is a local martingale, there exists a sequence of stopping times T_n, $n\in\mathbb{N}$, increasing to ∞

$$\begin{aligned}\mathbb{E}\left[k_{\gamma_{1\wedge T_n}} X^+_{1\wedge T_n}\right] &= \mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s} dX_s^+\right] \\ &= \mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}} dX_{s+}\right] + \frac{1}{2}\mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s} dL_s^0\right] + \\ &\quad +\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\Delta X_s^+ + k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\Delta X_s^-\right] + \\ &\quad +\mathbb{E}\left[\sum_{s< 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\Delta^+ X_s^+ + k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\Delta^+ X_s^-\right]\end{aligned}$$

since (dA_t^d) is carried by the set $\{t : X_{t-}=0\}$ and (dA_{t+}^g) is carried by the set $\{t : X_t = 0\}$, we get

$$\begin{aligned}\mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}} dX_s\right] &= \mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}} dM_s\right] + \mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}} dA_s^d\right] + \\ &\quad +\mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_s>0\}} dA_{s+}^g\right] \\ &= 0\end{aligned} \tag{4.2}$$

by the same arguments and since A^g is the càglàd process, we have $\Delta A_s^g = 0$, therefore

$$\begin{aligned}\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\Delta X_s^-\right] &= -\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\mathbf{1}_{\{X_s<0\}}\Delta X_s\right] \\ &= -\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\mathbf{1}_{\{X_s<0\}}\Delta M_s\right] + \\ &\quad -\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\mathbf{1}_{\{X_s<0\}}\Delta A_s^d\right] \\ &= 0\end{aligned} \tag{4.3}$$

since M and A^d are the càdlàg processes, then $\Delta^+ M_s = 0$ and $\Delta^+ A^d_s = 0$, hence

$$\begin{aligned}\mathbb{E}\left[\sum_{s<1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\Delta^+ X^-_s\right] &= -\mathbb{E}\left[\sum_{s<1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\mathbf{1}_{\{X_s<0\}}\Delta^+ X_s\right]\\ &= -\mathbb{E}\left[\sum_{s<1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}>0\}}\mathbf{1}_{\{X_s<0\}}\Delta A^g_s\right]\\ &= 0 \end{aligned} \tag{4.4}$$

we also have

$$\begin{aligned}\mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\Delta X^+_s\right] &= \mathbb{E}\left[\sum_{s\leq 1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\mathbf{1}_{\{X_s>0\}}\Delta X_s\right]\\ &= 0\end{aligned} \tag{4.5}$$

since X has only no positive jumps, so $\mathbf{1}_{\{X_{s-}\leq 0\}}\mathbf{1}_{\{X_s>0\}} = 0$, and we have

$$\begin{aligned}\mathbb{E}\left[\sum_{s<1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\Delta^+ X^+_s\right] &= \mathbb{E}\left[\sum_{s<1\wedge T_n} k_{\gamma_s}\mathbf{1}_{\{X_{s-}\leq 0\}}\mathbf{1}_{\{X_s>0\}}\Delta^+ X_s\right]\\ &= 0\end{aligned} \tag{4.6}$$

Therefore from (4.2),(4.3),(4.4),(4.5) and (4.6), we get

$$\mathbb{E}\left[k_{\gamma_{1\wedge T_n}} X_{1\wedge T_n}\right] = \frac{1}{2}\mathbb{E}\left[\int_0^{1\wedge T_n} k_{\gamma_s} dL^0_s\right]$$

for all $n \in \mathbb{N}$. By monotone convergence one obtains

$$\mathbb{E}\left[k_{\gamma_1} X_1\right] = \frac{1}{2}\mathbb{E}\left[\int_0^1 k_{\gamma_s} dL^0_s\right].$$

□

Acknowledgments

The authors would like to thank Professor Youssef Ouknine for various discussions on optional submartingales.

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