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A subclass of strongly close-to-convex functions associated with generalized Janowski functions

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ABSTRACT: In this paper, we introduce a new subclass of strongly close-to-convex functions by associating to generalized Janowski function. We establish coefficient estimates, distortion theorem, argument theorem, inclusion relations and radius of convexity for this class. The results established here will generalize various earlier known results.

Key Words: Analytic functions, Subordination, Janowski-type function, Close-to-convex functions, Coefficient bounds.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $E = \{z :| z |< 1\}$, which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in E.

By $\mathcal{U}$, we denote the class of Schwarzian functions w satisfying $w(0) = 0$ and $|w(z)| \leq 1$, which are analytic in E and have expansion of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in E. \tag{1.2}$$

For $0 \leq \alpha < 1$, the classes of starlike functions and convex functions of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively and defined as

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

In particular, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ which is the class of starlike functions and $\mathcal{K}(0) = \mathcal{K}$, the class of convex functions. For $\alpha = \frac{1}{2}$, $\mathcal{S}^*(\frac{1}{2})$ is the class of starlike functions of order $\frac{1}{2}$.

Kaplan [8] introduced the concept of close-to-convex functions. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of close-to-convex functions if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

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Sakaguchi [15] established the class $\mathcal{S}_s^*$ of the functions $f \in \mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > 0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Clearly, the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions, as $\dfrac{f(z)-f(-z)}{2}$ is a starlike function [4] in E.

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$.

Many interesting subclasses of the class $\mathcal{A}$ have been studied by various authors from different view points. We choose to give an overview of the classes which are closely related to our investigation, as follows:

Getting inspired from the class $\mathcal{S}_s^*$, Gao and Zhou [5] studied the class $\mathcal{K}_S$ given by:

$$\mathcal{K}_s = \left\{f : f \in \mathcal{A}, Re\left(\frac{-z^2f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E\right\},$$

where $\mathcal{S}^*\left(\frac{1}{2}\right)$ is the class of starlike functions of order $\frac{1}{2}$.
Kowalczyk and Les-Bomba [9] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma)(0 \leq \gamma < 1)$ mentioned below:

$$\mathcal{K}_s(\gamma) = \left\{f : f \in \mathcal{A}, Re\left(\frac{-z^2f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E\right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Further, Prajapat [12] established that, a function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$, if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that

$$Re\left[\frac{tz^2f'(z)}{g(z)g(tz)}\right] > \gamma.$$

In particular $\chi_{-1}(\gamma) \equiv \mathcal{K}_S(\gamma)$ and $\chi_{-1}(0) \equiv \mathcal{K}_S$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [11] introduced the class $\mathcal{P}(A, B; \alpha)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + [B + (A-B)(1-\alpha)]z}{1+Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [7].

Using the concept of subordination, Singh et al. [16] introduced the following class:

Let $\chi_t(A, B)(|t| \leq 1, t \neq 0)$ denote the class of functions $f \in \mathcal{A}$ and satisfying the conditions

$$\frac{tz^2f'(z)}{g(z)g(tz)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in E,$$

where $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$.
The following observations are obvious:
(i) $\chi_t(1-2\gamma, -1) \equiv \chi_t(\gamma)$.
(ii)$\chi_{-1}(1-2\gamma, -1) \equiv \mathcal{K}_S(\gamma)$.

(iii)$\chi_{-1}(1,-1) \equiv \mathcal{K}_S$.

Recently, Raina et al. [13] defined the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \left| arg \left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\},$$

or equivalently

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1+z}{1-z} \right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\}.$$

In particular $\mathcal{C}'_1 \equiv \mathcal{C}'$, the subclass of close-to-convex functions studied by Abdel-Gawad and Thomas [1].

Motivated by the above mentioned classes, now we introduce the following generalized subclass of strongly close-to-convex functions:

Let $\chi_t(A,B;\alpha;\beta)(|t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \left(\frac{1 + [B + (A-B)(1-\alpha)]z}{1+Bz} \right)^\beta, -1 \leq B < A \leq 1, z \in E, \tag{1.3}$$

where $g \in \mathcal{S}^* \left(\frac{1}{2}\right)$.

The following observations are obvious:

(i) $\chi_t(A,B;0;1) \equiv \chi_t(A,B)$.

(ii) $\chi_t(1-2\gamma,-1;0;1) \equiv \chi_t(\gamma)$.

(iii) $\chi_{-1}(1-2\gamma,-1;0,1) \equiv \mathcal{K}_s(\gamma)$.

(iv) $\chi_{-1}(1,-1;0,1) \equiv \mathcal{K}_s$.

By definition of subordination, it follows that $f \in \chi_t(A,B;\alpha;\beta)$ if and only if

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = \left(\frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1+Bw(z)} \right)^\beta, w \in \mathcal{U}. \tag{1.4}$$

In the present work, we obtain the coefficient estimates, inclusion relation, distortion theorem, argument theorem and radius of convexity for the functions in the class $\chi_t(A,B;\alpha;\beta)$. Our results extend the known results due to various authors.

Throughout our present discussion, to avoid repetition, we lay down once for all that $-1 \leq B < A \leq 1, 0 < |t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in E$.

2. Preliminary Lemmas

Lemma 2.1 *[3,14] Let,*

$$\left(\frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1+Bw(z)} \right)^\beta = (P(z))^\beta = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then

$$|p_n| \leq \beta(1-\alpha)(A-B), n \geq 1.$$

Lemma 2.2 *[17] Let $g \in S^* \left(\frac{1}{2}\right)$, then for*

$$G(z) = \frac{g(z)g(tz)}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*, \tag{2.2}$$

we have, $|d_n| \leq n$.

Lemma 2.3 *[13] Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ and $0 < \beta \leq 1$, then*

$$\left(\frac{1+A_1 z}{1+B_1 z}\right)^{\beta} \prec \left(\frac{1+A_2 z}{1+B_2 z}\right)^{\beta}.$$

Lemma 2.4 *[2,3] If $P(z) = \dfrac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, -1 \leq B < A \leq 1, w \in \mathcal{U}$, then for $|z| = r < 1$, we have*

$$Re\frac{zP'(z)}{P(z)} \geq \begin{cases} -\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \leq R_2, \\ 2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\frac{(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)}{(1-B)(1+Br^2)}}$ and $R_2 = \frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}$.

3. Main Results

Theorem 3.1 *If $f \in \chi_t(A, B; \alpha; \beta)$, then*

$$|a_n| \leq 1 + \frac{\beta(1-\alpha)(n-1)(A-B)}{2}. \tag{3.1}$$

Proof: As $f \in \chi_t(A, B; \alpha; \beta)$, therefore (1.4) can be expressed as

$$\frac{zf'(z)}{G(z)} = (P(z))^{\beta}. \tag{3.2}$$

Using (1.1), (2.1) and (2.2) in (3.2), it yields

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} d_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right). \tag{3.3}$$

Equating the coefficients of z^{n-1} in (3.3), we have

$$na_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + ... + d_2 p_{n-2} + p_{n-1}. \tag{3.4}$$

Therefore using Lemma 2.1 and Lemma 2.2, it gives

$$n|a_n| \leq n + \beta(1-\alpha)(A-B)[(n-1) + (n-2) + ... + 2 + 1]. \tag{3.5}$$

Hence from (3.5), (3.1) can be easily obtained.

□

For $\alpha = 0, \beta = 1$, Theorem 3.1 gives the following result due to Singh et al. [16].

Corollary 3.1 *If $f \in \chi_t(A, B)$, then*

$$|a_n| \leq 1 + \frac{(n-1)(A-B)}{2}.$$

On putting $A = 1-2\gamma$, $B = -1$, $\alpha = 0$ and $\beta = 1$ in Theorem 3.1, the following result due to Prajapat [12] is obvious:

Corollary 3.2 *If $f \in \chi_t(\gamma)$, then*

$$|a_n| \leq 1 + (n-1)(1-\gamma).$$

Theorem 3.2 *If $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then*

$$\chi_t(A_1, B_1; \alpha_1; \beta) \subset \chi_t(A_2, B_2; \alpha_2; \beta).$$

Proof: As $f \in \chi_t(A_1, B_1; \alpha_1; \beta)$, so

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \left(\frac{1 + [B_1 + (A_1 - B_1)(1-\alpha_1)]z}{1 + B_1 z}\right)^\beta.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$-1 \leq B_1 + (1-\alpha_1)(A_1 - B_1) \leq B_2 + (1-\alpha_2)(A_2 - B_2) \leq 1.$$

Thus by Lemma 2.3, it yields

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \left(\frac{1 + [B_2 + (A_2 - B_2)(1-\alpha_2)]z}{1 + B_2 z}\right)^\beta,$$

which implies $f \in \chi_t(A_2, B_2; \alpha_2; \beta)$.

□

Theorem 3.3 *If $f \in \chi_t(A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left(\frac{1 - [B + (A-B)(1-\alpha)]r}{1 - Br}\right)^\beta \cdot \frac{1}{(1+r)^2} \leq |f'(z)| \leq \left(\frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br}\right)^\beta \cdot \frac{1}{(1-r)^2} \quad (3.6)$$

and

$$\int_0^r \left(\frac{1 - [B + (A-B)(1-\alpha)]t}{1 - Bt}\right)^\beta \cdot \frac{1}{(1+t)^2} dt \leq |f(z)| \leq \int_0^r \left(\frac{1 + [B + (A-B)(1-\alpha)]t}{1 + Bt}\right)^\beta \cdot \frac{1}{(1-t)^2} dt. \quad (3.7)$$

Proof: From (3.2), we have

$$|f'(z)| = \frac{|G(z)|}{|z|}(P(z))^\beta. \quad (3.8)$$

Aouf [3] proved that

$$\frac{1 - [B + (A-B)(1-\alpha)]r}{1 - Br} \leq |P(z)| \leq \frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br},$$

which implies

$$\left(\frac{1 - [B + (A-B)(1-\alpha)]r}{1 - Br}\right)^\beta \leq |P(z)|^\beta \leq \left(\frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br}\right)^\beta. \quad (3.9)$$

Since by Lemma 2.2, G is a starlike function and so due to Mehrok [10], we have

$$\frac{r}{(1+r)^2} \leq |G(z)| \leq \frac{r}{(1-r)^2}. \quad (3.10)$$

(3.8) together with (3.9) and (3.10) yields (3.6). On integrating (3.6) from 0 to r, (3.7) follows.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.3, the following result due to Singh et al. [16] is obvious:

Corollary 3.3 *If $f \in \chi_t(A,B)$, then*

$$\frac{1-Ar}{(1-Br)(1+r)^2} \leq |f'(z)| \leq \frac{1+Ar}{(1+Br)(1-r)^2}$$

and

$$\int_0^r \frac{1-At}{(1-Bt)(1+t)^2}dt \leq |f(z)| \leq \int_0^r \frac{1+At}{(1+Bt)(1-t)^2}dt.$$

For $A = 1-2\gamma, B = -1, \alpha = 0, \beta = 1$, Theorem 3.3 gives the following result due to Prajapat [12]:

Corollary 3.4 *If $f \in \chi_t(\gamma)$, then*

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-(1-2\gamma)t}{(1+t)^3}dt \leq |f(z)| \leq \int_0^r \frac{1+(1-2\gamma)t}{(1-t)^3}dt.$$

Theorem 3.4 *If $f \in \chi_t(A,B;\alpha;\beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$|argf'(z)| \leq \beta sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right) + 2sin^{-1}r.$$

Proof: From (3.2), we have

$$f'(z) = \frac{G(z)}{z}(P(z))^\beta,$$

which implies

$$|argf'(z)| \leq \beta|argP(z)| + \left|arg\frac{G(z)}{z}\right|. \tag{3.11}$$

As G is a starlike function and so due to Mehrok [10], we have

$$\left|arg\frac{G(z)}{z}\right| \leq 2sin^{-1}r. \tag{3.12}$$

Aouf [3], established that,

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), the proof is obvious.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.4, the following result due to Singh et al. [16] is obvious:

Corollary 3.5 *If $f \in \chi_t(A,B)$, then*

$$|argf'(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

Theorem 3.5 *Let $f \in \chi_t(A, B; \alpha; \beta)$, then*

$$Re\frac{(zf'(z))'}{f'(z)} \geq \begin{cases} \frac{1-r}{1+r} - \beta\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \leq R_2, \\ \frac{1-r}{1+r} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)} & \\ +2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -2\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 2.4.

Proof: As $f \in \chi_t(A, B; \alpha; \beta)$, we have

$$zf'(z) = G(z)(P(z))^{\beta}.$$

On differentiating it logarithmically, we get

$$\frac{(zf'(z))'}{f'(z)} = \frac{zG'(z)}{G(z)} + \beta\frac{zP'(z)}{P(z)}. \tag{3.14}$$

As $G \in \mathcal{S}^*$, from [10], we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}. \tag{3.15}$$

Hence, using (3.15) and Lemma 2.4 in (3.14), the proof of Theorem 3.5 is obvious.

□

References

1. Abdel-Gawad, H. R. and Thomas, D. K., *A subclass of close-to-convex functions*, Publ. Inst. Math. (N.S.), 49(63)(1991), 61-66.
2. Anh, V. V. and Tuan, P. D., *On β-convexity of certain starlike functions*, Rev. Roum. Math. Pures et Appl., 25(1979), 1413-1424.
3. Aouf, M. K., *On a class of p-valent starlike functions of order α*, Int. J. Math. Math. Sci., 10(4)(1987), 733-744.
4. Das, R. N. and Singh, P., *On subclasses of schlicht mapping*, Indian J. Pure Appl. Math., 8(8)(1977), 864-872.
5. Gao, C. Y. and Zhou, S. Q., *On a class of analytic functions related to the starlike functions*, Kyungpook Math. J., 45(2005), 123-130.
6. Goel, R. M. and Mehrok, B. S., *A subclass of univalent functions*, Houston J. Math., 8(3)(1982), 343-357.
7. Janowski, W., *Some extremal problems for certain families of analytic functions*, Ann. Pol. Math., 28(1973), 297-326.
8. Kaplan, W., *Close-to-convex schlicht functions*, Michigan Math. J., 1(1952), 169-185.
9. Kowalczyk, J. and Les-Bomba, E., *On a subclass of close-to-convex functions*, Appl. Math. Letters, 23(2010), 1147-1151.
10. Mehrok, B. S., A subclass of close-to-convex functions, *Bull. Inst. Math. Acad. Sin.*, 10(4)(1982), 389-398.
11. Polatoglu, Y., Bolkal, M., Sen, A. and Yavuz, E., *A study on the generalization of Janowski function in the unit disc*, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis, 22(2006), 27-31.
12. Prajapat, J. K., *A new subclass of close-to-convex functions*, Surveys in Math. and its Appl., 11(2016), 11-19.
13. Raina, R. K., Sharma, P. and Sokol, J., *A class of strongly close-to-convex functions*, Bol. Soc. Paran. Mat., 38(6)(2020), 9-24.
14. Rogosinski, W., *On the coefficients of subordinate functions*, Proc. Lond. Math. Soc., 48(2)(1943),48-82.
15. Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan, 11(1959), 72-75.
16. Singh, G., Singh, G. and Singh, G., *A new subclass of univalent functions*, Ufa Math. J., 11(1)(2019), 133-140.
17. Soni, A. and Kant, S., *A new subclass of close-to-convex functions with Fekete-Szegö problem*, J. Rajasthan Acad. Phy. Sci., 12(2)(2013), 1-14.

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