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On the Periodic Solutions for a class of Partial Differential Equations with unbounded Delay

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ABSTRACT: Trough this work we investigate the existence of periodic solutions for the following partial differential equations with infinite delay of the form $\dot{w}(t) = \mathcal{L}w(t) + \mathcal{D}(w_t) + \mathcal{H}(t)$. We assume that the operator $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ is generally nondensely defined operator and verifies the Hille-Yosida condition. Using the theory of perturbation of semi-Fredholm operators, we propose some sufficient conditions on the linear operators $\mathcal{L}$, $\mathcal{D}$ and the phase space $\mathscr{B}$ to guarantee the existence of periodic solutions for this class of partial differential equations from bounded ones on the positive real half-line without considering the compactness of the semigroup generated by the part of $\mathcal{L}$ on the closure of it's domain. At the end, an application with numerical simulations, is given to confirm the applicability of the obtained theoretical results.

Key Words: Semigroup, Hille-Yosida condition, integral solutions, semi-Fredholm operators, Poincaré map, periodic solution.

Contents

1. Introduction and preliminary

Along this work we establish the existence of periodic solutions for the following partial functional differential equation with infinite delay

$$\begin{cases} \dot{w}(t) &= \mathcal{L}w(t) + \mathcal{D}(w_t) + \mathcal{H}(t) \quad for \ t \geq 0, \\ w_0 = & \varphi \quad \in \mathscr{B}, \end{cases} \tag{1.1}$$

where $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ is a linear operator on a Banach space $(\mathbb{X}, \|.\|)$, $\mathcal{D} : \mathscr{B} \to \mathbb{X}$ is a bounded linear operators, $\mathcal{H} : \mathbb{R}^+ \to \mathbb{X}$ is a continuous function and w_t, the history function, is an element of $\mathscr{B}$ defined by $w_t(\theta) = w(t + \theta)$, $\theta \leq 0$, where $\mathscr{B}$ is the space of all functions mapping from $(-\infty, 0]$ to $\mathbb{X}$ endowed with a norm $|.|_{\mathscr{B}}$ and complies with the followings axioms proposed in [10]:

(A) There is $c > 0$ and functions $M(.), K(.) : \mathbb{R}^+ \to \mathbb{R}^+$ with K is continuous and M is locally bounded function such that for $\varsigma \in \mathbb{R}$, $r > 0$, if a function $w : (-\infty, \varsigma + r] \to \mathbb{X}$ is continuous on $[\varsigma, \varsigma + r]$ and $w_\varsigma \in \mathscr{B}$, then for $\varsigma \leq t \leq \varsigma + r$ the statements below hold:

(i) $w_t \in \mathscr{B}$,

(ii) $\|w(t)\| \leq c|w_t|_{\mathscr{B}}$,

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(iii) $|w_t|_{\mathscr{B}} \leq K(t-\varsigma) \sup_{\varsigma \leq s \leq t} \|w(t)\| + M(t-\varsigma)|w_\varsigma|_{\mathscr{B}}$,

(iv) The function $t \to w_t$ is continuous from $[\varsigma, \varsigma + r]$ into $\mathscr{B}$,

(v) The space $\mathscr{B}$ is complete.

The problem of periodicity is widely investigated by several researchers in different directions concern ordinary and partial differential equations. The must popular approach that they employed was primarily based on the use of fixed points theory. There is a large literature on these topics, see for instance previous studies [2,3,6,7,8,13,14,15,16,17,18,19,20,22]. The authors, in [6], proved the periodicity of solutions by applying Horn's fixed points theorem to the Poincaré map and under the ultimate boundedness condition on the solutions. Recently, in [19], proceeded by leray-Shauder fixed points theorem to establish the periodicity for a class of partial differential equations, the authors, in [20], the authors investigated the existence of periodic solutions of a class of semilinear differential equations by using Banach fixed point theorem when the semigroup generated by the linear operator is not compact. Whereas, Schauder fixed point theorem is used when the semigroup is compact. Moreover, the authors in [22] investigate the existence of periodic solutions for some class of linear partial functional differential equations with delay by using some properties of semi-Fredholm operators in the case where $\mathbb{X} = \overline{\mathscr{D}(\mathcal{L})}$.

The present work would be a continuation and generalization of the work [22] in the case where the operator $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ is not necessarily densely defined. Moreover, we assume that $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ verifying the following Hille-Yosida condition:

$(\mathbf{C_0})$ There are constants $\widehat{K} \geq 1$ and $\widehat{\nu} \in \mathbb{R}$ such that $\rho(\mathcal{L}) \supset (\widehat{\nu}, +\infty)$ and

$$|(\eta I - \mathcal{L})^{-n}| \quad \leq \frac{\widehat{K}}{(\eta - \widehat{\nu})^n} \quad \text{for} \quad \eta > \widehat{\nu} \quad \text{and} \quad n \in \mathbb{N},$$

Trough this work, we denote by $\mathcal{HY}(\mathbb{X})$ the set of all operators defined from $\mathscr{D}(\mathcal{L})$ to $\mathbb{X}$ and verifies the Hille-Yosida condition. Our approach is essentially based on the combination between the perturbation theory of semi-Fredholm operators and the following modified Show and Hale fixed point theorem:

Theorem 1.1 *Let $\mathcal{F}$ be a linear affine map on $\mathbb{X}$ with $\mathcal{F}x = \overline{\mathcal{F}}x + y$ for $x \in \mathbb{X}$. If $I - \overline{\mathcal{F}} \in \mathcal{SF}_+(\mathbb{X})$ and $\{\mathcal{F}^k x_0, k \in \mathbb{N}\}$ is a bounded set in $\mathbb{X}$ for some $x_0 \in \mathbb{X}$. Then, $P_{\mathcal{F}} \neq \varnothing$, where $P_{\mathcal{F}}$ means the set of all fixed points of the operator $\mathcal{F}$.*

First, we need to introduce the definition and the principal theorem of perturbation of semi-Fredholm operators used in this work. A semi-Fredholm operators is the class of bounded linear operators $F \in \mathscr{L}(\mathbb{X}, \mathbb{F})$ such that $\ker(F)$ is of finite dimension and the range of F denoted by $Im(F)$ is closed in $\mathbb{F}$. Through this work we denote the set of all semi-Fredholm operators mapping from $\mathbb{X}$ to $\mathbb{X}$ by $\mathcal{SF}_+(\mathbb{X})$. Now, we need the following results taken from [21,22]. Let $F \in \mathscr{L}(\mathbb{X})$, then the quotient space $\mathbb{X}/\ker(F)$ is a Banach space equipped with the norm

$$|\overline{u}| = dist(u, \ker(F)),$$

where $\overline{u} = u + \ker(F)$ is an element of $\mathbb{X}/\ker(F)$.
Furthermore, if $\dim \ker(F) < \infty$, then there exists a closed subspace T of $\mathbb{X}$ such that

$$\mathbb{X} = \ker(F) \oplus T.$$

In addition, $S_T := F|_T$ the restriction of F to T has a bounded inverse.

Proposition 1.1 *[21] Let F be a bounded linear operator in $\mathbb{X}$. Then, $Im(F)$ is closed if and only if there exists a constant δ such that*

$$|\overline{u}| \leq \delta \, \|Fu\| \qquad for\ all \qquad u \in \mathbb{X}.$$

It is well know in the operator theory that if F is an element of $\mathcal{SF}_+(\mathbb{X})$, then the operator F perturbed by a small linear bounded perturbation remains also in $\mathcal{SF}_+(\mathbb{X})$. More precisely, we have the following theorem.

Theorem 1.2 *[22] Assume that $F \in \mathcal{SF}_+(\mathbb{X})$. If $G \in \mathscr{L}(\mathbb{X})$ and satisfies the following inequality:*

$$|G| < \frac{1}{2\delta(1+\sqrt{dim\ker(F)})},$$

where δ is the constant given in Proposition 1.1. Then,

$$F+G \in \mathcal{SF}_+(\mathbb{X}) \quad and \quad \dim\ker(F+G) \leq \dim\ker(F).$$

Moreover, we need to introduce some definitions and results concerning the integral solution of Eq. (1.1).

Definition 1.1 *[1] A function w defined from $[0,+\infty)$ to $\mathbb{X}$ is called an integral solution of Eq. (1.1) if:*

(i) The function w is continuous, and $w_0 = \varphi$,

(ii) $\displaystyle\int_0^t w(s)\,ds \in \mathscr{D}(\mathcal{L}) \quad for \quad t \in [0,+\infty),$

(iii) $\displaystyle w(t) = \varphi(0) + \mathcal{L}\int_0^t w(s)\,ds + \int_0^t (\mathcal{D}(w_s) + \mathcal{H}(s))\,ds \quad for \quad t \in [0,+\infty).$

As a consequence, from the continuity of the integral solution w one has that for all $t \geq 0$, $w(t)$ is an element of $\overline{\mathscr{D}(\mathcal{L})}$. Now, let $\mathcal{L}_0$ be the part of the operator $\mathcal{L}$ defined on $\overline{\mathscr{D}(\mathcal{L})}$ by:

(i) $\mathscr{D}(\mathcal{L}_0) = \{v \in \mathscr{D}(\mathcal{L}) : \mathcal{L}v \in \overline{\mathscr{D}(\mathcal{L})}\}$,

(ii) $\mathcal{L}_0 v = \mathcal{L}v$ for $v \in \mathscr{D}(\mathcal{L}_0)$.

Then, $\mathcal{L}_0$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{S}_0(t), t \geq 0\}$ on $\overline{\mathscr{D}(\mathcal{L})}$. Let $\nu_0 \in \mathbb{R}$ and $K_0 \geq 1$ such that $|S_0(t)| \leq K_0 e^{\nu_0 t}$ for $t \geq 0$.

Theorem 1.3 *[1] Suppose that $(\mathbf{C_0})$ holds. Then, for $\varphi(0) \in \overline{\mathscr{D}(\mathcal{L})}$, Eq. (1.1) admits a unique integral solution $w : (-\infty,+\infty) \to \mathbb{X}$ satisfies*

$$w(t) = \mathcal{S}_0(t)\varphi(0) + \lim_{\mu\to+\infty}\int_0^t \mathcal{S}_0(t-s)\mu(\mu I - \mathcal{L})^{-1}(\mathcal{D}(w_s) + \mathcal{H}(s))\,ds \qquad for \quad t \geq 0.$$

Through this work, let

$$\mathscr{B}_0 = \{\varphi \in \mathscr{B} : \varphi(0) \in \overline{\mathscr{D}(\mathcal{L})}\}.$$

be the phase space of Eq. (1.1). Moreover, we denote by solution the integral solution of Eq. (1.1). Let $w(.,\varphi,\mathscr{D},\mathcal{H})$ be the solution of Eq. (1.1) and let $\mathcal{M}(t)\varphi = w_t(.,\varphi,0,0)$, $t \geq 0$ be the linear operator defined on $\mathscr{B}_0$ where w is the solution of the equation:

$$\begin{cases} \dot{w}(t) = & \mathcal{L}w(t) \quad for \quad t \geq 0, \\ w_0 = \varphi. \end{cases} \tag{1.2}$$

Theorem 1.4 *[2] The bounded linear operator $(\mathcal{M}(t))_{t\geq 0}$ is a C_0-semigroup defined on $\mathscr{B}_0$, that is*
i) $\mathcal{M}(0) = I$ and $\mathcal{M}(t+s) = \mathcal{M}(t)\mathcal{M}(s)$ for $t,s \geq 0$,
ii) for $\varphi \in \mathscr{B}_0$, the function $\mathcal{M}(t)\varphi$ is continuous from $[0,+\infty)$ to $\mathscr{B}_0$,
iii) for $t \geq 0$ and $\theta \in (-\infty,0]$, the operator $(\mathcal{M}(t))_{t\geq 0}$ verifies

$$[\mathcal{M}(t)\varphi](\theta) = \begin{cases} (\mathcal{M}(t+\theta)\varphi)(0) & for \quad t \geq -\theta, \\ \varphi(t+\theta) & for \quad t \leq -\theta. \end{cases}$$

Theorem 1.5 *[2] Suppose that* $(\mathbf{C_0})$ *holds. Then, the solution of Eq. (1.1) with* $\mathcal{H}=0$ *denoted by* $\mathcal{W}(t)\varphi = w_t(.,\varphi,\mathcal{D},0)$*,* $t\geq 0$ *is decomposed as:*

$$\mathcal{W}(t)\varphi = \mathcal{M}(t)\varphi + \mathcal{R}(t)\varphi, \tag{1.3}$$

where $\mathcal{R}(t)$*,* $t\geq 0$ *is given for* $\varphi \in \mathscr{B}_0$ *by*

$$[\mathcal{R}(t)\varphi](\theta) = \begin{cases} \lim\limits_{\mu\to+\infty}\displaystyle\int_0^{t+\theta} \mathcal{S}_0(t+\theta-s)\mu(\mu I-\mathcal{L})^{-1}\mathcal{D}(w_s(\varphi))\,ds & t\geq -\theta, \\ 0 & t\leq -\theta. \end{cases}$$

Through this work, we investigate the τ-periodicity of solutions of Eq. (1.1) without considering the compactness condition on the semigroup $\{\mathcal{S}_0(t), t\geq 0\}$ generates by the part of $\mathcal{L}_0$ of $\mathcal{L}$ on $\overline{\mathscr{D}(\mathcal{L})}$. First, we discuss the τ-periodicity of Eq. (1.1) in the general case where $\mathcal{L}\in\mathcal{HY}(\mathbb{X})$ and the phase space $\mathscr{B}$ is a fading memory space. Second, in order to establish the τ-periodicity of solutions of Eq. (1.1) in the case when the semigroup $\{\mathcal{S}_0(t), t\geq 0\}$ is not necessarily exponentially stable, when the phase space is a fading memory space and also uniform fading memory space, we propose to treat the case where $\mathcal{L}$ is a sum of two operators, the first one, denoted by $\widetilde{\mathcal{L}}$, is an element of $\mathcal{HY}(\mathbb{X})$ and the second one, denoted by $\widehat{\mathcal{L}}$, is an element of $\mathscr{L}(\mathbb{X})$. Recall that the case where $\widehat{\mathcal{L}}=0$ has already treated in [12,22] and the τ-periodicity of the solution is obtained in the particular case where $\overline{\mathscr{D}(A)}=\mathbb{X}$. We give some appriori estimates on $\widehat{\mathcal{L}}$ to get that $I-\mathcal{M}(\tau)$ is a semi-Fredholm operator by using Theorem 1.2. This property allows us to prove, by using Theorem 1.1, that the Poincaré map has a fixed point, which yields the τ-periodic solution of Eq. (1.1). To achieve this objective, we propose in section 2, to introduce some useful estimations and some semi-Fredholm properties for the operator $I-\mathcal{M}(\tau)$ in general case. Moreover, in section 3, we present some semi-Fredholm properties for the operator $I-\mathcal{M}(\tau)$ in the special case of Eq. (1.2) where $\mathcal{L}=\widetilde{\mathcal{L}}+\widehat{\mathcal{L}}$ with $\widetilde{\mathcal{L}}\in\mathcal{HY}(\mathbb{X})$ and $\widehat{\mathcal{L}}\in\mathscr{L}(\mathbb{X})$. In section 4, we propose to examine the τ-periodicity of Eq. (1.1) in the global phase space $\mathscr{B}$. To achieve this objectives, we need some additional properties summarize as follows:

Let $\mathcal{C}_{00}:(-\infty,0]\to\mathbb{X}$ be the space of continuous function with compact supports. We assume that $\mathscr{B}$ verifies the following hypothesis:

(C) Let $(\psi_n)_{n\geq 0}\in\mathcal{C}_{00}$ be an uniformly bounded sequence, if $(\psi_n)_{n\geq 0}$ converges compactly to ψ on $(-\infty,0]$, then $\psi\in\mathscr{B}$ and $|\psi_n-\psi|_{\mathscr{B}}\to 0$.

For $\psi\in\mathscr{B}$ and $\theta\leq 0$, we define the linear operator $X(t), t\geq 0$ as:

$$[X(t)\psi](\theta) = \begin{cases} \psi(0), & t+\theta\geq 0, \\ \psi(t+\theta), & t+\theta\leq 0. \end{cases}$$

Then the semigroup $(X(t))_{t\geq 0}$ satisfies the following equation :

$$\begin{cases} \dot{w}(t)=0, \\ w_0=\psi. \end{cases}$$

Let $X_0(t)=X(t)/\mathscr{B}_0$ such that $\mathscr{B}_0:=\{\psi\in\mathscr{B}:\psi(0)=0\}$. Let $\mathcal{BC}$ be the space of all bounded continuous functions maps from $(-\infty,0]$ to $\mathbb{X}$, equipped with the supremum norm, then we introduce the following proposition.

Definition 1.2 $\mathscr{B}$ *is called a fading memory space if it satisfies the axioms* (A)*,* (C) *and* $X_0(t)\psi\to 0$ *as* $t\to+\infty$*, for all* $\psi\in\mathscr{B}_0$*.*

Proposition 1.2 *Assume that* $\mathscr{B}$ *is a fading memory space. then the space* $\mathcal{BC}$ *included in* $\mathscr{B}$ *and there is* $\beta>0$ *verifying* $|\psi|_{\mathscr{B}}\leq\beta|\psi|_{\mathcal{BC}}$*. Moreover*

$$|w_t|_{\mathscr{B}}\leq\beta\sup_{\varsigma\leq s\leq t}|w(s)|+(1+c\beta)\ |X_0(t-\varsigma)|\,|w_\varsigma|\quad for\quad \varsigma\geq 0,$$

for any function w *satisfying axiom* (A)*.*

Definition 1.3 *The space $\mathscr{B}$ is called a uniform fading memory space, if hypothesis (C) holds with $|X_0(t)| \to 0$ as $t \to +\infty$.*

Section 5 is devoted to control the value of the constant δ appears in Theorem 1.2. In addition, we propose to get the results given in section 4, in the special uniform fading memory space $\mathscr{B} = \mathcal{UC}_\gamma(\mathbb{X})$, $\gamma < 0$. Finally, we confirm, in section 6, the applicability of the obtained theoretical results by a mathematical model with some numerical simulations.

2. Several estimations and semi-Fredholm properties of the operator $I - \mathcal{M}(\zeta)$ in general case:

Before announcing theorems concern the τ-periodicity of Eq. (1.1), we need to introduce the following useful results.

Proposition 2.1 *Suppose that condition $(\mathbf{C_0})$ holds. If $\mathscr{B}$ is a fading memory space, then, for $t > 0$, the operator $\mathcal{R}(t)$ satisfies*

$$|\mathcal{R}(t)| \leq (\beta K_0 c + M)e^{\nu_0^+ t}\left(e^{\beta K_0 \widehat{K}|\mathcal{D}|t} - 1\right),$$

where $\nu_0^+ = \max\{\nu_0, 0\}$ and $M = (1 + c\beta)\sup_{t\geq 0}|X_0(t)|$.

Proof: For $t \geq 0$ and $t + \theta \geq 0$

$$(\mathcal{R}(t)\varphi)(\theta) = \lim_{\mu\to+\infty}\int_0^{t+\theta} \mathcal{S}_0(t+\theta-s)\mu(\mu I - \mathcal{L})^{-1}\mathcal{D}(w_s(\varphi))\, ds.$$

From the decomposition (1.3) in Theorem 1.5 and the fact that $\mathscr{B}$ is a fading memory space, one can see that

$$\begin{aligned}
|\mathcal{R}(t)\varphi|_{\mathscr{B}} &\leq \beta \sup_{0\leq s\leq t}\|(\mathcal{R}(s)\varphi)(0)\| \\
&\leq \beta K_0 \widehat{K}\sup_{0\leq s\leq t}\int_0^s e^{\nu_0(s-\xi)}\|\mathcal{D}(\mathcal{W}(\xi)(\varphi))\|\, d\xi \\
&\leq \beta K_0 \widehat{K}|\mathcal{D}|\sup_{0\leq s\leq t}\int_0^s e^{\nu_0(s-\xi)}\left(|\mathcal{M}(\xi)\varphi|_{\mathscr{B}} + |\mathcal{R}(\xi)\varphi|_{\mathscr{B}}\right) d\xi \\
&\leq \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{\nu_0^+(t-\xi)}\left(|\mathcal{M}(\xi)\varphi|_{\mathscr{B}} + |\mathcal{R}(\xi)\varphi|_{\mathscr{B}}\right) d\xi.
\end{aligned}$$

Then,

$$e^{-\nu_0^+ t}|\mathcal{R}(t)\varphi|_{\mathscr{B}} \leq \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{M}(\xi)\varphi|_{\mathscr{B}}\, d\xi + \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{R}(\xi)\varphi|_{\mathscr{B}}\, d\xi.$$

Applying Gronwall's inequality one has

$$\begin{aligned}
e^{-\nu_0^+ t}|\mathcal{R}(t)\varphi|_{\mathscr{B}} &\leq \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{M}(\xi)\varphi|_{\mathscr{B}}\, d\xi \\
&\quad + (\beta K_0 \widehat{K}|\mathcal{D}|)^2\int_0^t e^{(\beta K_0 \widehat{K}|\mathcal{D}|)(t-\xi)}\int_0^\xi e^{-\nu_0^+\sigma}|\mathcal{M}(\sigma)\varphi|_{\mathscr{B}}\, d\sigma\, d\xi \\
&\leq \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{M}(\xi)\varphi|_{\mathscr{B}}\, d\xi \\
&\quad + (\beta K_0 \widehat{K}|\mathcal{D}|)^2\int_0^t\int_\sigma^t e^{(\beta K_0 \widehat{K}|\mathcal{D}|)(t-\xi)}\, d\xi\, e^{-\nu_0^+\sigma}|\mathcal{M}(\sigma)\varphi|_{\mathscr{B}}\, d\sigma \\
&\leq \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{M}(\xi)\varphi|_{\mathscr{B}}\, d\xi + \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{\beta K_0 \widehat{K}|\mathcal{D}|(t-\sigma)}e^{-\nu_0^+\sigma}|\mathcal{M}(\sigma)\varphi|_{\mathscr{B}}\, d\sigma \\
&\quad - \beta K_0 \widehat{K}|\mathcal{D}|\int_0^t e^{-\nu_0^+\xi}|\mathcal{M}(\xi)\varphi|_{\mathscr{B}}\, d\xi.
\end{aligned}$$

Which implies that

$$|\mathcal{R}(t)\varphi|_{\mathscr{B}} \leq \beta K_0 \widehat{K}|\mathcal{D}| \int_0^t e^{(\beta K_0 \widehat{K}|\mathcal{D}|+\nu_0^+)(t-\sigma)}|\mathcal{M}(\sigma)\varphi|_{\mathscr{B}}\, d\sigma.$$

On the other hand, Since $\mathscr{B}$ is a fading memory space, one can see that

$$\begin{aligned} |\mathcal{M}(\sigma)\varphi|_{\mathscr{B}} &\leq \beta \sup_{0\leq s\leq \sigma} \|\mathcal{S}_0(s)\varphi(0)\| + M|\varphi|_{\mathscr{B}} \\ &\leq \beta K_0 c e^{\nu_0^+ \sigma}|\varphi|_{\mathscr{B}} + M|\varphi|_{\mathscr{B}}. \end{aligned}$$

Then,

$$|\mathcal{M}(\sigma)\varphi|_{\mathscr{B}} \leq (\beta K_0 c + M)e^{\nu_0^+\sigma}|\varphi|_{\mathscr{B}}.$$

Then, it follows that

$$|\mathcal{R}(t)\varphi|_{\mathscr{B}} \leq \beta K_0 \widehat{K}|\mathcal{D}|(\beta K_0 c + M)e^{(\beta K_0 \widehat{K}|\mathcal{D}|+\nu_0^+)\zeta} \int_0^t e^{-\beta K_0 \widehat{K}|\mathcal{D}|\sigma}\, d\sigma\ |\varphi|_{\mathscr{B}}.$$

consequently,

$$|\mathcal{R}(t)\varphi|_{\mathscr{B}} \leq (\beta K_0 c + M)e^{\nu_0^+ t}\left(e^{\beta K_0 \widehat{K}|\mathcal{D}|t} - 1\right)|\varphi|_{\mathscr{B}}.$$

which complete the proof of our Proposition. □

Theorem 2.1 *Suppose that condition* $(\mathbf{C_0})$ *holds. Let* $\mathscr{B}$ *be a fading memory space. Let* $\zeta > 0$ *and* n *be a positive integer with* $\dim\ker(I - \mathcal{S}_0(\zeta)) = n$ *and there is a constant* $\delta > 0$ *such that*

$$|\overline{\varphi}| \leq \delta\,|(I - \mathcal{M}(\zeta))\varphi|_{\mathscr{B}} \qquad for \quad \varphi \in \mathscr{B}_0.$$

If the operator $\mathcal{D}$ *satisfies the following estimate*

$$|\mathcal{D}| < \frac{1}{\beta K_0 \widehat{K}\zeta} \ln\left(1 + \frac{e^{-\nu_0^+\zeta}}{2\delta(1+\sqrt{n})(\beta K_0 c + M)}\right).$$

Then

$$I - \mathcal{W}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0) \quad and \quad \dim\ker(I - \mathcal{W}(\zeta)) \leq n.$$

To prove the above Theorem, we need the following proposition taken from [22].

Proposition 2.2 *Suppose that condition* $(\mathbf{C_0})$ *hold. If the phase space* $\mathscr{B}$ *satisfies axioms* (A) *and* (C)*. Then, for* $\zeta > 0$,

$$dim\ \ker(I - \mathcal{M}(\zeta)) = dim\ \ker(I - \mathcal{S}_0(\zeta)).$$

Proof of Theorem 2.1. The proof follows immediately from Theorem 1.2, Proposition 2.1 and Proposition 2.2. □

Now, to control the value of the positive constant δ appear in the previous Theorem, we need to introduce one proposition proved in [22] in the densely case $\overline{\mathscr{D}(\mathcal{L})} = \mathbb{X}$. Notice that the same proposition hold true in the non densely case $\overline{\mathscr{D}(\mathcal{L})} \neq \mathbb{X}$ and the proof is omitted here. Suppose that $\mathscr{B}$ verifies axioms (A) and (C). If $I - \mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$, then from Theorem 1.2, $\overline{\mathscr{D}(\mathcal{L})}$ can be decomposed as

$$\overline{\mathscr{D}(\mathcal{L})} = \ker(I - \mathcal{S}_0(\zeta)) \oplus T(\nu).$$

such that $T(\nu)$ is a closed subset of $\overline{\mathscr{D}(\mathcal{L})}$. We denote by $S_{T(\nu)}$ the restriction of $I-\mathcal{S}_0(\zeta)$ to $T(\nu)$ given by

$$S_{T(\nu)}: T(\nu) \to Im(I-\mathcal{S}_0(\zeta)),$$

then, $S_{T(\nu)}$ is continuous and bijective linear operator satisfying $Im(S_{T(\nu)}) = Im(I-\mathcal{S}_0(\zeta))$.
Consider $S^{-1}_{T(\nu)}$ the inverse operator of $S_{T(\nu)}$, then $S^{-1}_{T(\nu)}$ is continuous, if $\varphi \in \mathscr{B}_0$ such that $\varphi(0) \in Im(I-\mathcal{S}_0(\zeta))$, then $S^{-1}_{T(\nu)}\varphi(0)$ is well defined and we get

$$(I-\mathcal{S}_0(\zeta))S^{-1}_{T(\nu)}\varphi(0) = \varphi(0). \tag{2.1}$$

Moreover, let $\mathcal{P}\varphi$ be the $\mathscr{B}_0$-valued function given by:

$$(\mathcal{P}\varphi)(\theta) = \sum_{j=0}^{k-1}\varphi(\theta+j\zeta) + \mathcal{S}_0(\theta+k\zeta)S^{-1}_{T(\nu)}\varphi(0), \quad \theta \in I_k = [-k\zeta, -(k-1)\zeta);\ k \geq 1$$

and

$$\mathscr{D}(\mathcal{P}) = \{\varphi \in \mathscr{B}_0 : \varphi(0) \in Im(I-\mathcal{S}_0(\zeta))\}.$$

Then, we have the following proposition.

Proposition 2.3 *Assume that $I-\mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$. Let δ be a positive constant such that*

$$|\mathcal{P}\varphi|_{\mathscr{B}} \leq \delta|\varphi|_{\mathscr{B}} \qquad for \quad \mathcal{P}\varphi \in \mathscr{B}_0,$$

then,

$$|\overline{\psi}| \leq \delta|(I-\mathcal{M}(\zeta)\psi|_{\mathscr{B}} \ \ for \ \ \psi \in \mathscr{B}_0,$$

equivalently the subspace $Im(I-\mathcal{M}(\zeta))$ is closed.

3. Semi-Fredholm properties for the operator $I-\mathcal{M}(\zeta)$ in the case where $\mathcal{L} = \widetilde{\mathcal{L}} + \widehat{\mathcal{L}}$ with $\widetilde{\mathcal{L}} \in \mathcal{HY}(\mathbb{X})$ and $\widehat{\mathcal{L}} \in \mathscr{L}(\mathbb{X})$:

Firstly, in order to introduce a sufficient conditions which guarantee that $I-\mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$, we consider the following differential equation:

$$\begin{cases} \dot{v}(t) = \mathcal{L}v(t), \\ v(0) = x. \end{cases} \tag{3.1}$$

such that $\mathcal{L} = \widetilde{\mathcal{L}} + \widehat{\mathcal{L}}$ and suppose that $\widetilde{\mathcal{L}}$ verifies the following Hille-Yosida condition:
$(\mathbf{C_0'})$ There are constants $\widetilde{K} \geq 1$ and $\widetilde{\nu} \in \mathbb{R}$ such that $\rho(\mathcal{L}) \supset (\widetilde{\nu}, +\infty)$ and

$$|(\eta I - \mathcal{L})^{-n}| \quad \leq \frac{\widetilde{K}}{(\eta-\widetilde{\nu})^n} \quad \text{for} \quad \eta > \widetilde{\nu} \quad \text{and} \quad n \in \mathbb{N},$$

and $\widehat{\mathcal{L}} \in \mathscr{L}(\mathbb{X})$. Moreover, let $\widetilde{\mathcal{L}}_0$ be the part of $\widetilde{\mathcal{L}}$ on $\overline{\mathscr{D}(\mathcal{L})}$. Then, $\widetilde{\mathcal{L}}_0$ generates a C_0-semigroup $\{\mathcal{T}_0(t), t \geq 0\}$ on $\overline{\mathscr{D}(\mathcal{L})}$. If $x \in \overline{\mathscr{D}(\mathcal{L})}$, then Eq. (3.1) has a unique solution $v(t,x)$ satisfying:

$$v(t,x) = \mathcal{T}_0(t)x + \lim_{\mu\to+\infty}\int_0^t \mathcal{T}_0(t-s)\mu(\mu I - \widetilde{\mathcal{L}})^{-1}\widehat{\mathcal{L}}v(s,x)\,ds. \tag{3.2}$$

Proposition 3.1 *Suppose that condition $(\mathbf{C_0'})$ holds. Let $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ for $t \geq 0$, $\widetilde{K}_0 \geq 1$ and $\widetilde{\nu}_0 > 0$, then, the operator $\mathcal{Q}(t)$ defined on $\overline{\mathscr{D}(\mathcal{L})}$ by*

$$\mathcal{Q}(t)x = \lim_{\mu\to+\infty}\int_0^t \mathcal{T}_0(t-s)\mu(\mu I - \widetilde{\mathcal{L}})^{-1}\widehat{\mathcal{L}}v(s,x)\,ds,$$

satisfies

$$|\mathcal{Q}(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0\zeta}\left(e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|t} - 1\right) \quad for \ \ t > 0.$$

Proof: Let $\zeta \geq 0$. Then,

$$\|\mathcal{Q}(t)x\| \leq \widetilde{K}\widetilde{K}_0|\widehat{\mathcal{L}}| \int_0^t e^{-\widetilde{\nu}_0(t-s)}\|v(s,x)\|\,ds.$$

Since

$$\|v(s,x)\| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 s}\|x\| + \widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}| \int_0^s e^{-\widetilde{\nu}_0(s-\sigma)}\|v(\sigma,x)\|\,d\sigma.$$

and

$$e^{\widetilde{\nu}_0 s}\|v(s,x)\| \leq \widetilde{K}_0\|x\| + \widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}| \int_0^s e^{\widetilde{\nu}_0\sigma}\|v(\sigma,x)\|\,d\sigma.$$

Applying Gronwall's inequality, we obtain that

$$e^{\widetilde{\nu}_0 s}\|v(s,x)\| \leq \widetilde{K}_0 e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|s}\|x\|,$$

hence,

$$\|v(s,x)\| \leq \widetilde{K}_0 e^{(\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|-\widetilde{\nu}_0)s}\|x\|.$$

Then,

$$\begin{aligned}\|\mathcal{Q}(t)x\| &\leq \widetilde{K}_0^2\widetilde{K}|\widehat{\mathcal{L}}|e^{-\widetilde{\nu}_0 t}\int_0^t e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|s}\,ds\ \|x\| \\ &\leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}\left(e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|t}-1\right)\|x\|,\end{aligned}$$

Which prove the Proposition. □

Lemma 3.1 *Suppose that* $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \geq 0$, $\widetilde{K}_0 \geq 1$ *and* $\widetilde{\nu}_0 > 0$. *Then, for* $\zeta > 0$,

$$I - \mathcal{T}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})}) \quad and \quad \dim\ker(I - \mathcal{T}_0(\zeta)) = 0.$$

Proof: Let us introduce the following new norm on $\mathbb{X}$ given by

$$|z|_{\widetilde{\nu}_0} = \sup_{t\geq 0}\|e^{\widetilde{\nu}_0 t}\mathcal{T}_0(t)z\|,$$

Then, we have that $\|z\| \leq |z|_{\widetilde{\nu}_0} \leq \widetilde{K}_0\|z\|$, which implies that the norms $\|.\|$ and $|.|_{\widetilde{\nu}_0}$ are equivalent. Moreover, for every $z \in \overline{\mathscr{D}(\mathcal{L})}$ and $\zeta \geq 0$, one has

$$\begin{aligned}|\mathcal{T}_0(\zeta)z|_{\widetilde{\nu}_0} &= \sup_{s\geq 0}\|e^{\widetilde{\nu}_0 s}\mathcal{T}_0(s+\zeta)z\| = e^{-\widetilde{\nu}_0\zeta}\sup_{s\geq 0}\|e^{\widetilde{\nu}_0(s+\zeta)}\mathcal{T}_0(s+\zeta)z\| \\ &= e^{-\widetilde{\nu}_0\zeta}\sup_{\sigma\geq\zeta}\|e^{\widetilde{\nu}_0\sigma}\mathcal{T}_0(\sigma)z\| \leq e^{-\widetilde{\nu}_0\zeta}\sup_{\sigma\geq 0}\|e^{\widetilde{\nu}_0\sigma}\mathcal{T}_0(\sigma)z\| \leq e^{-\widetilde{\nu}_0\zeta}|z|_{\widetilde{\nu}_0},\end{aligned}$$

then $I - \mathcal{T}_0(\zeta)$ is invertible. Which means that $I - \mathcal{T}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$ and $\dim\ker(I - \mathcal{T}_0(\zeta))$. □

Proposition 3.2 *Suppose that condition* $(\mathbf{C_0'})$ *holds. Let* $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \geq 0$, $\widetilde{K}_0 \geq 1$ *and* $\widetilde{\nu}_0 > 0$. *Let* $\zeta > 0$. *If the operator* $\widehat{\mathcal{L}}$ *verifies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0\widetilde{K}\zeta}\ln\left(1 + \frac{e^{\widetilde{\nu}_0\zeta}}{2\delta_0\widetilde{K}_0}\right), \tag{3.3}$$

where $\delta_0 > 0$ *be such that* $|(I - \mathcal{T}_0(\zeta))^{-1}v|_{\widetilde{\nu}_0} \leq \delta_0|v|_{\widetilde{\nu}_0}$ *for* $v \in \overline{\mathscr{D}(\mathcal{L})}$. *Then,*

$$I - \mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})}) \quad and \quad \dim\ker(I - \mathcal{S}_0(\zeta)) = 0.$$

Proof: Formula (3.2) implies that for $\zeta > 0$,

$$I - \mathcal{S}_0(\zeta) = I - \mathcal{T}_0(\zeta) - \mathcal{Q}(\zeta). \tag{3.4}$$

From Lemma 3.1, we get that $I - \mathcal{T}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$. On the other hand, the estimation (3.3) implies that

$$\widetilde{K}_0 e^{-\widetilde{\nu}_0 \zeta}\left(e^{\widetilde{K}_0 \widetilde{K} |\widehat{\mathcal{L}}| \zeta} - 1\right) < \frac{1}{2\delta_0},$$

and consequently, Proposition 3.1 gives that

$$|\mathcal{Q}(\zeta)| < \frac{1}{2\delta_0}.$$

Finally, by applying Theorem 1.2 to the formula (3.4), we get that $I - \mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$ and $\dim\ker(I - \mathcal{S}_0(\zeta)) = \dim\ker(I - \mathcal{T}_0(\zeta)) = 0$. □

Proposition 3.3 *Suppose that condition* $(\mathbf{C_0'})$ *holds. Let* $|\mathcal{T}_0(t)| \le \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \ge 0$, $\widetilde{K}_0 \ge 1$ *and* $\widetilde{\nu}_0 > 0$. *Let* $\zeta > 0$. *If the operator* $\widehat{\mathcal{L}}$ *satisfies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0 \widetilde{K} \zeta} \ln\left(1 + \frac{e^{\widetilde{\nu}_0 \zeta} - 1}{2\widetilde{K}_0}\right). \tag{3.5}$$

Then

$$I - \mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})}) \quad \textit{and} \quad \dim\ker(I - \mathcal{S}_0(\zeta)) = 0.$$

Proof: It suffices to compute the value δ_0 in the estimation (3.3). To do this, let $v \in \overline{\mathscr{D}(A)}$, then

$$\begin{aligned} |v|_{\widetilde{\nu}_0} &\le |v - \mathcal{T}_0(\zeta)v|_{\widetilde{\nu}_0} + |\mathcal{T}_0(\zeta)v|_{\widetilde{\nu}_0} \\ &\le |v - \mathcal{T}_0(\zeta)v|_{\widetilde{\nu}_0} + e^{-\widetilde{\nu}_0 \zeta}|v|_{\widetilde{\nu}_0}. \end{aligned}$$

Then,

$$(1 - e^{-\widetilde{\nu}_0 \zeta})|v|_{\widetilde{\nu}_0} \le |v - \mathcal{T}_0(\zeta)v|_{\widetilde{\nu}_0} \quad \text{for all} \quad v \in \overline{\mathscr{D}(\mathcal{L})}$$

which implies that

$$|v|_{\widetilde{\nu}_0} \le \frac{1}{1 - e^{-\widetilde{\nu}_0 \zeta}} |v - \mathcal{T}_0(\zeta)v|_{\widetilde{\nu}_0}.$$

and hence,

$$|(I - \mathcal{T}_0(\zeta))^{-1} v|_{\widetilde{\nu}_0} \le \frac{1}{1 - e^{-\widetilde{\nu}_0 \zeta}} |v|_{\widetilde{\nu}_0}.$$

Then, we choice the value of δ_0 such that

$$\delta_0 \le \frac{1}{1 - e^{-\widetilde{\nu}_0 \zeta}}.$$

Finally, from estimation (3.5), we deduce that the estimation (3.3) is satisfy. Then it follows from Proposition 3.2 that $I - \mathcal{S}_0(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$. □

Proposition 3.4 *Suppose that condition* $(\mathbf{C_0'})$ *holds. Let* $|\mathcal{T}_0(t)| \le \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \ge 0$, $\widetilde{K}_0 \ge 1$ *and* $\widetilde{\nu}_0 > 0$. *Let* $\zeta > 0$. *If the operator* $\widehat{\mathcal{L}}$ *satisfies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0 \widetilde{K} \zeta} \ln\left(1 + \frac{e^{\widetilde{\nu}_0 \zeta} - 1}{2\widetilde{K}_0}\right)$$

and if $Im(I - \mathcal{M}(\zeta))$ *is closed. Then,*

$$I - \mathcal{M}(\zeta) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})}) \quad \textit{and} \quad \dim\ker(I - \mathcal{M}(\zeta)) = 0.$$

Proof: The proof follows immediately from Proposition 2.2 and Proposition 3.3. □

Now, the solution $w(t,\varphi)$ of Eq. (1.2) is given by the following formula:

$$w(t,\varphi) = \mathcal{T}_0(t)\varphi(0) + \lim_{\mu\to+\infty}\int_0^t \mathcal{T}_0(t-s)\mu(\mu I - \widetilde{\mathcal{L}})^{-1}\widehat{\mathcal{L}}w(s,\varphi)\,ds \qquad for \quad t\geq 0. \tag{3.6}$$

Then, the operator $(\mathcal{M}(t))_{t\geq 0}$ is decomposed on $\mathscr{B}_0$ as follows:

$$\mathcal{M}(t) = \mathcal{Y}(t) + \mathcal{Z}(t) \quad \text{for} \quad t\geq 0, \tag{3.7}$$

where $\{\mathcal{Y}(t), t\geq 0\}$ is the strongly continuous semigroup on $\mathscr{B}_0$ defined by

$$(\mathcal{Y}(t)\phi)(\theta) = \begin{cases} \mathcal{T}_0(t+\theta)\phi(0) & if \quad t\geq -\theta, \\ \phi(t+\theta) & if \quad t\leq -\theta, \end{cases}$$

and the operator $\mathcal{Z}(t), t\geq 0$ is defined on $\mathscr{B}_0$, by

$$(\mathcal{Z}(t)\varphi)(\theta) = \begin{cases} \lim_{\mu\to+\infty}\int_0^{t+\theta} \mathcal{T}_0(t+\theta-s)\mu(\mu I - \widetilde{\mathcal{L}})^{-1}\widehat{\mathcal{L}}w(s,\varphi)\,ds & for \quad t+\theta\geq 0, \\ 0 & for \quad t+\theta\leq 0. \end{cases}$$

Proposition 3.5 *Suppose that* $(\mathbf{C_0'})$ *holds. Let* $\mathscr{B}$ *be a fading memory space. If* $|\mathcal{T}_0(t)|\leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t\geq 0$, $\widetilde{K}_0\geq 1$ *and* $\widetilde{\nu}_0 > 0$. *Then, the operator* $\mathcal{Z}(t)$ *satisfies*

$$|\mathcal{Z}(t)\varphi|_{\mathscr{B}} \leq \beta\widetilde{K}_0 c\left(e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|t} - 1\right)|\varphi|_{\mathscr{B}}. \tag{3.8}$$

Proof: Let $t+\theta\geq 0$, Since

$$\begin{aligned} \|(\mathcal{Z}(t)\varphi)(\theta)\| &\leq \widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|\int_0^{t+\theta} e^{-\widetilde{\nu}_0(t+\theta-s)}\|w(s,\varphi)\|\,ds \\ &\leq \widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|\int_0^{t} e^{\widetilde{\nu}_0 s}\|w(s,\varphi)\|\,ds. \end{aligned}$$

which implies that $\mathcal{Z}(t)\varphi\in\mathcal{BC}$ and from axiom (C) one has

$$|\mathcal{Z}(t)\varphi|_{\mathscr{B}} \leq \beta|\mathcal{Z}(t)\varphi|_{\mathcal{BC}} \leq \beta\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|\int_0^t e^{\widetilde{\nu}_0 s}\|w(s,\varphi)\|\,ds.$$

On the other hand, from the variation of constant formula (3.6), we obtain that

$$\|w(s,\varphi)\| \leq \widetilde{K}_0 c e^{-\widetilde{\nu}_0 s}|\varphi|_{\mathscr{B}} + \widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|\int_0^s e^{-\widetilde{\nu}_0(s-\sigma)}\|w(\sigma,\varphi)\|\,d\sigma.$$

From Gronwall's inequality, we obtain that

$$e^{\widetilde{\nu}_0 s}\|w(s,\varphi)\| \leq \widetilde{K}_0 c e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|s}|\varphi|_{\mathscr{B}},$$

then

$$\begin{aligned} |\mathcal{Z}(t)\varphi|_{\mathscr{B}} &\leq \beta c\widetilde{K}_0^2\widetilde{K}|\widehat{\mathcal{L}}|\int_0^t e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|s}\,ds\,|\varphi|_{\mathscr{B}} \\ &\leq \beta\widetilde{K}_0 c\left(e^{\widetilde{K}_0\widetilde{K}|\widehat{\mathcal{L}}|t} - 1\right)|\varphi|_{\mathscr{B}}. \end{aligned}$$

□

Proposition 3.6 *Let $\mathscr{B}$ be a uniform fading memory space. Suppose that $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ for $t \geq 0$, $\widetilde{K}_0 \geq 1$ and $\widetilde{\nu}_0 > 0$. Then, $\{\mathcal{Y}(t), t \geq 0\}$ is exponentially stable, which means that is there exists $\widetilde{N}_0 \geq 1$ and $\widetilde{\eta}_0 > 0$ such that $|\mathcal{Y}(t)| \leq \widetilde{N}_0 e^{-\widetilde{\eta}_0 t}$ for $t \geq 0$.*

To prove this proposition, we need the following lemma.

Lemma 3.2 *[22] Let ξ be the set of continuous functions $x : (-\infty, a) \to \mathbb{X}, 0 < a \leq +\infty$, such that $x_0 \in \mathcal{B}$ and x is continuous on $[0, a)$. If ξ_0 and $\xi_{|[0,t]}$ are bounded in $\mathcal{B}$ and $\mathcal{C}[0, t]$, respectively. Then,*

(i) $H^{-1}\alpha(\xi(t)) \leq \alpha(\xi_t) \leq K(t)\alpha(\xi_{|[0,t]}) + M(t)\alpha(\xi_0)$.

(ii) If the phase space $\mathcal{B}$ satisfies the axiom (C), then

$$\alpha(\xi_t) \leq \beta\alpha(\xi_{|[0,t]}) + (1 + c\beta)\alpha(W_0(t))\alpha(\xi_0).$$

Proof of proposition 3.6

Let Ω be a bounded set in $\mathcal{B}$. Then for $t \geq 0$, it follows from lemma 3.2, that

$$\alpha(\mathcal{V}_1(t)\Omega) \leq \beta\alpha\left(T_1(.)\Omega(0)_{|[0,t]}\right) + (1 + c\beta)|W_0(t)|_\alpha\ \alpha(\Omega),$$

where

$$\Omega(0) = \{\varphi(0) : \varphi \in \Omega\}.$$

Since $\mathcal{V}_1(t)\Omega = \mathcal{V}_1(\frac{t}{2})\left(\mathcal{V}_1(\frac{t}{2})\Omega\right)$, then

$$\alpha(\mathcal{V}_1(t)\Omega) \leq \beta\alpha\left(T_1(.)(\mathcal{V}_1(\frac{t}{2})\Omega)(0)_{|[0,\frac{t}{2}]}\right) + (1 + c\beta)|W_0(\frac{t}{2})|_\alpha\ \alpha((\mathcal{V}_1(\frac{t}{2})\Omega).$$

By the translation property of $(\mathcal{V}_1(t))_{t\geq 0}$, we get that

$$\begin{aligned}\alpha(\mathcal{V}_1(t)\Omega) &\leq \beta\alpha\left(T_1(.)T_1(\frac{t}{2})\Omega(0)_{|[0,\frac{t}{2}]}\right)\\ &\quad +(1 + c\beta)|W_0(\frac{t}{2})|_\alpha\ \left(\beta\alpha\left(T_1(.)\Omega(0)_{|[0,\frac{t}{2}]}\right) + (1 + c\beta)|W_0(\frac{t}{2})|_\alpha\ \alpha(\Omega)\right)\\ &\leq \beta \sup_{\frac{t}{2}\leq s\leq t} |T_1(s)|\alpha(\Omega(0)) + \beta(1 + c\beta)\alpha\left(T_1(.)\Omega(0)_{|[0,\frac{t}{2}]}\right)|W_0(\frac{t}{2})|_\alpha\\ &\quad +(1 + c\beta)^2|(W_0(\frac{t}{2})|^2_\alpha\ \alpha(\Omega).\end{aligned}$$

From the axiom (A)-ii), we get $\alpha(\Omega(0)) \leq c\alpha(\Omega)$, which implies that

$$\begin{aligned}\alpha(\mathcal{V}_1(t)\Omega) &\leq c\beta \sup_{\frac{t}{2}\leq s\leq t} |T_1(s)|\alpha(\Omega) + c\beta(1 + c\beta) \sup_{0\leq s\leq \frac{t}{2}} |T_1(s)||W_0(\frac{t}{2})|_\alpha\ \alpha(\Omega)\\ &\quad +(1 + c\beta)^2|W_0(\frac{t}{2})|^2_\alpha\ \alpha(\Omega),\end{aligned}$$

and consequently,

$$\alpha(\mathcal{V}_1(t)\Omega) \leq \left(c\beta\widetilde{M}e^{-\frac{\widetilde{\omega}t}{2}} + c\beta(1 + c\beta)\widetilde{M}|W_0(\frac{t}{2})| + (1 + c\beta)^2|W_0(\frac{t}{2})|^2\right)\alpha(\Omega).$$

Moreover, since $\mathcal{B}$ is a uniform fading memory space, we have $|W_0(t)| \to 0$ as $t \to +\infty$, which implies that

$$\lim_{t\to+\infty} \frac{1}{t} \log |W_0(t)| < 0,$$

and hence

$$w_0(W_0) < 0,$$

consequently, there exist positive constants $N \geq 1$ and $\eta > 0$ such that

$$|W_0(t)| \leq Ne^{-\eta t} \quad for\ all \quad t \geq 0.$$

We deduce that

$$\begin{aligned} \alpha(\mathcal{V}_1(t)\Omega) &\leq \left(\beta\widetilde{M}e^{-\frac{\widetilde{\omega}t}{2}} + c\beta(1+c\beta)\widetilde{M}Ne^{-\frac{\eta t}{2}} + (1+c\beta)^2N^2e^{-\eta t}\right)\alpha(\Omega) \\ &\leq e^{-\min(\widetilde{\omega},\eta)\frac{t}{2}}\left(c\beta\widetilde{M} + c\beta(1+c\beta)\widetilde{M}N + (1+c\beta)^2N^2\right)\alpha(\Omega), \end{aligned}$$

which implies that

$$\omega_{ess}(\mathcal{V}_1) = \lim_{t\to+\infty}\frac{1}{t}\log\alpha(\mathcal{V}_1(t)) \leq -\frac{1}{2}\min(\widetilde{\omega},\eta).$$

□

Let us introduce the following new norm on $\mathscr{B}_0$ given by

$$|\psi|_{\widetilde{\eta}_0} = \sup_{t\geq 0}|e^{\widetilde{\eta}_0 t}\mathcal{Y}(t)\psi|_{\mathscr{B}} \quad \text{with} \quad \widetilde{\eta}_0 = \frac{1}{2}\min(\widetilde{\nu}_0,\eta),$$

Then, we have that $|\psi|_{\mathscr{B}} \leq |\psi|_{\widetilde{\eta}_0} \leq \widetilde{N}_0|\psi|_{\mathscr{B}}$, which implies that the norms $|.|_{\mathscr{B}}$ and $|.|_{\widetilde{\eta}_0}$ are equivalent. Moreover, for every $\psi \in \mathscr{B}_0$ and $\zeta \geq 0$, one has

$$\begin{aligned} |\mathcal{Y}(\zeta)\psi|_{\widetilde{\eta}_0} &= \sup_{s\geq 0}|e^{\widetilde{\eta}_0 s}\mathcal{Y}(s+\zeta)\psi|_{\mathscr{B}} = e^{-\widetilde{\eta}_0\zeta}\sup_{s\geq 0}|e^{\widetilde{\eta}_0(s+\zeta)}\mathcal{Y}(s+\zeta)\psi|_{\mathscr{B}} \\ &= e^{-\widetilde{\eta}_0\zeta}\sup_{\sigma\geq\zeta}|e^{\widetilde{\eta}_0\sigma}\mathcal{Y}(\sigma)\psi|_{\mathscr{B}} \leq e^{-\widetilde{\eta}_0\zeta}\sup_{\sigma\geq 0}|e^{\widetilde{\eta}_0\sigma}\mathcal{Y}(\sigma)\psi|_{\mathscr{B}} \leq e^{-\widetilde{\eta}_0\zeta}|\psi|_{\widetilde{\eta}_0}, \end{aligned}$$

then $I - \mathcal{Y}(\zeta)$ is invertible and we have the following lemma:

Lemma 3.3 *Suppose that $|\mathcal{T}_0(t)| \leq \widetilde{K}_0e^{-\widetilde{\nu}_0 t}$ for $t \geq 0$, $\widetilde{K}_0 \geq 1$ and $\widetilde{\nu}_0 > 0$. Then, for $\zeta > 0$,*

$$I - \mathcal{Y}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0) \quad and \quad \dim\ker(I - \mathcal{Y}(\zeta)) = 0.$$

Proposition 3.7 *Suppose that condition $(\mathbf{C_0'})$ holds. Let $\mathscr{B}$ be a uniform fading memory space. Assume that $|\mathcal{T}_0(t)| \leq \widetilde{K}_0e^{-\widetilde{\nu}_0 t}$ for $t \geq 0$, $\widetilde{K}_0 \geq 1$ and $\widetilde{\nu}_0 > 0$. For $\zeta > 0$, if $\widehat{\mathcal{L}}$ verifies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0\widetilde{K}\zeta}\ln\left(1 + \frac{1}{2\delta_1\beta\widetilde{K}_0 c}\right), \tag{3.9}$$

where $\delta_1 > 0$ be such that $|(I - \mathcal{Y}(\zeta))^{-1}\varphi|_{\widetilde{\eta}_0} \leq \delta_1|\varphi|_{\widetilde{\eta}_0}$ for $\varphi \in \mathscr{B}_0$. Then,

$$I - \mathcal{M}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0) \quad and \quad \dim\ker(I - \mathcal{M}(\zeta)) = 0.$$

Proof: From the decomposition (3.7), it follows that

$$I - \mathcal{M}(\zeta) = I - \mathcal{Y}(\zeta) - \mathcal{Z}(\zeta).$$

Moreover, Lemma 3.3 implies that $I - \mathcal{Y}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0)$. The objective now is to apply Theorem 1.2. Then, to show that $I - \mathcal{M}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0)$, it suffice to get the following estimation

$$|\mathcal{Z}(\zeta)| < \frac{1}{2\delta_1}.$$

Clearly, the above inequality is satisfied by considering inequalities (3.8) and (3.9) and hence $I - \mathcal{M}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0)$. Moreover, Theorem 1.2 and Lemma 3.3 imply that

$$\dim\ker(I - \mathcal{M}(\zeta)) = \dim\ker(I - \mathcal{Y}(\zeta)) = 0.$$

Which complete the proof of the Proposition. □

Applying the same approach used in the proof of Proposition 3.3, one can choice the value of δ_1 such that

$$\delta_1 \leq \frac{1}{1 - e^{-\frac{1}{2}\min(\widetilde{\nu}_0, \eta)\zeta}}.$$

Then, from Proposition 3.7, we get the following result.

Proposition 3.8 *Suppose that condition* $(\mathbf{C_0'})$ *holds. Let* $\mathscr{B}$ *be a uniform fading memory space. Assume that* $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \geq 0$, $\widetilde{K}_0 \geq 1$ *and* $\widetilde{\nu}_0 > 0$. *For* $\zeta > 0$, *if* $\widehat{\mathcal{L}}$ *verefies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0 \widetilde{K} \zeta} \ln \left(1 + \frac{1 - e^{-\frac{1}{2}\min(\widetilde{\nu}_0, \eta)\zeta}}{2\beta \widetilde{K}_0 c} \right).$$

Then,

$$I - \mathcal{M}(\zeta) \in \mathcal{SF}_+(\mathscr{B}_0) \quad \text{and} \quad \dim\ker(I - \mathcal{M}(\zeta)) = 0.$$

4. The τ-periodicity of solutions for Eq. (1.1) in the general phase space $\mathscr{B}$:

Now, we say that Eq. (1.1) verifies property $(\mathscr{BP})$ if the following equivalence holds:

there exist a τ-periodic solution of Eq. (1.1) if and only if it has a bounded ones on the positive real half-line.

Now, to discuss the periodicity of solutions of Eq. (1.1), we need to suppose that:

$(\mathbf{C_1})$ $\mathcal{H}$ is τ-periodic.

We establish the following first result binding between the boundedness of solution on the positive real half-line and the τ-periodicity of solutions for Eq. (1.1) .

Theorem 4.1 *Suppose that* $(\mathbf{C_0})$ *and* $(\mathbf{C_1})$ *hold. Let* $\mathscr{B}$ *be a fading memory space. Assume that* $\dim\ker(I - \mathcal{S}_0(\tau)) = n$ *and* $Im(I - \mathcal{M}(\tau))$ *is closed, which means that there is* $\delta > 0$ *such that*

$$|\overline{\varphi}| \leq \delta\, |(I - \mathcal{M}(\tau))\varphi|_{\mathscr{B}} \qquad \text{for all} \quad \varphi \in \mathscr{B}_0.$$

If the operator $\mathcal{D}$ *satisfies*

$$|\mathcal{D}| < \frac{1}{\beta K_0 \widehat{K} \tau} \ln \left(1 + \frac{e^{-\nu_0^+ \tau}}{2\delta(1 + \sqrt{n})(\beta K_0 c + M)} \right). \tag{4.1}$$

Then, Eq. (1.1) verifies the property $(\mathscr{BP})$.

Proof: It's enough to prove that the Poincaré map P_τ defined by $P_\tau \varphi = w_\tau(., \varphi, \mathcal{H})$ on $\mathscr{B}_0$ has a fixed point, where $w_\tau(., \varphi, \mathcal{H})$ is the integral solution of Eq. (1.1).
From the uniqueness property of the solution, the Poincaré map is decomposed as

$$P_\tau \varphi = w_\tau(., \varphi, 0) + w_\tau(., 0, \mathcal{H}),$$

where $w(., \varphi, 0)$ denotes the integral solution of Eq. (1.1) such that $\mathcal{H} = 0$ and $w(., 0, \mathcal{H})$ denotes the integral solution of Eq. (1.1) such that $\varphi = 0$, then, P_τ is given by $P_\tau \varphi = \widehat{P}_\tau \varphi + \phi$, where $\widehat{P}_\tau \varphi = w_\tau(., \varphi, 0)$ and $\phi = w_\tau(., 0, \mathcal{H})$. By decomposition (1.3) given in Theorem 1.5, $\widehat{P}_\tau$ is given by $\widehat{P}_\tau = \mathcal{W}(\tau) = \mathcal{M}(\tau) + \mathcal{R}(\tau)$. Furthermore, from estimation (4.1), Proposition 2.1 implies that $I - \widehat{P} \in \mathcal{SF}_+(\mathscr{B}_0)$. Let $w(., \varphi, \mathcal{H})$ be a bounded integral solution of Eq. (1.1) . Then,

$$\{P_\tau^n \varphi, n \in \mathbb{N}\} = \{w_{n\tau}(., \varphi, \mathcal{H}), n \in \mathbb{N}\},$$

which implies that $(P_\tau^n\varphi)_{n\geq 0}$ is bounded in $\mathscr{B}_0$. From Theorem 1.1, we conclude that $\mathcal{F}_{P_\tau} \neq \varnothing$, which implies that Eq. (1.1) verifies the property $(\mathscr{B}\mathscr{P})$. □

In addition, we obtain the following theorem if $\mathcal{L} = \widetilde{\mathcal{L}} + \widehat{\mathcal{L}}$ with $\widetilde{\mathcal{L}} \in \mathcal{HY}(\mathbb{X})$ and $\widehat{\mathcal{L}} \in \mathscr{L}(\mathbb{X})$.

Theorem 4.2 *Suppose that conditions* $(\mathbf{C_0'})$ *and* $(\mathbf{C_1})$ *hold. Let* $\mathscr{B}$ *be a fading memory space. Assume that* $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \geq 0$, $\widetilde{K}_0 \geq 1$ *and* $\widetilde{\nu}_0 > 0$. *Suppose that* $Im(I - \mathcal{M}(\tau))$ *is closed, equivalently there exists* $\delta > 0$ *with*

$$|\overline{\varphi}| \leq \delta\,|(I - \mathcal{M}(\tau))\varphi|_{\mathscr{B}} \qquad \textit{for all} \quad \varphi \in \mathscr{B}_0.$$

If $\widehat{\mathcal{L}}$ *verifies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0\widetilde{K}\tau}\ln\left(1 + \frac{e^{\widetilde{\nu}_0\tau} - 1}{2\widetilde{K}_0}\right).$$

Moreover, if the operator $\mathcal{D}$ *verifies*

$$|\mathcal{D}| < \frac{1}{\beta K_0\widehat{K}\tau}\ln\left(1 + \frac{e^{-\nu_0^+\tau}}{2\delta(\beta K_0 c + M)}\right).$$

Then, Eq. (1.1) verifies the property $(\mathscr{B}\mathscr{P})$.

Proof: The proof is a combination between Proposition 3.4 and Theorem 4.1. □

Theorem 4.3 *Suppose that conditions* $(\mathbf{C_0'})$ *and* $(\mathbf{C_1})$ *hold. Let* $\mathscr{B}$ *be a uniform fading memory space. Assume that* $|\mathcal{T}_0(t)| \leq \widetilde{K}_0 e^{-\widetilde{\nu}_0 t}$ *for* $t \geq 0$, $\widetilde{K}_0 \geq 1$ *and* $\widetilde{\nu}_0 > 0$. *If* $\widehat{\mathcal{L}}$ *verifies*

$$|\widehat{\mathcal{L}}| < \frac{1}{\widetilde{K}_0\widetilde{K}\tau}\ln\left(1 + \frac{1 - e^{-\frac{1}{2}\min(\widetilde{\nu}_0,\eta)\tau}}{2\beta\widetilde{K}_0 c}\right).$$

Moreover, if the operator $\mathcal{D}$ *verifies*

$$|\mathcal{D}| < \frac{1}{\beta K_0\widehat{K}\tau}\ln\left(1 + \frac{e^{-\nu_0^+\tau}}{2\delta(\beta K_0 c + M)}\right).$$

Then, Eq. (1.1) verifies the property $(\mathscr{B}\mathscr{P})$.

Proof: Since the condition $(\mathbf{C_0'})$ hold, it follows from [9] that the condition $(\mathbf{C_0})$ is verified with $\widehat{\nu} = \widetilde{\nu} + \widetilde{K}|\widehat{\mathcal{L}}|$ and $\widehat{K} = \widetilde{K}$. Then, the proof follows immediately from Proposition 2.2, Proposition 3.8 and Theorem 4.1. □

Remark 4.1 Clearly, if $\beta c \geq 1$, then

$$\ln\left(1 + \frac{1 - e^{-\frac{1}{2}\min(\widetilde{\nu}_0,\eta)\tau}}{2\beta\widetilde{K}_0 c}\right) \leq \ln\left(1 + \frac{e^{\widetilde{\nu}_0\tau} - 1}{2\widetilde{K}_0}\right).$$

Consequently, if $I - \mathcal{M}(\tau)$ is closed, then Theorem 4.2 is more general than Theorem 4.3.

5. The τ-periodicity of Eq. (1.1) in the phase space $\mathscr{B} = \mathcal{UC}_\gamma(\mathbb{X}),\ \gamma < 0$:

Let us introduce the phase space $\mathcal{UC}_\gamma(\mathbb{X})$ with $\gamma < 0$ as follows:

$$\mathcal{UC}_\gamma(\mathbb{X}) := \left\{\varphi \in \mathcal{UC}((-\infty,0];\mathbb{X}) : \frac{\|\varphi(\theta)\|}{e^{\gamma\theta}} \text{ is bounded and uniformly continuous on } (-\infty,0]\right\}$$

endowed with the norm

$$|\varphi|_\gamma = \sup_{\theta\in(-\infty,0]} \frac{\|\varphi(\theta)\|}{e^{\gamma\theta}}.$$

Then, from [11], $\mathcal{UC}_\gamma(\mathbb{X})$ with $\gamma < 0$, is a uniform fading memory space with $\beta = 1$, $c = 1$ and $X_0(t) = e^{\gamma t}$ for $t \geq 0$.

Proposition 5.1 *Let $\mathscr{B} = \mathcal{UC}_\gamma(\mathbb{X})$ with $\gamma < 0$. Suppose that $I - \mathcal{S}_0(\tau) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$, then $Im(I - \mathcal{M}(\tau))$ is closed and $I - \mathcal{M}(\tau) \in \mathcal{SF}_+(\mathscr{B}_0)$. Moreover, one can take the value of δ such that*

$$\delta \leq \frac{1}{1 - e^{\gamma\tau}} + \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|,$$

Proof: It is enough to verify the hypothesis of Proposition 2.3 to guarantee the result. Let $\varphi \in \mathscr{B}_0$ and $\theta \in I_k$, then

$$\begin{aligned}
\frac{\|\mathcal{P}\varphi(\theta)\|}{e^{\gamma\theta}} &\leq \frac{1}{e^{\gamma\theta}} \sum_{j=0}^{k-1} \|\varphi(\theta + j\tau)\| + \frac{1}{e^{\gamma\theta}} |\mathcal{S}_0(\theta + k\tau)|\ |S^{-1}_{T(\nu)}|\ \|\varphi(0)\| \\
&\leq \sum_{j=0}^{k-1} \frac{e^{\gamma(\theta+j\tau)}}{e^{\gamma\theta}} \frac{1}{e^{\gamma(\theta+j\tau)}} \|\varphi(\theta + j\tau)\| + \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|\ |\varphi|_\gamma.
\end{aligned}$$

Since $\gamma < 0$, then

$$\frac{\|\mathcal{P}\varphi(\theta)\|}{e^{\gamma\theta}} \leq \left(\frac{1}{1 - e^{\gamma\tau}} + \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|\right) |\varphi|_\gamma.$$

Which implies that

$$|\mathcal{P}\varphi|_\gamma \leq \left(\frac{1}{1 - e^{\gamma\tau}} + \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|\right) |\varphi|_\gamma.$$

Consequently, all hypothesis of Proposition 2.3 are satisfied and $I - \mathcal{M}(\tau)$ is closed by taking the value of the constant δ such that

$$\delta \leq \frac{1}{1 - e^{\gamma\tau}} + \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|.$$

Moreover, using Proposition 2.2, we get that $Im(I - \mathcal{M}(\zeta)) \in \mathcal{SF}_+(\mathscr{B}_0)$. □

Theorem 5.1 *Suppose that conditions $(\mathbf{C_0})$ and $(\mathbf{C_1})$ hold. Let $\mathscr{B} = \mathcal{UC}_\gamma(\mathbb{X})$ with $\gamma < 0$. Assume that $I - \mathcal{S}_0(\tau)) \in \mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$. If the operator $\mathcal{D}$ satisfies*

$$|\mathcal{D}| < \frac{1}{K_0 \widehat{K} \tau} \ln\left(1 + \frac{e^{-\nu_0^+\tau}(1 - e^{\gamma\tau})}{2(1 + \sqrt{n})(K_0 + 2)\left(1 + (1 - e^{\gamma\tau}) \sup_{0\leq t\leq\tau} |\mathcal{S}_0(t)|\ |S^{-1}_{T(\nu)}|\right)}\right).$$

Then, Eq. (1.1) verifies the property $(\mathscr{BP})$.

Proof: Since $I-\mathcal{S}_0(\tau))\in\mathcal{SF}_+(\overline{\mathscr{D}(\mathcal{L})})$, it follows from Proposition 5.1 that all conditions of Theorem 4.1 are satisfied by taking

$$\delta\leq\frac{1}{1-e^{\gamma\tau}}+\sup_{0\leq t\leq\tau}|\mathcal{S}_0(t)|\;|S^{-1}_{T(\nu)}|$$

and $\beta=c=1$ and $M=2$. Then, Eq. (1.1) verifies the property $(\mathscr{BP})$. Which complete the proof. □

Now, we introduce the fundamental theorem in the case where the operator $\mathcal{L}$ is decomposed as $\mathcal{L}=\widetilde{\mathcal{L}}+\widehat{\mathcal{L}}$ with $\widetilde{\mathcal{L}}\in\mathcal{HY}(\mathbb{X})$ and $\widehat{\mathcal{L}}\in\mathscr{L}(\mathbb{X})$. Since $\beta=c=1$ and according to Remark 4.1, we have the following result.

Theorem 5.2 *Suppose that* $(\mathbf{C_0'})$ *and* $(\mathbf{C_1})$ *hold. Let* $\mathscr{B}=\mathcal{UC}_\gamma(\mathbb{X})$ *with* $\gamma<0$. *Assume that* $|\mathcal{T}_0(t)|\leq\widetilde{K}_0e^{-\widetilde{\nu}_0t}$ *for* $t\geq0$, $\widetilde{K}_0\geq1$ *and* $\widetilde{\nu}_0>0$. *If the operator* $\widehat{\mathcal{L}}$ *satisfies the following inequality*

$$|\widehat{\mathcal{L}}|<\frac{1}{\widetilde{K}_0\widetilde{K}\tau}\ln\left(1+\frac{e^{\widetilde{\nu}_0\tau}-1}{2\widetilde{K}_0}\right). \tag{5.1}$$

In addition, if $\mathcal{D}$ verifies the following inequality

$$|\mathcal{D}|<\frac{1}{K_0\widehat{K}\tau}\ln\left(1+\frac{e^{-\nu_0^+\tau}(1-e^{\gamma\tau})}{2(K_0+2)\left(1+(1-e^{\gamma\tau})\sup\limits_{0\leq t\leq\tau}|\mathcal{S}_0(t)|\;|S^{-1}_{T(\nu)}|\right)}\right).$$

Then, Eq. (1.1) verifies the property $(\mathscr{BP})$.

Proof: Estimation (5.1) implies that Proposition 3.3 holds, then, from Proposition 5.1, we deduce that all conditions of Theorem 5.1 are satisfied by taking $n=0$ and

$$\delta\leq\frac{1}{1-e^{\gamma\tau}}+\sup_{0\leq t\leq\tau}|\mathcal{S}_0(t)|\;|S^{-1}_{T(\nu)}|.$$

Then, Eq. (1.1) verifies the property $(\mathscr{BP})$. □

6. Application:

$$\begin{cases}\dfrac{\partial}{\partial t}w(t,x)=\dfrac{\partial}{\partial x}w(t,x)-aw(t,\xi)+\displaystyle\int_0^{+\infty}\vartheta(x,y)w(t,y)\,dy+\int_{-\infty}^0\eta(\theta)w(t+\theta,x)\,d\theta+\mathcal{G}(t,x),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in\mathbb{R}^+\quad\text{and}\quad x\in\mathbb{R}^+,\\ w(t,0)=\lim\limits_{x\to+\infty}w(t,x)=0,\quad t\in\mathbb{R}^+,\\ w(\theta,x)=w_0(\theta,x),\qquad \theta\in(-\infty,0]\quad\text{and}\quad x\in\mathbb{R}^+,\end{cases} \tag{6.1}$$

such that $a>0$, $\eta:(-\infty,0]\to\mathbb{R}^+$, $\vartheta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ satisfies $\vartheta(x,.)\in L^1(\mathbb{R}^+)$, $\mathcal{G}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a continuous function and $w_0:(-\infty,0]\times[0,+\infty)\to\mathbb{R}$. Let $\mathbb{X}=\mathbf{C}([0,+\infty])$ where $\mathbf{C}([0,+\infty])$ is the space of continuous functions on $[0,+\infty)$ such that $\lim\limits_{x\to+\infty}w(x)$ exists. Then, $\mathbb{X}$ is a Banach space provided with the norm

$$\|z\|_\infty=\sup_{0\leq x\leq+\infty}|z(x)|.$$

To put problem (6.1) into abstract form , we define the operator $\mathcal{L}_1$ from $\mathscr{D}(\mathcal{L}_1)\subset\mathbb{X}$ to $\mathbb{X}$ by:

$$\begin{cases}\mathscr{D}(\mathcal{L}_1)&=&\{w\in\mathbf{C}^1([0,+\infty]):w(0)=\lim\limits_{x\to+\infty}w(x)=0\},\\ \mathcal{L}_1w&=&w'.\end{cases}$$

Using the existing study in [5], we have

$$\rho(\mathcal{L}_1) \supset (0,+\infty) \text{ and } |(\mu I - \mathcal{L}_1)^{-1}| \leq \frac{1}{\mu} \text{ for } \mu > 0.$$

Furthermore,

$$\overline{\mathscr{D}(\mathcal{L}_1)} = \{w \in \mathbf{C}([0,+\infty]) : w(0) = \lim_{x\to+\infty} w(x) = 0\} \neq \mathbb{X}.$$

In addition, $\overline{\mathcal{L}}_1$ the part of the operator $\mathcal{L}_1$ in $\overline{\mathscr{D}(\mathcal{L}_1)}$ is expressed by

$$\begin{cases} \mathscr{D}(\overline{\mathcal{L}}_1) & = & \{w \in \mathbf{C}^1([0,+\infty]) : w(0) = w'(0) = \lim_{x\to+\infty} w(x) = \lim_{x\to+\infty} w'(x) = 0\}, \\ \overline{\mathcal{L}}_1 w & = & w'. \end{cases}$$

Lemma 6.1 *[5] $\overline{\mathcal{L}}_1$ generate a strongly continuous semigroup $\{\mathcal{T}_{\overline{\mathcal{L}}_1}(t), t \geq 0\}$ on $\overline{\mathscr{D}(\mathcal{L}_1)}$. Moreover,*

$$|\mathcal{T}_{\overline{\mathcal{L}}_1}(t)| = 1 \quad for \quad t \geq 0.$$

Define the operator $\widetilde{\mathcal{L}} : \mathscr{D}(\widetilde{\mathcal{L}}) \subset \mathbb{X} \to \mathbb{X}$ by:

$$\begin{cases} \mathscr{D}(\widetilde{\mathcal{L}}) & = & \{w \in \mathbf{C}^1([0,+\infty]) : w(0) = \lim_{x\to+\infty} w(x) = 0\}, \\ \widetilde{\mathcal{L}} w & = & \mathcal{L}_1 w - aw. \end{cases}$$

Then, it is clear that

$$\rho(\widetilde{\mathcal{L}}) \supset (-a,+\infty) \text{ and } |(\mu I - \widetilde{\mathcal{L}})^{-1}| \leq \frac{1}{\mu + a} \text{ for } \mu > -a.$$

Hence, hypothesis $(\mathbf{C_0'})$ is satisfied with $\widetilde{K} = 1$ and $\widetilde{\nu} = -a$. Moreover, $\widetilde{\mathcal{L}}_0$ the part of $\widetilde{\mathcal{L}}$ on $\overline{\mathscr{D}(\widetilde{\mathcal{L}})}$ generate a C_0-semigroup $\{\mathcal{T}_0(t), t \geq 0\}$ such that

$$|\mathcal{T}_0(t)| \leq e^{-at}, \quad t \geq 0$$

Now, let the operator $\widehat{\mathcal{L}}$ defined on $\mathbb{X}$ by

$$(\widehat{\mathcal{L}} w)(x) = \int_0^{+\infty} \vartheta(x,y) w(y)\, dy \text{ for all } x \in [0,+\infty).$$

Since $\vartheta(x,.) \in L^1(\mathbb{R}^+)$, the operator $\widehat{\mathcal{L}}$ is well defined from $\mathbb{X}$ to $\mathbb{X}$. Moreover, if $\varrho = \sup_{x\in[0,+\infty)} \int_0^{+\infty} \vartheta(x,y)\, dy < \infty$, clearly $\widehat{\mathcal{L}} \in \mathscr{L}(\mathbb{X})$ and satisfies $|\widehat{\mathcal{L}}| \leq \varrho$.

Lemma 6.2 *[9] The part $\mathcal{L}_0$ of the operator $\mathcal{L} = \widetilde{\mathcal{L}} + \widehat{\mathcal{L}}$ generate a C_0-semigroup $\{\mathcal{S}_0(t), t \geq 0\}$ satisfying:*

$$|\mathcal{S}_0(t)| \leq e^{(\varrho - a)t}, \ t \geq 0.$$

Now, Let $\mathcal{UC}_\gamma(\mathbb{X})$, $\gamma < 0$. Suppose that $w_0(0,0) = w_0(0,x) = 0$ such that $\dfrac{w_0(\theta,x)}{e^{\gamma\theta}}$ is bounded and uniformly continuous on $(-\infty,0]$. Then, $\varphi(0) \in \overline{\mathscr{D}(\mathcal{L})}$. Furthermore, consider the following notations:

$$\begin{cases} w(t)(x) & = & w(t,x), \quad t \geq 0 \text{ and } x \in \mathbb{R}^+, \\ \varphi(\theta)(x) & = & w_0(\theta,x), \quad \theta \leq 0 \text{ and } \mathbb{R}^+, \\ \mathcal{H}(t)(x) & = & \mathcal{G}(t,x), \quad t \geq 0 \text{ and } \mathbb{R}^+. \end{cases}$$

Let $\mathcal{D} : \mathscr{C} := \mathscr{C}((-\infty,0],\mathbb{X}) \to \mathbb{X}$ be defined by

$$\mathcal{D}(\phi)(x) = \int_{-\infty}^0 \eta(\theta)\varphi(\theta)(x)\, d\theta, \quad x \geq 0.$$

It is clear that if $\int_{-\infty}^{0} e^{\gamma\theta}\eta(\theta)\,d\theta < \infty$, then $\mathcal{D} : \mathcal{UC}_\gamma(\mathbb{X}) \to \mathbb{X}$ is a bounded linear operator. Furthermore, one has that

$$|\mathcal{D}| < \int_{-\infty}^{0} e^{\gamma\theta}\eta(\theta)\,d\theta.$$

So, Eq. (6.1) takes the following form:

$$\begin{cases} \dot{w}(t) &= \mathcal{L}w(t) + \mathcal{D}(w_t) + \mathcal{H}(t) \qquad for \quad t \geq 0, \\ w_0 &= \varphi. \end{cases} \tag{6.2}$$

To examine the boundedness of solutions for Eq. (6.2), we introduce the variation of constant formula associate to Eq. (1.1):

$$w(t) = \mathcal{T}_0(t)\varphi(0) + \lim_{\mu\to+\infty} \int_0^t \mathcal{T}_0(t-s)\mu(\mu I - \widetilde{\mathcal{L}})^{-1}(\widehat{L}w(s) + \mathcal{D}(w_s) + \mathcal{H}(s))\,ds.$$

Moreover, suppose that:

$(\mathbf{C_2})$ there is a constant $l \in (0,1)$ with $\varrho + \int_{-\infty}^{0} \eta(\theta)\,d\theta < a(1-l)$.

If we put $\lambda = 1 + \frac{|\mathcal{H}|_\infty}{a\,l}$, such that $|\mathcal{H}|_\infty = \sup_{0\leq t\leq\tau} |\mathcal{H}(t)|$. Then we obtain the following result

Lemma 6.3 *Under the above assumption* $(\mathbf{C_2})$ *and let* $\varphi \in \mathcal{UC}_\gamma$ *with* $\gamma < 0$ *such that* $|\varphi| < \lambda$. *Then, the integral solution* $w(t, w_0)$ *of Eq. (6.2) satisfies* $|w(t, w_0)| \leq \lambda$ *for* $t \geq 0$.

Proof: Let $t_1 = \inf\{t > 0 : |w(t,\varphi)| > \lambda\}$. The continuity of w implies that

$$|w(t_0,\varphi)| = \lambda,$$

and there is $\epsilon > 0$ with

$$|w(t,\varphi)| > \lambda \qquad for \qquad t \in (t_1, t_1+\epsilon).$$

Hence, for $t_1 \geq 0$

$$|w(t_1,\varphi)| \leq |\mathcal{T}_0(t_1)|\,|\varphi(0)| + \int_0^{t_1} e^{-a(t_1-s)}\left(|\widehat{L}w(s)| + |\mathcal{D}(w_s)| + |\mathcal{H}(s)|\right) ds,$$

Since $-\infty < s+\theta \leq s+t_1 \leq t_1$ for $\theta \leq 0$, one has that

$$|\mathcal{D}(w_s(.,\varphi))| = \int_{-\infty}^{0} \eta(\theta)|w(s+\theta,\varphi)|\,d\theta \leq \lambda \int_{-\infty}^{0} \eta(\theta)\,d\theta,$$

it follows that

$$\begin{aligned} |w(t_1,\varphi)| &\leq \lambda e^{-at_1} + |\mathcal{H}|_\infty \int_0^{t_1} e^{-a(t_1-s)}\,ds + \lambda\left(|\widehat{L}| + \int_{-\infty}^{0}\eta(\theta)\,d\theta\right)\int_0^{t_1} e^{-a(t_1-s)}\,ds \\ &\leq \lambda e^{-at_1} + \frac{(1-e^{-at_1})}{a}\left(|\mathcal{H}|_\infty + \lambda\left(|\widehat{L}| + \int_{-\infty}^{0}\eta(\theta)\,d\theta\right)\right) \\ &\leq \lambda e^{-at_1} + \frac{(1-e^{-at_1})}{a}\left(|\mathcal{H}|_\infty + \lambda\left(\varrho + \int_{-\infty}^{0}\eta(\theta)\,d\theta\right)\right) \\ &\leq \lambda e^{-at_1} + \frac{(1-e^{-at_1})}{a}\left(|\mathcal{H}|_\infty + a\,\lambda(1-l)\right), \\ &\leq \lambda e^{-at_1} + (\lambda - l)\,(1-e^{-at_1}) \\ &\leq \lambda - l(1-e^{-at_1}) \\ &\leq \lambda, \end{aligned}$$

which contradict the above definition of t_1, then

$$|w(t,\varphi)| \leq \lambda \qquad for \quad each \quad t \geq 0.$$

□

In order to examine the problem of τ-periodicity of solutions for Eq. (6.2), we suppose that:

$(\mathbf{C_3})$ $\mathcal{H}$ is τ-periodic.

Theorem 6.1 *Suppose that* $(\mathbf{C_2})$ *and* $(\mathbf{C_3})$ *hold. If*

$$\varrho < \frac{1}{\tau}\ln\left(\frac{1+e^{a\tau}}{2}\right), \tag{6.3}$$

and

$$\int_{-\infty}^{0} e^{\gamma\theta}\eta(\theta)\,d\theta < \frac{1}{\tau}\ln\left(1+\frac{(1-e^{\gamma\tau})(1-e^{(\varrho-a)\tau})}{12-6e^{(\varrho-a)\tau}-6e^{\gamma\tau}}\right).$$

Then, Eq. (6.2) admits a τ-periodic solution.

Proof: Inequality (6.3) implies that

$$|\widehat{\mathcal{L}}| < \frac{1}{\tau}\ln\left(\frac{1+e^{a\tau}}{2}\right).$$

On the other hand, hypothesis $(\mathbf{C_2})$ implies that $|\mathcal{S}_0(\tau)| < 1$ and hence, by the inequality (2.1), we obtain that $|S_{T(\nu)}^{-1}| \leq \dfrac{1}{1-e^{(\varrho-a)\tau}}$. Moreover, since $K_0 = \widehat{K} = 1$, we get that

$$\frac{1}{\tau}\ln\left(1+\frac{(1-e^{\gamma\tau})(1-e^{(\varrho-a)\tau})}{12-6e^{(\varrho-a)\tau}-6e^{\gamma\tau}}\right)$$

$$\leq \frac{1}{\tau}\ln\left(1+\frac{e^{-\nu_0^+\tau}(1-e^{\gamma\tau})}{6\left(1+(1-e^{\gamma\tau})\sup\limits_{0\leq t\leq\tau}|\mathcal{S}_0(t)|\;|S_{T(\nu)}^{-1}|\right)}\right).$$

Thus

$$|\mathcal{D}| < \frac{1}{\tau}\ln\left(1+\frac{e^{-\nu_0^+\tau}(1-e^{\gamma\tau})}{6\left(1+(1-e^{\gamma\tau})\sup\limits_{0\leq t\leq\tau}|\mathcal{S}_0(t)|\;|S_{T(\nu)}^{-1}|\right)}\right).$$

Consequently, From hypothesis $(\mathbf{C_3})$ and Proposition 6.3, it follows that all condition of Theorem 5.2 are verified with $\widetilde{K} = \widetilde{K}_0 = 1$. Finally, Eq. (6.2) has a τ-periodic solution.

□

The objective now is checking the validity of our theoretical results by presenting some numerical simulations. In the example of application, we consider the phase space $\mathcal{UC}_{-1}$ and the following quantities:

$$a = 1,\ \vartheta(x,y) = 0.71\ e^{-y} \quad \text{and} \quad \eta(\theta) = 2.3\times 10^{-2}e^{2\theta},$$

the initial condition is given by

$$w(\theta, x) = \frac{1}{\pi^2} x e^{\theta - x},$$

the functions $\mathcal{G}$ is given by

$$\mathcal{G}(t, x) = \sin(\pi t + x).$$

clearly:

- $\mathcal{G}$ is 2-periodic function,
- $\varrho = \sup_{x \in [0,+\infty)} \int_0^{+\infty} \vartheta(x, y)\, dy = 0.71,$
- $\int_{-\infty}^{0} e^{-\theta} \eta(\theta)\, d\theta = 0.023.$

Then, for $\varrho = 0.71$ and $\int_{-\infty}^{0} e^{-\theta} \eta(\theta)\, d\theta = 0.023$, all conditions of Theorem 6.1 are satisfied and consequently, Equation (6.1) has a 2-periodic solution. This result is illustrated by some numerical simulations given in figure 1 in $3D$ and figures 2 and 3 in $2D$ for $x = 0.5$ and 0.9 respectively.

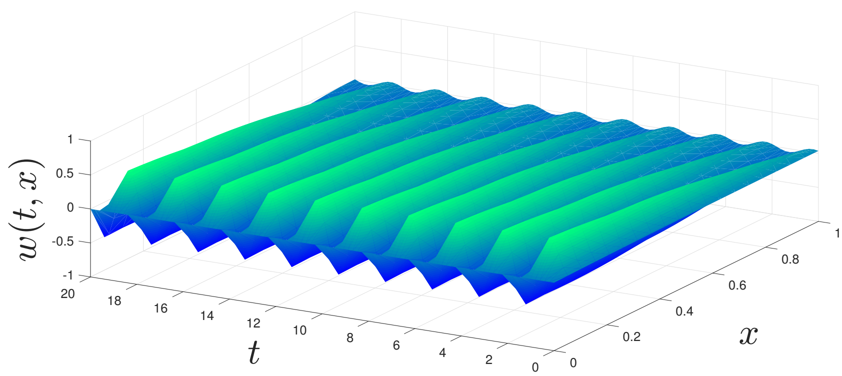

Figure 1: The graph of the 2-periodic solution of Equation (6.1) in $3D$. (the solution w with respect to t and x).

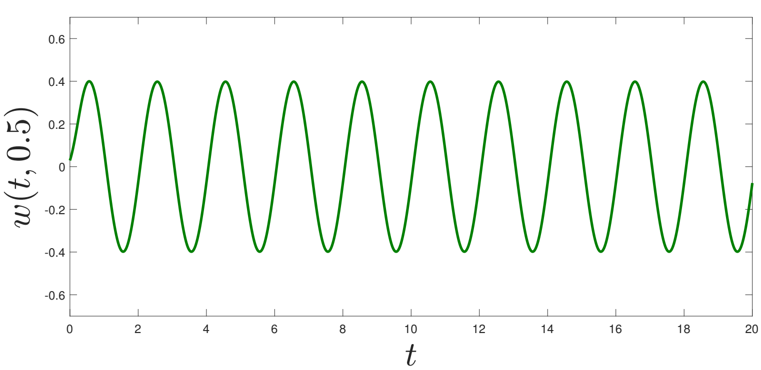

Figure 2: The graph of the 2-periodic solution of Equation (6.1) in $2D$ for $x = 0.5$ (the solution w with respect to t).

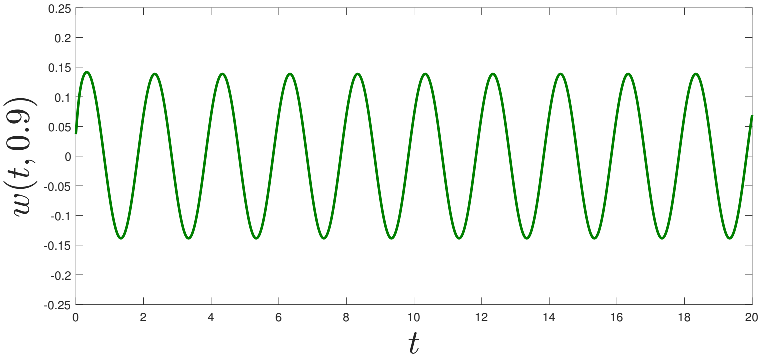

Figure 3: The graph of the 2-periodic solution of Equation (6.1) in $2D$ for $x = 0.9$ (the solution w with respect to t).

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