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On a New Generalized Class of p-Valent Functions and its Mapping Conditions by a Wright's Generalized Hypergeometric Operator

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ABSTRACT: A generalized class $R_{\gamma}^{\tau}(m,p,A,B)$ involving $(m-1)^{th}$ and m^{th} derivatives of p-valent functions is defined and its integral representation, coefficient condition and a sufficient condition for a function to be in this class are obtained in this work. Further, certain mapping properties of a Wright's generalized hypergeometric operator $Wf(z)$ related to the class $R_{\gamma}^{\tau}(m,p,A,B)$ are investigated with the known classes $S_{p,\lambda}$ and $C_{p,\lambda}$.

Key Words: Starlike functions, convex functions, convolution, Wright's generalized hypergeometric functions, p-valent analytic functions.

Contents

1. Introduction and Preliminaries

Special functions are found to be very useful due to their numerous applications in different branches of mathematics, engineering, physics etc. [13]. One of the most important function among them is hypergeometric function which has made a major contribution in the development of analysis and has been extensively used in geometric function theory. There have been several studies conducted thus far in geometric function theory. Mapping properties involving some subclasses of functions are one of the significant characterizations in the theory.

Gangadharan et al. [8] investigated mapping properties concerning some subclasses of univalent analytic functions and established connections between these classes under Dziok-Srivastava operator. Mapping and inclusion properties of Dziok-Srivastava operator were also obtained in [15] (see also [14], [18] and [19]) for some other subclasses of univalent analytic functions. Study of mapping properties have been made not only for analytic univalent functions but also for some harmonic univalent functions concerning the hypergeometric functions (see in [1,2,3,4]), [10]. In [12], Raina and Sharma studied harmonic univalent maps which involves Wright's generalized hypergeometric functions and investigated several properties associated with the mappings. Recently Li-Mei Wang [21] investigated mapping attributes of certain hypergeometric functions. For more related works one may refer to [5], [16] and [23].

Denote by $\mathcal{A}_p$ the function class having p-valent functions which are analytic in $\mathbb{D}=\{z:|z|<1\}$ and have series representation

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\ (p\in\mathbb{N};z\in\mathbb{D}). \tag{1.1}$$

We denote the class $\mathcal{A}\equiv\mathcal{A}_1$.

Clearly the m^{th} derivative of $f(z)$ having the series representation (1.1) is

$$f^{(m)}(z)=\frac{p!}{(p-m)!}z^{p-m}+\sum_{n=1}^{\infty}\frac{(n+p)!}{(n+p-m)!}a_{n+p}z^{n+p-m}. \tag{1.2}$$

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Suppose f and g are analytic functions in unit disk $\mathbb{D}$ then g is subordinate to f $(g \prec f)$ in $\mathbb{D}$ if there exist an analytic function h in unit disk $\mathbb{D}$ satisfying $|h(z)| < 1$, $h(0) = 0$ such that $g(z) = f(h(z))$ for all $z \in \mathbb{D}$. Further, if f is univalent, then $g \prec f \iff g(\mathbb{D}) \subset f(\mathbb{D})$ and $g(0) = f(0)$.

For the functions $f \in \mathcal{A}$, Sharma and Raina in [19] investigated the class $R_\gamma^\tau(\phi)$ $(0 \le \gamma \le 1, 0 \ne \tau \in \mathbb{C})$ which involves the first, second derivative of functions $f \in \mathcal{A}$ and obtained mapping structures for some known classes of functions. Motivated with the work in [19], we here define a new generalized class $R_\gamma^\tau(m, p, A, B)$ as follows:

$$R_\gamma^\tau(m,p,A,B) := \left\{ f \in \mathcal{A}_p : 1 + \frac{1}{\tau}\left(F\left(p,m,\gamma;z\right) - 1\right) \prec \frac{1+Az}{1+Bz} \ \ (-1 \le B < A \le 1) \right\}, \tag{1.3}$$

where the function $F\left(p,m,\gamma;z\right)$ is given by

$$F\left(p,m,\gamma;z\right) = \frac{(p+1-m)!z^{m-p-1}}{p!(1+\gamma(p+1-m))}\left\{f^{(m-1)}(z) + \gamma z f^{(m)}(z)\right\} \tag{1.4}$$

$$\left(0 \le \gamma \le 1, 0 \ne \tau \in \mathbb{C}\right).$$

Clearly,

$$f \in R_\gamma^\tau(m,p,A,B) \iff \left|\frac{F\left(p,m,\gamma;z\right) - 1}{\tau(A-B) - B\left(F\left(p,m,\gamma;z\right) - 1\right)}\right| < 1, \tag{1.5}$$

where the function $F\left(p,m,\gamma;z\right)$ is given by (1.4).
Moreover $f \in R_\gamma^\tau(m,p,A,B)$ satisfies

$$\mathrm{Re}\left(1 + \frac{1}{\tau}\left(F\left(p,m,\gamma;z\right) - 1\right)\right) > \frac{1-A}{1-B},$$

where the function $F\left(p,m,\gamma;z\right)$ is given by (1.4).

We now mention some special forms of the function class $R_\gamma^\tau(m,p,A,B)$ studied earlier in several papers for different values of the parameters involved. Some of them are as follows:

(1) For $p = 1$ and $m = 2$, the function class $R_\gamma^\tau(m,p,A,B)$ transforms to the form $R_\gamma^\tau(A,B)$ [19].

(2) For $p = 1$, $m = 2$ and $\gamma = 0$, the function class $R_\gamma^\tau(m,p,A,B)$ transforms to $R^\tau(A,B)$ [7].

(3) For $p = 1$, $m = 2$, $\gamma = 0$ and $\tau = e^{-i\alpha}\cos\alpha$ for $|\alpha| < \frac{\pi}{2}$, the function class $R_\gamma^\tau(m,p,A,B)$ was studied in [6].

(4) For $p = 1$, $m = 2$, $\gamma = 0$, $\tau = e^{-i\alpha}\cos\alpha$ for $|\alpha| < \frac{\pi}{2}$, $A = 1-2\beta$, $B = -1$ $(0 \le \beta < 1)$, the function class $R_\gamma^\tau(m,p,A,B)$ becomes $R_\alpha(\beta)$ [11].

(5) For $p = 1$, $m = 2$, $A = 1-2\beta$, $B = -1$ $(0 \le \beta < 1)$, the function class $R_\gamma^\tau(m,p,A,B)$ becomes $R_\gamma^\tau(\beta)$ [20].

(6) For $p = 1$, $m = 2$, $\tau = 1$, $\gamma = 0$ and $A = -B = \beta$ (where $0 \le \beta < 1$), the function class $R_\gamma^\tau(m,p,A,B)$ was studied in [10].

Consider the Wright's generalized hypergeometric function ${}_r\psi_s(z)$ (also known as Wgh function) [22] (see also [9], [13]), which is defined as

$${}_r\psi_s(z) \equiv {}_r\psi_s\left(\begin{array}{c}(a_i, A_i)_{1,r} \\ (b_i, B_i)_{1,s}\end{array}; z\right) = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^r \Gamma(a_i + nA_i)}{\prod_{i=1}^s \Gamma(b_i + nB_i)}\frac{z^n}{n!}, \tag{1.6}$$

for $A_i > 0 (i = 1, ..., r), B_i > 0 (i = 1, ..., s)$ such that
$1 + \sum_{i=1}^s B_i - \sum_{i=1}^r A_i \ge 0$, $a_i \in \mathbb{C}\left(\frac{a_i}{A_i} \ne 0, -1, -2, ... \text{ for } i = 1, ..., r\right)$, $b_i \in \mathbb{C}\left(\frac{b_i}{B_i} \ne 0, -1, -2, ... \text{ for } i =\right.$

$1, ..., s\Bigg)$.

Note that the Wgh function ${}_r\psi_s(z)$ is entire if $1+\sum_{i=1}^{s} B_i-\sum_{i=1}^{s} A_i > 0$ and is analytic in $|z| < \frac{\prod_{i=1}^{s} {B_i}^{B_i}}{\prod_{i=1}^{r} {A_i}^{A_i}}$ if $1+\sum_{i=1}^{s} B_i-\sum_{i=1}^{s} A_i = 0$. Further, if $1+\sum_{i=1}^{s} B_i-\sum_{i=1}^{s} A_i = 0$ and $|z| = \frac{\prod_{i=1}^{s} {B_i}^{B_i}}{\prod_{i=1}^{r} {A_i}^{A_i}}$, then the Wgh function ${}_r\psi_s(z)$ is an analytic function for

$$\operatorname{Re}\left(\sum_{i=1}^{s} b_i - \sum_{i=1}^{r} a_i\right) + \frac{r-s}{2} > \frac{1}{2}.$$

A convolution $*$ between $f \in \mathcal{A}_p$ having series representation (1.1) and $h \in \mathcal{A}_p$ whose series representation is

$$h(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$$

is written as $(f * h)(z)$=$f(z) * h(z)$ and is given by

$$f(z) * h(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}.$$

A Wright's generalized hypergeometric operator $W : \mathcal{A}_p \to \mathcal{A}_p$ is defined by

$$Wf(z) = z^p \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\prod_{i=1}^{r}\Gamma(a_i)} {}_r\psi_s(z) * f(z), \tag{1.7}$$

where ${}_r\psi_s(z)$ is defined by (1.6).
The series expression of $Wf(z)$ (for $f \in \mathcal{A}_p$ having the representation (1.1)) is given by

$$Wf(z) = z^p + \sum_{n=1}^{\infty} \theta_n a_{n+p} z^{n+p}, \tag{1.8}$$

where

$$\theta_n = \frac{\prod_{i=1}^{r} \frac{\Gamma(a_i+A_i n)}{\Gamma(a_i)}}{\prod_{i=1}^{s} \frac{\Gamma(b_i+B_i n)}{\Gamma(b_i)}} \frac{1}{n!}. \tag{1.9}$$

2. Main Results

In this section, we first find an integral representation of $f^{(m-1)}(z)$, coefficient inequality and a sufficient condition for the function $f \in R_{\gamma}^{\tau}(m,p,A,B)$.

Theorem 2.1 *If $f \in \mathcal{A}_p$ given by (1.1), then $f \in R_{\gamma}^{\tau}(m,p,A,B)$ if and only if there exist an analytic function $w(z)$ for which $|w(z)| < 1$, $w(0) = 0$ such that for $0 \neq z \in \mathbb{D}$, when $\gamma \neq 0$,*

$$f^{(m-1)}(z) = \frac{1}{z^{\frac{1}{\gamma}}} \int_0^z \frac{(1-\gamma(p+1-m))p!}{\gamma t^{m-p-\frac{1}{\gamma}}(p+1-m)!} \left(1 + \frac{\tau(A-B)w(t)}{1+Bw(t)}\right) dt \tag{2.1}$$

and for $\gamma = 0$,

$$f^{(m-1)}(z) = \frac{p!}{z^{m-p-1}(p+1-m)!} \left(1 + \frac{\tau(A-B)w(z)}{1+Bw(z)}\right). \tag{2.2}$$

Proof: Let $f \in R_{\gamma}^{\tau}(m,p,A,B)$. Then there exists an analytic function $w(z)$ satisfying $w(0) = 0$, $|w(z)| < 1$, such that from class condition (1.3), we have

$$1 + \frac{1}{\tau}\left(F(p,m,\gamma;z) - 1\right) = \frac{1+Aw(z)}{1+Bw(z)} \ \ (z \in \mathbb{D}),$$

where the function $F\left(p,m,\gamma;z\right)$ is given by (1.4), hence for $z\neq 0$,

$$f^{(m-1)}(z)+\gamma z f^{(m)}(z)=\frac{(1+\gamma(p+1-m))p!}{z^{m-p-1}(p+1-m)!}\left(1+\frac{\tau(A-B)w(z)}{1+Bw(z)}\right). \tag{2.3}$$

In case $\gamma\neq 0$, equation (2.3) may be given by

$$\frac{1}{\gamma}z^{\frac{1}{\gamma}-1}f^{(m-1)}(z)+z^{\frac{1}{\gamma}}f^{(m)}(z)=\frac{(1+\gamma(p+1-m))p!}{\gamma z^{m-p-\frac{1}{\gamma}}(p+1-m)!}\left(1+\frac{\tau(A-B)w(z)}{1+Bw(z)}\right)$$

or,

$$\left(z^{\frac{1}{\gamma}}f^{(m-1)}(z)\right)'=\frac{(1+\gamma(p+1-m))p!}{\gamma z^{m-p-\frac{1}{\gamma}}(p+1-m)!}\left(1+\frac{\tau(A-B)w(z)}{1+Bw(z)}\right). \tag{2.4}$$

Integrating equation (2.4), we obtain equation (2.1), which proves the result for $\gamma\neq 0$. For the case when $\gamma=0$, we obtain from equation (2.3), the expression (2.2). □

We now give coefficient condition for $f\in R^{\tau}_{\gamma}(m,p,A,B)$.

Theorem 2.2 *If $f\in R^{\tau}_{\gamma}(m,p,A,B)$ given by (1.1), then*

$$|a_{n+p}|\leq\frac{p!(1+\gamma(p+1-m))(n+p+1-m)!|\tau|(A-B)}{(p+1-m)!(n+p)!(1+\gamma(n+p+1-m))}\ \ (n=1,2,....). \tag{2.5}$$

Proof: Suppose that $f\in R^{\tau}_{\gamma}(m,p,A,B)$ and let

$$h(z)=1+\frac{1}{\tau}\left(F\left(p,m,\gamma;z\right)-1\right)=1+\sum_{n=1}^{\infty}h_n z^n, \tag{2.6}$$

where $F\left(p,m,\gamma;z\right)$ is given by (1.4). From the class condition (1.3), it follows that

$$1+\sum_{n=1}^{\infty}h_n z^n\prec\frac{1+Az}{1+Bz}=1+(A-B)z-(A-B)Bz^2+\dots\ . \tag{2.7}$$

Application of the Rogosinski result [17] to the above equation (2.7) yields

$$|h_n|\leq A-B\ \ (n=1,2,\cdots).$$

In view of (1.2) and (1.4), we obtain from (2.6), that

$$h_n=\frac{(p+m-1)!}{\tau p!(1+\gamma(p+1-m))}\frac{(n+p)!(1+\gamma(n+p+1-m))}{(n+p+1-m)!}a_{n+p}.$$

Hence, we get

$$|a_{n+p}|\leq\frac{p!(1+\gamma(p+1-m))(n+p+1-m)!|\tau|(A-B)}{(p+1-m)!(n+p)!(1+\gamma(n+p+1-m))}\ \ (n=1,2,....)$$

which proves the desired result (2.5). □

Theorem 2.3 *Let $f\in\mathcal{A}_p$ of the form (1.1) satisfies*

$$\sum_{n=1}^{\infty}\frac{(n+p)!(1+\gamma(n+p+1-m))}{(n+p+1-m)!}|a_{n+p}|\leq\frac{(A-B)|\tau|p!(1+\gamma(p+1-m))}{(1+|B|)(p+1-m)!}, \tag{2.8}$$

for $0\leq\gamma<1$, $0\neq\tau\in\mathbb{C}$ and $m,p\in\mathbb{N}$. Then $f\in R^{\tau}_{\gamma}(m,p,A,B)$.
Result is sharp for the function f_n given by

$$f_n(z)=z^p+\frac{(A-B)|\tau|p!(1+\gamma(p+1-m))(n+p+1-m)!}{(n+p)!(p+1-m)!(1+\gamma(n+p+1-m))(1+|B|)}z^{n+p}\ \ (n\in\mathbb{N}). \tag{2.9}$$

Proof: To prove $f \in R_{\gamma}^{\tau}(m,p,A,B)$, it is sufficient to verify the condition (1.5). Let us suppose that

$$\kappa = \left|F\left(p,m,\gamma;z\right)-1\right| - \left|\left(A-B\right)\tau - \left(F\left(p,m,\gamma;z\right)-1\right)B\right|,$$

where $F\left(p,m,\gamma;z\right)$ is given by (1.4) and hence,

$$\begin{aligned}\left|F\left(p,m,\gamma;z\right)-1\right| &< \frac{(p+1-m)!}{p!(1+\gamma(p+1-m))} \\ &\times \sum_{n=1}^{\infty}\frac{(n+p)!(1+\gamma(n+p+1-m))}{(n+p+1-m)!}\left|a_{n+p}\right|.\end{aligned}$$

Thus,

$$\begin{aligned}\kappa &\le \left|F\left(p,m,\gamma;z\right)-1\right| - \left(A-B\right)\left|\tau\right| + \left|\left(F\left(p,m,\gamma;z\right)-1\right)\right|\left|B\right| \\ &= \left|F\left(p,m,\gamma;z\right)-1\right|\left(1+|B|\right) - |\tau|(A-B) \\ &< \frac{(1+|B|)(p+1-m)!}{p!(1+\gamma(p+1-m))} \\ &\times \sum_{n=1}^{\infty}\frac{(n+p)!(1+\gamma(n+p+1-m))}{(n+p+1-m)!}\left|a_{n+p}\right| - (A-B)|\tau| \le 0.\end{aligned}$$

Since the inequality (2.8) holds, which proves the desired result. The sharpness of the result (2.8) can easily be verified for the function $f_n(z)$ given by (2.9). □

Next we discuss various mapping results involving the class $R_{\gamma}^{\tau}(m,p,A,B)$. For this we define classes $S_{p,\lambda}$ and $C_{p,\lambda}$ as follows:

$$S_{p,\lambda} = \left\{f \in \mathcal{A}_p : \left|\frac{zf'(z)}{f(z)} - p\right| < \lambda \;\; (\lambda > 0; z \in \mathbb{D})\right\}.$$

Let $C_{p,\lambda}$ denotes the class having the functions $f \in \mathcal{A}_p$ such that whenever $f \in C_{p,\lambda}$, $\frac{zf'}{p} \in S_{p,\lambda}$. Note that the sufficient conditions for $f \in \mathcal{A}_p$ to be in the function classes $S_{p,\lambda}$ and $C_{p,\lambda}$ are

$$\sum_{n=1}^{\infty}(\lambda+n)|a_{n+p}| \le \lambda \tag{2.10}$$

and

$$\sum_{n=1}^{\infty}(\lambda+n)(n+p)|a_{n+p}| \le \lambda, \tag{2.11}$$

respectively.

Note that the class $S_{p,\lambda}$ $(C_{p,\lambda})$ is a class of starlike (convex) functions $f \in \mathcal{A}_p$ of the order $p-\lambda$, where $p-1 < \lambda \le p$.

Next result provides a mapping result for the operator W given by (1.7), to map $R_{\gamma}^{\tau}(m,p,A,B)$ onto itself.

Theorem 2.4 *Let W be given by (1.7). If $A_i > 0 (i=1,...,r)$, $B_i > 0 (i=1,...,s)$ such that $1+\sum_{i=1}^{s}B_i - \sum_{i=1}^{s}A_i = 0$, $\frac{\prod_{i=1}^{s}{B_i}^{B_i}}{\prod_{i=1}^{r}{A_i}^{A_i}} = 1$ and $0 \ne a_i \in \mathbb{C}\left(i=1,...,r\right), b_i \in \mathbb{C}\left(\frac{\mathrm{Re}(b_i)}{B_i} \ne 0,-1,-2,...; i=1,...,s\right)$ with the condition*

$$\sum_{i=1}^{s}\mathrm{Re}(b_i) - \sum_{i=1}^{r}|a_i| + \frac{r-s}{2} > \frac{1}{2},$$

the inequality

$$\frac{\prod_{i=1}^{s}\Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r}\Gamma(|a_i|)}\; {}_r\psi_s\left(\begin{array}{c}\left(|a_i|, A_i\right)_{1,r} \\ (\mathrm{Re}(b_i), B_i)_{1,s}\end{array};1\right) \le \frac{2+|B|}{1+|B|} \tag{2.12}$$

holds, then $W : R_{\gamma}^{\tau}(m,p,A,B) \to R_{\gamma}^{\tau}(m,p,A,B)$.

Proof: Consider $f \in R^{\tau}_{\gamma}(m,p,A,B)$ be of the form (1.1). To show $Wf(z)$ given by (1.8) belongs in the class $R^{\tau}_{\gamma}(m,p,A,B)$, in view of the condition (2.8) of Theorem 2.3, we need to show that

$$S_1 := \sum_{n=1}^{\infty} \frac{(p+1-m)!(n+p)!(1+\gamma(n+p+1-m))}{p!(1+\gamma(p+1-m))(n+p+1-m)!\,(A-B)\,|\tau|} |\theta_n a_{n+p}| \leq \frac{1}{1+|B|}.$$

where θ_n is given by (1.9) and in view of (see in [3])

$$\frac{\Gamma(\mathrm{Re}(\mathbf{a})+nA)}{\Gamma(\mathrm{Re}(\mathbf{a}))} \leq \left|\frac{\Gamma(\mathbf{a}+nA)}{\Gamma(\mathbf{a})}\right| \leq \frac{\Gamma(|\mathbf{a}|+nA)}{\Gamma(|\mathbf{a}|)}$$

we write,

$$|\theta_n| \leq \frac{\prod_{i=1}^{r} \frac{\Gamma(|a_i|+A_i n)}{\Gamma(|a_i|)}}{\prod_{i=1}^{s} \frac{\Gamma(\mathrm{Re}(b_i)+B_i n)}{\Gamma(\mathrm{Re}(b_i))}} \frac{1}{n!}. \tag{2.13}$$

Since $f \in R^{\tau}_{\gamma}(m,p,A,B)$, we have from (2.5) and under the parametric conditions with the inequality (2.12)

$$\begin{aligned} S_1 \leq \sum_{n=1}^{\infty} |\theta_n| = \sum_{n=0}^{\infty} |\theta_n| - 1 \quad &\leq \quad \frac{\prod_{i=1}^{s}\Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r}\Gamma(|a_i|)} {}_r\psi_s \left(\begin{array}{c} (|a_i|\,, A_i)_{1,r} \\ (\mathrm{Re}(b_i), B_i)_{1,s} \end{array} ;1 \right) - 1 \\ &\leq \quad \frac{1}{1+|B|} \end{aligned}$$

which proves the desired result. □

Our next result provides a mapping result for the operator W (given by (1.7)) to map $R^{\tau}_{\gamma}(m,p,A,B)$ to the class $S_{p,\lambda}$.

Theorem 2.5 *Let the operator W be given by (1.7). If $A_i > 0 (i = 1,...,r), B_i > 0 (i = 1,...,s)$ such that $1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{s} A_i = 0$, $\frac{\prod_{i=1}^{s} {B_i}^{B_i}}{\prod_{i=1}^{r} {A_i}^{A_i}} = 1$ and $0 \neq a_i \in \mathbb{C}\,(i = 1,...,r)\,, b_i \in \mathbb{C}\, \left(\frac{\mathrm{Re}(b_i)}{B_i} \neq 0,-1,-2,...;\right.$ $\left. i = 1,...,s\right)$ with the condition*

$$\sum_{i=1}^{s} \mathrm{Re}(b_i) - \sum_{i=1}^{r} |a_i| + \frac{r-s}{2} > \frac{3}{2} - m,$$

the inequality

$$\begin{aligned} &\frac{p!(1+\gamma(p+1-m))}{\lambda(p+1-m)!} \frac{\prod_{i=1}^{s}\Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r}\Gamma(|a_i|)} \times \\ &{}_{m+r+1}\psi_{m+s+1} \left(\begin{array}{c} (|a_i|\,, A_i)_{1,r}, (\lambda+1,1), (p,1), ..., (p-m+2,1), \\ (\mathrm{Re}(b_i), B_i)_{1,s}, (\lambda,1), (p+1,1), ..., (p-m+3,1), \end{array} \right. \\ &\qquad\qquad \left. \begin{array}{c} (1+\gamma(p+1-m),\gamma) \\ (2+\gamma(p+1-m),\gamma) \end{array} ;1 \right) \leq \frac{1}{|\tau|(A-B)} + 1. \end{aligned} \tag{2.14}$$

holds, then the operator W maps $R^{\tau}_{\gamma}(m,p,A,B)$ to the class $S_{p,\lambda}$ that is, $W : R^{\tau}_{\gamma}(m,p,A,B) \to S_{p,\lambda}$.

Proof: Consider $f \in R^{\tau}_{\gamma}(m,p,A,B)$ be of the form (1.1). To show $Wf(z)$ given by (1.8) is in the class $S_{p,\lambda}$, in view of the (2.10), we only need to show that

$$S_2 := \sum_{n=1}^{\infty} (\lambda+n)|\theta_n a_{n+p}| \leq \lambda,$$

where $|\theta_n|$ is given by (2.13). Since $f \in R_\gamma^\tau(m,p,A,B)$, so using inequality (2.5), we obtain

$$\begin{aligned}
S_2 &\leq \frac{|\tau|(A-B)p!(1+\gamma(p+1-m))}{(p+1-m)!} \\
&\times \sum_{n=1}^{\infty} \frac{(\lambda+n)(n+p+1-m)!}{(n+p)!(1+\gamma(n+p+1-m))} \frac{\prod_{i=1}^{r} \frac{\Gamma(|a_i|+A_i n)}{\Gamma(|a_i|)}}{\prod_{i=1}^{s} \frac{\Gamma(\mathrm{Re}(b_i)+B_i n)}{\Gamma(\mathrm{Re}(b_i))}} \frac{1}{n!} \\
&= |\tau|(A-B)\Bigg[\frac{p!(1+\gamma(p+1-m))}{(p+1-m)!} \\
&\times \sum_{n=0}^{\infty} \frac{(\lambda+n)(n+p+1-m)!}{(n+p)!(1+\gamma(n+p+1-m))} \frac{\prod_{i=1}^{r} \frac{\Gamma(|a_i|+A_i n)}{\Gamma(|a_i|)}}{\prod_{i=1}^{s} \frac{\Gamma(\mathrm{Re}(b_i)+B_i n)}{\Gamma(\mathrm{Re}(b_i))}} \frac{1}{n!} - \lambda\Bigg] \\
&= |\tau|(A-B)\Bigg[\frac{p!(1+\gamma(p+1-m))}{(p+1-m)!} \frac{\prod_{i=1}^{s}\Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r}\Gamma(|a_i|)} \\
&\quad {}_{m+r+1}\psi_{m+s+1}\left(\begin{array}{l} (|a_i|, A_i)_{1,r}, (\lambda+1,1), (p,1), ..., (p-m+2,1), \\ (\mathrm{Re}(b_i), B_i)_{1,s}, (\lambda,1), (p+1,1), ..., (p-m+3,1), \end{array}\right. \\
&\qquad \left.\begin{array}{l} (1+\gamma(p+1-m),\gamma) \\ (2+\gamma(p+1-m),\gamma) \end{array}; 1\right) - \lambda\Bigg] \leq \lambda,
\end{aligned}$$

if the condition (2.14) holds. This proves the theorem. $\square$

We next prove a mapping result for the operator W given by (1.7), to map $R_\gamma^\tau(m,p,A,B)$ to the class $C_{p,\lambda}$.

Theorem 2.6 *Let the operator W be given by (1.7). If $A_i > 0 (i=1,...,r), B_i > 0 (i=1,...,s)$ such that $1+\sum_{i=1}^{s} B_i - \sum_{i=1}^{s} A_i = 0$, $\frac{\prod_{i=1}^{s} {B_i}^{B_i}}{\prod_{i=1}^{r} {A_i}^{A_i}} = 1$ and $0 \neq a_i \in \mathbb{C}\,(i=1,...,r), b_i \in \mathbb{C}\,\left(\frac{\mathrm{Re}(b_i)}{B_i} \neq 0,-1,-2,...;\right.$ $\left. i=1,...,s\right)$ with the condition*

$$\sum_{i=1}^{s} \mathrm{Re}(b_i) - \sum_{i=1}^{r} |a_i| + \frac{r-s}{2} > \frac{5}{2} - m,$$

the inequality

$$\begin{aligned}
&\frac{(p-1)!(1+\gamma(p+1-m))}{\lambda(p+1-m)!} \frac{\prod_{i=1}^{s}\Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r}\Gamma(|a_i|)} \times \\
&{}_{m+r}\psi_{m+s}\left(\begin{array}{l} (|a_i|, A_i)_{1,r}, (\lambda+1,1), (p-1,1), ..., (p-m+2,1), \\ (\mathrm{Re}(b_i), B_i)_{1,s}, (\lambda,1), (p,1), ..., (p-m+3,1), \end{array}\right. \\
&\qquad \left.\begin{array}{l} (1+\gamma(p+1-m),\gamma) \\ (2+\gamma(p+1-m),\gamma) \end{array}; 1\right) \leq \frac{1}{|\tau|(A-B)} + 1 \qquad (2.15)
\end{aligned}$$

holds, then the operator W maps $R_\gamma^\tau(m,p,A,B)$ to the class $C_{p,\lambda}$, that is, $W : R_\gamma^\tau(m,p,A,B) \to C_{p,\lambda}$.

Proof: Consider $f \in R_\gamma^\tau(m,p,A,B)$ be of the form (1.1). To show $Wf(z)$ given by (1.8) belongs in the class $C_{p,\lambda}$, in view of the (2.11), we only need to show that

$$S_3 := \sum_{n=1}^{\infty} (\lambda+n)(n+p)|\theta_n a_{n+p}| \leq \lambda p,$$

where $|\theta_n|$ is given by (2.13). Since $f \in R^{\tau}_{\gamma}(m,p,A,B)$, so using inequality (2.5), we obtain on using condition (2.15) with the hypothesis of Theorem 2.6,

$$
\begin{aligned}
S_3 &\leq \frac{p!|\tau|(A-B)(1+\gamma(p+1-m))}{(p+1-m)!} \\
&\times \sum_{n=1}^{\infty} \frac{(\lambda+n)(n+p+1-m)!}{(n+p-1)!(1+\gamma(n+p+1-m))} \frac{\prod_{i=1}^{r} \frac{\Gamma(|a_i|+A_i n)}{\Gamma(|a_i|)}}{\prod_{i=1}^{s} \frac{\Gamma(\mathrm{Re}(b_i)+B_i n)}{\Gamma(\mathrm{Re}(b_i))}} \cdot \frac{1}{n!} \\
&= \lambda p|\tau|(A-B)\left[\frac{(p-1)!(1+\gamma(p+1-m))}{\lambda(p+1-m)!} \frac{\prod_{i=1}^{s} \Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r} \Gamma(|a_i|)}\right. \\
&\quad \left.\sum_{n=0}^{\infty} \frac{(\lambda+n)(n+p+1-m)!}{(n+p-1)!(1+\gamma(n+p+1-m))n!} \frac{\Gamma(|a_i|+A_i n)}{\prod_{i=1}^{s} \Gamma(\mathrm{Re}(b_i)+B_i n)} - 1\right] \\
&= \lambda p|\tau|(A-B)\left[\frac{(p-1)!(1+\gamma(p+1-m))}{\lambda(p+1-m)!} \frac{\prod_{i=1}^{s} \Gamma(\mathrm{Re}(b_i))}{\prod_{i=1}^{r} \Gamma(|a_i|)}\right. \\
&\quad {}_{m+r}\psi_{m+s}\left(\begin{array}{c} (|a_i|, A_i)_{1,r}, (\lambda+1,1), (p-1,1), ..., (p-m+2,1), \\ (\mathrm{Re}(b_i), B_i)_{1,s}, (\lambda,1), (p,1), ..., (p-m+3,1), \end{array}\right. \\
&\quad\quad \left.\left. \begin{array}{c} (1+\gamma(p+1-m),\gamma) \\ (2+\gamma(p+1-m),\gamma) \end{array} ;1\right) - 1\right] \\
&\leq \lambda p,
\end{aligned}
$$

which proves Theorem 2.6. □

Remark 2.1 Note that if we set $p=1$ and $m=2$ in Theorem 2.4, 2.5 and 2.6, we get the mapping results proved earlier by Sharma and Raina in [19] which generalizes several more results mentioned in [19].

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References

1. Ahuja, O. P., *Planar harmonic convolution operators generated by hypergeometric functions*, Integral Transforms Spec. Funct. **18** (2007), no. 3-4, 165–177.
2. Ahuja, O. P., *Harmonic starlikeness and convexity of integral operators generated by hypergeometric series*, Integral Transforms Spec. Funct. **20** (2009), no. 7-8, 629–641.
3. Ahuja, O. P., Sharma, P., *Inclusion theorems involving Wright's generalized hypergeometric functions and harmonic univalent functions*, Acta Univ. Apulensis Math. Inform. No. 32 (2012), 109–126.
4. Ahuja, O. P., Silverman, H., *Inequalities associating hypergeometric functions with planar harmonic mappings*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 4, Article 99, 10 pp.
5. Chandrasekran, K., Prabhakaran, D. J., *New Hohlov Type Integral Operator involving Clausen's Hypergeometric Functions*, arXiv preprint arXiv:2205.13389, 2022.
6. Dashrath, *On some classes related to spiral-like univalent and multivalent functions*, Ph. D. Thesis, Kanpur University, Kanpur, 1984.
7. Dixit, K. K., Pal, S. K., *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math. **26** (1995), no. 9, 889–896.
8. Gangadharan, A., Shanmugam, T. N., and Srivastava, H. M., *Generalized hypergeometric functions associated with k-uniformly convex functions*, Comput. Math. Appl. **44** (2002), no. 12, 1515–1526.
9. Kilbas, A. A., Saigo, M. and Trujillo, J. J., *On the generalized Wright function*, Fract. Calc. Appl. Anal. **5** (2002), no. 4, 437–460.

10. Padmanabhan, K. S. *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math. **23** (1970/71), 73–81.

11. Ponnusamy,S. S., Rønning, F., *Starlikeness properties for convolutions involving hypergeometric series*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **52** (1998), no. 1, 141–155.

12. Raina, R. K., Sharma, P., *Harmonic univalent functions associated with Wright's generalized hypergeometric functions*, Integral Transforms Spec. Funct. **22** (2011), no. 8, 561–572.

13. Srivastava, H. M., Manocha, H. L., *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood limited, Chichester) (1984).

14. Sharma, A. K., Porwal, S. and Dixit, K. K., *Class mappings properties of convolutions involving certain univalent functions associated with hypergeometric functions*, Electronic J. Math. Anal. Appl 1.2 (2013): 326–333.

15. Srivastava, H. M., Murugusundaramoorthy, G. and Sivasubramanian, S., *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transforms Spec. Funct. **18** (2007), no. 7-8, 511–520.

16. Panigrahi, T., El-Ashwah, R., *Mapping properties of certain linear operator associated with hypergeometric functions*, Bol. Soc. Parana. Mat. (3) **39** (2021), no. 2, 223–236.

17. Rogosinski, W. W., *On the coefficients of subordinate functions*, Proc. London Math. Soc. (2) **48** (1943), 48–82.

18. Sharma, P., *Wgh inequalities for certain mappings of a linear operator involving Wright's generalized hypergeometric (Wgh) functions*, An. Univ. Oradea Fasc. Mat. **19** (2012), no. 1, 217–230.

19. Sharma, P., Raina, R. K., *On a generalized hypergeometric operator and its mapping structures*, Proc. Jangjeon Math. Soc. **19** (2016), no. 3, 545–562.

20. Swaminathan, A. *Sufficient conditions for hypergeometric functions to be in a certain class of analytic functions*, Comput. Math. Appl. **59** (2010), no. 4, 1578–1583.

21. Wang, L.-M., *Mapping properties of the zero-balanced hypergeometric functions*, J. Math. Anal. Appl. **505** (2022), no. 1, Paper No. 125448, 13 pp.

22. Wright, E. M., *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. (2) **46** (1940), 389–408.

23. Yee, T.-L., Ho, K.-P., *Hardy's inequalities and integral operators on Herz-Morrey spaces*, Open Math. **18** (2020), no. 1, 106–121.

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