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Common Solution for a Finite Family of Equilibrium Problems, Finite Family of Inclusion Problems and Fixed Points of a Nonexpansive Mapping in Hadamard Manifolds

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ABSTRACT: The purpose of this paper is to showcase an iterative algorithm and demonstrate that the sequence produced by it converges robustly to a common solution of a finite collection of equilibrium problems, a finite collection of quasi-variational inclusion problems, and a set of fixed points of a nonexpansive mapping.

Key Words: Hadamard Manifolds, equilibrium problems, inclusion problems.

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1. Introduction

In 1976, Rockafellar [21] studied the inclusion problem of finding

$$\eta^\dagger \in S^{-1}(0) \tag{1.1}$$

where S is a maximal monotone set-valued mapping, the author has devised a method called the proximal point method to tackle the inclusion problem (1.1) in a Hilbert space M. Over the years, due to its practical uses in various fields such as science, engineering, management, and social sciences, the inclusion problem has been extended and generalized in many ways, as seen in references [5,11,22,17,18,19,20]. Additionally, in recent times, several authors have expanded the outcomes obtained through the proximal point algorithm from classical spaces to Hadamard manifolds, as demonstrated in references [13,1].

In 2019, Al-Homidan et. al. [1] considered the problem of finding

$$\eta^\dagger \in F(S)\bigcap(G+H)^{-1}(0),$$

in Hadamard manifold, where S, H and G are nonexpansive, set-valued maximal monotone and single-valued continues and monotone mappings, respectively.

Let M be a Hilbert space, $\mathcal{E} \neq \emptyset$ a subset of M and $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction. A wide range of optimization problems, including variational inequality, convex minimization, fixed point, and Nash equilibrium problems, can be formulated as an equilibrium problem linked with the bifunction F and the set $\mathcal{E}$ [3,16]

$$\text{find } \eta \in \mathcal{E} \text{ such that } F(\eta, \varsigma) \geq 0 \text{ for all } \varsigma \in \mathcal{E}.$$

A point $\eta \in \mathcal{E}$ solving this problem is called an equilibrium point. The set of equilibrium points is denoted by $EP(F)$. There is a vast number of algorithms present in the literature that examine the existence and approximation of solutions to equilibrium problems in linear spaces. Colao et. al. [7] and Khammahawong et. al. [12] recently studied the equilibrium theory in Hadamard manifolds and, under appropriate conditions, demonstrated the existence of equilibrium points for a bifunction. They also provided applications to variational inequality, fixed point, and Nash equilibrium problems.

Zhu et. al. [24] presented an iterative algorithm for finding a common solution for a finite family of equilibrium problems, quasi-variational inclusion problems and fixed points of a nonexpansive mapping

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on Hadamard Manifolds and presented some strong convergence results they considered the following problem of finding

$$\eta^{\dagger} \in \bigcap_{i=1}^{m} EP\left(F_i\right) \bigcap (G+H)^{-1}(0) \bigcap F(S) \tag{1.2}$$

in a Hadamard manifold, where G, H and S are same as defined above.

In this paper, we consider the following common solution for a finite family of equilibrium problems, finite family of inclusion problems and fixed points of a nonexpansive mapping in Hadamard Manifolds, i.e. to obtain $\eta^{\dagger} \in \mathcal{E}$ such that

$$\eta^{\dagger} \in \bigcap_{i=1}^{m} EP(F_i) \bigcap_{i=1}^{m} (G_i+H_i)^{-1}(0) \bigcap F(S). \tag{1.3}$$

In this paper, we present an algorithm and prove that the sequence generated by the algorithm converges strongly which is the common solution of problem (1.3). In this way, we extend some results in the literature.

2. Preliminaries

Assume that Σ is a finite dimensional differentiable manifold, then for any $k \in \Sigma$ we denote by $T_k\Sigma$ the tangent space of Σ at k which is a vector space of the same dimension as Σ and we denote the tangent bundle of Σ by $T\Sigma = \bigcup_{k \in \Sigma} T_k\Sigma$. An inner product $\mathcal{R}_k(\cdot,\cdot)$ defined on the tangent space $T_k\Sigma$ is said to be a Riemannian metric on the tangent space $T_k\Sigma$. To become a Riemannian manifold we assume that Σ can be endowed with a Riemannian metric $\mathcal{R}_k(\cdot,\cdot)$. We denote the corresponding norm to the inner product on $T_k\Sigma$ by $\|\cdot\|_k$. A manifold Σ is said to be a Riemannian manifold if it is differentiable endowed with a Riemannian metric $\mathcal{R}(\cdot,\cdot)$. We define the length of piecewise smooth curve $\Lambda : [0,1] \to \Sigma$ joining k to l (i.e. $\Lambda(0) = k$ and $\Lambda(1) = l$) by $L(\Lambda) = \int_0^1 \|\Lambda^{'}(t)\| dt$. The minimal length over the set of all such curves joining k to l, which includes the original topology on Σ is called the Riemannian distance $d(k,l)$. A Riemannian manifold Σ is said to be complete if for all $k \in \Sigma$, all the geodesic emerging from k are defined $\forall\ t \in \mathbb{R}$. We say that a geodesic joining k to l in Σ is minimal if its length is equal to $d(k,l)$. A Riemannian manifold Σ with the Riemannian distance d is a metric space (Σ, d). According to the Hopf-Rinow Theorem [23], if the Riemannian manifold Σ is complete, then all the pair of points in Σ can be joined by a minimal geodesic. Moreover, the metric space (Σ, d) is complete and its closed and bounded subsets are compact.

Definition 2.1 *Suppose Σ is a given complete Riemannian manifold. We define the exponential map $\exp_k : T_k\Sigma \to \Sigma$ at point $k \in \Sigma$ by $\exp_k v = \Lambda_v(1,k)\ \forall\ v \in T_k\Sigma$, where $\Lambda_v(\cdot,k)$ is the geodesic with the velocity v and starting from the point k i.e. $\Lambda_v'(0,k) = v$ and $\Lambda_v(0,k) = k$.*

It is also known that for any $t \in \mathbb{R}$ the exponential map $\exp_k tv = \Lambda_v(t,k)$. Hence one can easily see that for any zero tangent vector 0, the exponential map $\exp_k 0 = \Lambda_v(0,k) = k$. We also note that $\exp_k$ is differentiable on the tangent space $T_k\Sigma$ for all $k \in \Sigma$. Also, $d(k,l) = \|\exp_k^{-1} l\|$ for all $k,l \in \Sigma$.

Definition 2.2 *A Riemannian manifold of non positive sectional curvature is said to be a Hadamard Manifold if it is simply connected and complete.*

Proposition 2.1 [23]. *Suppose Σ be any given Hadamard manifold. Then $\exp_k : T_k\Sigma \to \Sigma$ is a diffeomorphism for any $k \in \Sigma$, and for any pair of points $k,l \in \Sigma$, $\exists$ a unique normalized geodesic $\Lambda : [0,1] \to \Sigma$ joining points $k = \Lambda(0)$ to $l = \Lambda(1)$, in fact which is a minimal geodesic defined as*

$$\Lambda(t) = \exp_k t \exp_k^{-1} l \text{ for all } 0 \le t \le 1.$$

Lemma 2.1 [4]. *Suppose Σ be any given finite dimensional Hadamard manifold.*

(1) *Suppose* $\Lambda : [0,1] \to \Sigma$ *be any geodesic joining points* η *to* ς. *Then*

$$d(\Lambda(t_1), \Lambda(t_2)) = |t_1 - t_2| d(\eta, \varsigma) \ \forall t_1, t_2 \in [0,1].$$

(2) *The following hold true for any* $z, u, \eta, \varsigma, w \in \Sigma$, $0 \leq t \leq 1$:

$$d(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, z) \leq t d(\eta, z) + (1-t) d(\varsigma, z);$$

$$d^2(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, z) \leq t d^2(\eta, z) + (1-t) d^2(\varsigma, z) - t(1-t) d^2(\eta, \varsigma);$$

$$d(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, \exp_u (1-t) \exp_u^{-1} \zeta) \leq t d(\eta, u) + (1-t) d(\varsigma, \zeta).$$

A subset $\mathcal{E}$ of a Hadamard manifold Σ is geodesic convex if for all $\eta, \varsigma \in \mathcal{E}$, the geodesic joining points η to ς is also contained in $\mathcal{E}$.

Now onwards we assume that the Hadamard manifold Σ is finite dimensional, and $\mathcal{E}$ is a geodesic convex, bounded, nonempty, and closed subset in Σ and $F(S)$ is fixed point set of the mapping S.

Any function $h : \mathcal{E} \to (-\infty, \infty)$ is called geodesic convex if, $\forall\ \nu \in [0,1]$ the geodesic $\Lambda(\nu)$ joining points $\eta, \varsigma \in \mathcal{E}$, the function $h \circ \Lambda$ is convex, i.e.

$$h(\Lambda(\nu)) \leq \nu h(\Lambda(0)) + (1-\nu) h(\Lambda(1)) = \nu h(\eta) + (1-\nu) h(\varsigma).$$

Definition 2.3 *Suppose* M *be any given complete metric space and* $\mathcal{E} \neq \emptyset$ *a subset of* M. *The sequence* $\{\eta_n\}$ *is said to be Fejer monotone with respect to the subset* $\mathcal{E}$ *if* $\forall\ \varsigma \in \mathcal{E}$, $0 \leq n$,

$$d(\eta_n, \varsigma) \geq d(\eta_{n+1}, \varsigma).$$

Lemma 2.2 [10]. *Suppose* M *be any given complete metric space and* $\mathcal{E} \neq \emptyset$ *a subset of* M. *If* $\{\eta_n\} \subset M$ *be a Fejer monotone with respect to* $\mathcal{E}$, *then* $\{\eta_n\}$ *is bounded. Furthermore, if a cluster point* η *of the sequence* $\{\eta_n\}$ *belongs to* $\mathcal{E}$, *then* $\{\eta_n\}$ *converges to* η.

Definition 2.4 *A mapping* $S : \mathcal{E} \to \mathcal{E}$ *is said to be*

1. *nonexpansive if*

$$d(S(\eta), S(\varsigma)) \leq d(\eta, \varsigma) \text{for all } \eta, \varsigma \in \mathcal{E},$$

2. *firmly nonexpansive, if* $\forall\ \eta, \varsigma \in \mathcal{E}$, *the function* $\phi : [0,1] \to [0, \infty]$ *defined as*

$$\phi(t) = d\left(\exp_\eta t \exp_\eta^{-1} S(\eta), \exp_\varsigma t \exp_\varsigma^{-1} S(\varsigma)\right) \text{ for all } 0 \leq t \leq 1$$

is nonincreasing [23].

Proposition 2.2 [14]. *Suppose* $S : \mathcal{E} \to \mathcal{E}$ *be any mapping, then these following are equivalent.*

(1) S *is a firmly nonexpansive mapping;*

(2) $\forall\ \eta, \varsigma \in \mathcal{E}$, $0 \leq t \leq 1$

$$d(S(\eta), S(\varsigma)) \leq d\left(\exp_\eta t \exp_\eta^{-1} S(\eta), \exp_\varsigma t \exp_\varsigma^{-1} S(\varsigma)\right);$$

(3) $\forall\ \eta, \varsigma \in \mathcal{E}$

$$\mathcal{R}\left(\exp_{S(\eta)}^{-1} S(\varsigma), \exp_{S(\eta)}^{-1} \eta\right) + \mathcal{R}\left(\exp_{S(\varsigma)}^{-1} S(\eta), \exp_{S(\varsigma)}^{-1} \varsigma\right) \leq 0.$$

Lemma 2.3 [6]. *Suppose* $S : \mathcal{E} \to \mathcal{E}$ *be any given firmly nonexpansive mapping with* $F(S) \neq \emptyset$, *then for all* $\eta \in \mathcal{E}$, $k \in F(S)$ *the following condition holds true,*

$$d^2(S(\eta), k) \leq d^2(\eta, k) - d^2(S(\eta), \eta).$$

In the continuation, suppose $G : \Sigma \to T\Sigma$ be a single valued vector field in such a way that $G(\eta) \in T_\eta\Sigma$, $\forall\ \eta \in \Sigma$, we denote the set of all single valued vector fields by $\Theta(\Sigma)$. Suppose that domain $D(G)$ of vector field G is defined as

$$D(G) = \{\eta \in \Sigma : G(\eta) \in T_\eta\Sigma\}.$$

Definition 2.5 [15]. *A single valued vector field $G : \Sigma \to T\Sigma$ is monotone if*

$$\langle G(\varsigma), -\exp_\varsigma^{-1}\eta\rangle \geq \langle G(\eta), \exp_\eta^{-1}\varsigma\rangle \text{ for all } \eta, \varsigma \in \Sigma.$$

Suppose $H : \Sigma \to 2^{T\Sigma}$ is a set valued vector field in such a way that $H(\eta) \subset T_\eta\Sigma$, $\forall\ \eta \in \Sigma$ and we denote the set of all set valued vector fields by $\mathcal{X}(\Sigma)$. Suppose that domain $D(H)$ of set valued vector field H is defined as

$$D(H) = \{\eta \in \Sigma : \emptyset \neq B(\eta)\}.$$

Definition 2.6 [9]. *A set valued vector field $H : \Sigma \to 2^{T\Sigma}$ is called*

1. *monotone if $\forall\ \eta, \varsigma \in D(H)$*

$$\mathcal{R}\left(v, -\exp_\varsigma^{-1}\eta\right) \geq \mathcal{R}\left(u, \exp_\eta^{-1}\varsigma\right) \text{ for all } v \in H(\varsigma), \text{ for all } u \in H(\eta);$$

2. *maximal monotone if the mapping is monotone and $\forall\ \eta \in D(H)$, $u \in T_\eta\Sigma$, following assumption*

$$\mathcal{R}\left(v, -\exp_\varsigma^{-1}\eta\right) \geq \mathcal{R}\left(u, \exp_\eta^{-1}\varsigma\right) \text{ for all } v \in H(\varsigma), \text{ for all } \varsigma \in D(H),$$

implies that $u \in H(\eta)$.

3. *For any given positive ν, the resolvent of set valued vector field H of the order ν is also a set valued mapping $J_\nu^H : \Sigma \to 2^{T\Sigma}$ defined as*

$$J_\nu^H(\eta) = \{z \in \Sigma : \eta \in \exp_z \nu H(z)\} \text{ for all } \eta \in \Sigma.$$

Theorem 2.1 [14]. *Suppose $H \in \mathcal{X}(\Sigma)$. The following statements hold for a given positive ν*

(1) *the given set valued vector field H is monotone iff J_ν^H is a single valued and firmly nonexpansive mapping;*

(2) *if $D(H) = \Sigma$, the set valued vector field H is maximal monotone iff J_ν^H is a single valued firmly nonexpansive mapping and domain $D(J_\nu^H) = \Sigma$.*

Proposition 2.3 [14]. *Suppose $\mathcal{E} \neq \emptyset$ be a subset of Σ, $S : \mathcal{E} \to \Sigma$ a firmly nonexpansive mapping. Then*

$$\mathcal{R}\left(\exp_{S(\varsigma)}^{-1}\eta, \exp_{S(\varsigma)}^{-1}\varsigma\right) \leq 0$$

holds $\forall\ \eta \in F(S)$, $\varsigma \in \mathcal{E}$.

Lemma 2.4 [1]. *Suppose $\mathcal{E} \neq \emptyset$ be a closed subset of Σ, $H : \Sigma \to 2^{T\Sigma}$ a maximal monotone set valued vector field. Suppose $\{\nu_n\}$ be a sequence of positive real numbers along with $\lim_{n\to\infty} \nu_n = \nu > 0$, a sequence $\{\eta_n\} \subset \mathcal{E}$ along with $\lim_{n\to\infty} \eta_n = \eta \in \mathcal{E}$ in such a way that $\lim_{n\to\infty} J_{\nu_n}^H(\eta_n) = \varsigma$. Then we get $\varsigma = J_\nu^H(\eta)$.*

Proposition 2.4 [2]. *Suppose $G : \Sigma \to T\Sigma$ be a given single valued monotone, $H : \Sigma \to 2^{T\Sigma}$ a given set valued maximal monotone vector field. Then $\forall\ \eta \in \mathcal{E}$, following conditions are equivalent*

(1) *$\eta \in (G + H)^{-1}(0)$;*

(2) *$\eta = J_\nu^H(\exp_\eta(-\nu G(\eta)))$ for all $\nu > 0$.*

Proposition 2.5 *Suppose $G_i : \Sigma \to T\Sigma$ be a given family of single valued monotone, $H_i : \Sigma \to 2^{T\Sigma}$ a given family of set valued maximal monotone vector fields. Then $\forall\ \eta \in \mathcal{E}$, following conditions are equivalent*

(1) $\eta \in (G_i + H_i)^{-1}(0)$;

(2) $\eta = J_{\nu}^{H_i}(\exp_{\eta}(-\nu G_i(\eta)))$ *for all* $\nu > 0$.

Proof: (1) $\iff$ (2)

$$\begin{aligned} &\eta = J_{\nu}^{H_i}(\exp_{\eta}(-\nu G_i(\eta))) \\ &\iff \exp_{\eta}(-\nu G_i(\eta)) \in \exp_{\eta}(\nu H_i(\eta)) \\ &\iff -\nu G_i(\eta) \in \nu H_i(\eta) \\ &\iff \eta \in (G_i + H_i)^{-1}(0). \end{aligned}$$

□

Suppose $\mathcal{E} \neq \emptyset$ be a geodesic convex and closed set in Σ, $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction satisfying given following suppositions:

(A_1) $\forall\, \eta \in \mathcal{E}$, $0 \leq F(\eta, \eta)$;

(A_2) F is monotone, i.e. $\forall\, \eta, \varsigma \in \mathcal{E}$, $F(\varsigma, \eta) + F(\eta, \varsigma) \leq 0$;

(A_3) $\forall\, \varsigma \in \mathcal{E}$, $\eta \mapsto F(\eta, \varsigma)$ is upper semicontinuous;

(A_4) $\forall\, \eta \in \mathcal{E}$, $\varsigma \mapsto F(\eta, \varsigma)$ is lower semicontinuous and geodesic convex;

(A_5) $\eta \mapsto F(\eta, \eta)$ is lower semicontinuous;

(A_6) $\exists$ a compact set $L \subseteq \Sigma$ in such a way that $\eta \in \mathcal{E}/L \implies \exists\, \varsigma \in \mathcal{E} \cap L$ in such a way that $F(\eta, \varsigma) < 0$.

Definition 2.7 [8]. *Suppose* $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be a given bifunction. Then the resolvent of bifunction* F *is a multivalued mapping* $T_r^F : \Sigma \to 2^{\mathcal{E}}$ *defined as*

$$T_r^F(\eta) = \{z \in \mathcal{E} : 0 \leq F(z, \varsigma) - \frac{1}{r}\langle \exp_z^{-1}\eta, \exp_z^{-1}\varsigma\rangle \text{ for all } \varsigma \in \mathcal{E}\}.$$

Theorem 2.2 [8]. *Suppose* $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be a given bifunction satisfying following assertions:*

(1) *the given bifunction* F *is monotone;*

(2) $\forall\, r > 0$, T_r^F *is properly defined, i.e. the domain* $D(T_r^F) \neq \emptyset$. *Then* $\forall\, r > 0$,

  (a) T_r^F *is a single valued mapping;*

  (b) T_r^F *is a firmly nonexpansive mapping;*

  (c) *the set of fixed points of mapping* T_r^F *is the set of equilibrium points of bifunction* F *i.e.,*

$$EP(F) = F(T_r^F).$$

Theorem 2.3 [14]. *Suppose* $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be a given bifunction satisfying the above suppositions* (A_1)*-*(A_3)*. Then* $D(T_r^F) = \Sigma$.

Theorem 2.4 [14]. *Suppose* $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be a given bifunction satisfying the above suppositions* (A_1),(A_3),(A_4),(A_5)*, and* (A_6)*. Then* $\exists\, z \in \mathcal{E}$ *in such a way that*

$$0 \leq F(z, \varsigma) - \frac{1}{r}\langle \exp_z^{-1}\eta, \exp_z^{-1}\varsigma\rangle \text{ for all } \varsigma \in \mathcal{E}$$

$\forall\, r > 0$, $\eta \in \Sigma$.

3. Main Results

In the sequel, we always assume that

1. $\mathcal{E} \neq \emptyset$ is a closed bounded geodesic convex subset of a Hadamard manifold Σ;

2. $H_i : \mathcal{E} \to 2^{T\Sigma},\ i = 1, 2, \cdots m$ is a family of maximal monotone setvalued vector fields;

3. $G_i : \mathcal{E} \to T\Sigma,\ i = 1, 2, \cdots m$ is a family of monotone and continuous single valued vector fields satisfying following condition:

$$d(\exp_\eta(-\nu G_i(\eta)), \exp_\varsigma(-\nu G_i(\varsigma))) \leq (1-\rho) d(\eta, \varsigma)\ \forall \eta, \varsigma \in \mathcal{E}, 0 < \nu, 0 \leq \rho \leq 1. \tag{3.1}$$

4. $S : \mathcal{E} \to \mathcal{E}$ is a nonexpansive mapping;

5. $F_i : \mathcal{E} \times \mathcal{E} \to \mathbb{R},\ i = 1, 2, \cdots, m,$ is a finite family of bifunctions satisfying the above suppositions (A_1) - (A_6), for any given $0 < r$, the resolvent of family of bifunctions F_i is multivalued mapping $T_r^{F_i} : \Sigma \to 2^{\mathcal{E}}$ in such a way that $\forall\ \eta \in \Sigma$

$$T_r^{F_i}(\eta) = \{z \in \mathcal{E} : 0 \leq F_i(z, \varsigma) - \frac{1}{r}\langle \exp_z^{-1} \eta, \exp_z^{-1} \varsigma\rangle, \forall\ \varsigma \in \mathcal{E}\}, i = 1, 2, \cdots, m.$$

6. Denote by

$$S_r^j = T_r^{F_j} \circ T_r^{F_{j-1}} \circ \cdots T_r^{F_2} \circ T_r^{F_1},\ j = 1, 2, \cdots, m.$$

Theorem 3.1 *Suppose $\mathcal{E}, \Sigma, G_i, H_i, \{F_i\}_{i=1}^m, \{S_r^j\}_{r=1}^m$ and S be the same as defined above. Suppose $\{\eta_n\}, \{u_n\}, \{\varsigma_n\}$ and $\{z_n\}$ are the sequences generated by $\eta_0 \in \mathcal{E}$*

$$\begin{cases} u_n^i = J_{\nu_n}^{H_i}(\exp_{\eta_n}(-\nu_n G_i(\eta_n))), \\ w_n \in \{u_n^i,\ i = 1, 2, \cdots m\} \text{ such that } d(w_n, \eta_n) = \max\limits_{1 \leq i \leq m} d(u_n^i, \eta_n), \\ \varsigma_n = \exp_{\eta_n} \vartheta_n \exp_{\eta_n}^{-1} S(w_n), \\ z_n = S_r^m(\varsigma_n), \\ \eta_{n+1} = \exp_{\eta_n} \Upsilon_n \exp_{\eta_n}^{-1}(z_n),\ \forall n \geq 0, \end{cases} \tag{3.2}$$

where $\forall\ n \in \mathbb{N}$, $\{\vartheta_n\}$, $\{\Upsilon_n\}$, $\{\nu_n\}$ are the given sequences satisfying these following conditions:

(a) $0 < a \leq \vartheta_n, \Upsilon_n \leq b < 1$,

(b) $0 < \hat{\nu} \leq \nu_n \leq \tilde{\nu} < \infty$,

(c) $\sum\limits_{n=1}^{\infty} \vartheta_n \Upsilon_n = \infty$.

If $\Theta = \bigcap\limits_{i=1}^m EP(F_i) \bigcap\limits_{i=1}^m (G_i + H_i)^{-1}(0) \bigcap F(S) \neq \emptyset$, then the sequence $\{\eta_n\}$ converges strongly to a solution of problem (1.3).

Proof: Suppose $\Lambda_n : [0, 1] \to \Sigma$ be the geodesic joining $\Lambda_n(0) = \eta_n$ to $\Lambda_n(1) = z_n$, and $\hat{\Lambda}_n : [0, 1] \to \Sigma$ be the geodesic joining $\hat{\Lambda}_n(0) = \eta_n$ to $\hat{\Lambda}_n(1) = S(u_n^i)$ then we can write $\{\eta_{n+1}\}$ as $\eta_{n+1} = \Lambda_n(\Upsilon_n)$, and $\varsigma_n = \hat{\Lambda}_n(\vartheta_n)$.

First we prove that Θ is geodesic convex and closed.

Since all the nonexpansive mappings are continuous, hence $F(S)$ is closed. Now we show $F(S)$ is geodesic convex.

Suppose $k, l \in F(S)$, to prove $F(S)$ is geodesic convex, we have to show that geodesic $\Lambda : [0,1] \to \Sigma$ joining points k and l is also contained in $F(S)$. We know that in a given Hadamard manifold Σ, $\forall$ $k, l \in \Sigma$, $0 \leq t \leq 1$, $\exists$ a unique point $\Lambda(t) = \exp_k t \exp_k^{-1} l = \zeta_t$ such that

$$d(k,l) = d(k,\zeta_t) + d(\zeta_t, l).$$

Using the geodesic convexity of Riemannian distance, nonexpansiveness of S we get

$$d(k, S(\zeta_t)) = d(S(k), S(\zeta_t)) \leq d(k,\zeta_t) = d(k,\Lambda(t)) \leq td(k,l).$$

Similarly we can also get

$$d(S(\zeta_t), l) \leq (1-t)d(k,l).$$

Using above equations we get

$$d(k,l) \leq d(k, S(\zeta_t)) + d(S(\zeta_t), l) \leq d(k,l).$$

Hence

$$d(k,l) = d(k, S(\zeta_t)) + d(S(\zeta_t), l).$$

Since ζ_t is unique, hence we have $\zeta_t = S(\zeta_t)$. Hence, $\Lambda(t) = \zeta_t \in F(S)$. Therefore $F(S)$ is geodesic convex.

Now using Proposition 2.5 we can say $(G_i + H_i)^{-1}(0) = F(J_\nu^{H_i}(\exp(-\nu G_i)))$. Since $J_\nu^{H_i}$ is nonexpansive mapping using this together assumption (3) we can easily get $J_\nu^{H_i}(\exp(-\nu G_i))$ is also a nonexpansive mapping. Hence $(G_i + H_i)^{-1}(0)$ is also geodesic convex and closed in Σ.

Now, using Theorem 2.2, we say $T_r^{F_i}$ is a firmly nonexpansive mapping and $F(T_r^{F_i}) = EP(F_i)$. Therefore, $EP(F_i)$ is also geodesic convex and closed in Σ, and hence Θ is geodesic convex and closed.

Now we show that the sequence $\{\eta_n\}$ is Fejer monotone with respect to Θ.

Suppose $\zeta \in \Theta$

$$\begin{aligned} d(u_n^i, \zeta) &= d(J_{\nu_n}^{H_i}(\exp_{\eta_n}(-\nu_n G_i(\eta_n))), \zeta) \\ &= d(J_{\nu_n}^{H_i}(\exp_{\eta_n}(-\nu_n G_i(\eta_n))), J_{\nu_n}^{H_i}(\exp_\zeta(-\nu_n G_i(\zeta)))) \\ &\leq d(\exp_{\eta_n}(-\nu_n G_i(\eta_n)), \exp_\zeta(-\nu_n G_i(\zeta))) \\ &\leq (1-\rho)d(\eta_n, \zeta) \leq d(\eta_n, \zeta). \end{aligned}$$

Since $\zeta \in \Theta$ using Theorem 2.2, we say $T_r^{F_i}$ is firmly nonexpansive and hence $T_r^{F_i}$ is nonexpansive therefore S_r^m is also nonexpansive, and $\zeta \in F(S_r^m)$, we get

Now,

$$\begin{aligned} d^2(\varsigma_n, \zeta) &= d^2(\exp_{\eta_n} \vartheta_n \exp_{\eta_n}^{-1} S(w_n), \zeta) \\ &\leq (1-\vartheta_n)d^2(\eta_n,\zeta) + \vartheta_n d^2(S(w_n),\zeta) - \vartheta_n(1-\vartheta_n)d^2(\eta_n, S(w_n)) \\ &\leq (1-\vartheta_n)d^2(\eta_n,\zeta) + \vartheta_n d^2(w_n,\zeta) - \vartheta_n(1-\vartheta_n)d^2(\eta_n, S(w_n)) \\ &\leq (1-\vartheta_n)d^2(\eta_n,\zeta) + \vartheta_n d^2(w_n,\zeta) \\ &\leq (1-\vartheta_n)d^2(\eta_n,\zeta) + \vartheta_n d^2(u_n^i,\zeta) \\ &\leq (1-\vartheta_n)d^2(\eta_n,\zeta) + \vartheta_n d^2(\eta_n,\zeta) = d^2(\eta_n,\zeta). \end{aligned}$$

Thus

$$d(\varsigma_n, \zeta) \leq d(\eta_n, \zeta).$$

$$d(z_n, \zeta) = d(S_r^m(\varsigma_n), S_r^m(\zeta)) \leq d(\varsigma_n, \zeta) \leq d(\eta_n, \zeta).$$

And

$$\begin{aligned}
d^2(\eta_{n+1},\zeta) &= d^2(\exp_{\eta_n}\Upsilon_n \exp_{\eta_n}^{-1} z_n,\zeta)\\
&\le (1-\Upsilon_n)d^2(\eta_n,\zeta)+\Upsilon_n d^2(z_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\le (1-\Upsilon_n)d^2(\eta_n,\zeta)+\Upsilon_n d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\le d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\le d^2(\eta_n,\zeta).
\end{aligned}$$

Thus

$$d(\eta_{n+1},\zeta)\le d(\eta_n,\zeta) \text{ for all } n\ge 0, \zeta\in\Theta,$$

and hence the sequence $\{\eta_n\}$ is Fejer monotone with respect to Θ. Using Lemma 2.2 implies that sequence $\{\eta_n\}$ is bounded alongwith $\{u_n^i\}$, $\{\varsigma_n\}$, $\{z_n\}$ and $\lim_{n\to\infty} d(\eta_n,\zeta)$ exists for any $\zeta\in\Theta$.

Now we show that $\lim_{n\to\infty} d(\eta_{n+1},\eta_n)=0$.

$$d^2(\eta_{n+1},\zeta)\le d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)$$

$$\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\le d^2(\eta_n,\zeta)-d^2(\eta_{n+1},\zeta)$$

$$a(1-b)d^2(\eta_n,z_n)\le \Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\le d^2(\eta_n,\zeta)-d^2(\eta_{n+1},\zeta)\to 0.$$

Since $a(1-b)>0$, it implies

$$d^2(\eta_n,z_n)\to 0 \implies d(\eta_n,z_n)\to 0.$$

Since $\eta_{n+1}=\Lambda_n(\Upsilon_n)$, we have

$$\begin{aligned}
d(\eta_{n+1},\eta_n) &= d(\Lambda_n(\Upsilon_n),\eta_n)\\
&\le (1-\Upsilon_n)d(\Lambda_n(0),\eta_n)+\Upsilon_n d(\Lambda_n(1),\eta_n)\\
&= (1-\Upsilon_n)d(\eta_n,\eta_n)+\Upsilon_n d(z_n,\eta_n)\\
&= \Upsilon_n d(z_n,\eta_n)\\
&\le bd(z_n,\eta_n).
\end{aligned}$$

Applying limit we get $\lim_{n\to\infty} d(\eta_{n+1},\eta_n)=0$.

Now, we prove that $\lim_{n\to\infty} d(S(u_n^i),\eta_n)=0$, $\lim_{n\to\infty} d(u_n^i,\eta_n)=0$, and $\lim_{n\to\infty} d(S_r^m(\eta_n),\eta_n)=0$.

Now,

$$\begin{aligned}
d(\eta_{n+1},\zeta) &= d(\Lambda_n(\Upsilon_n),\zeta)\\
&\le (1-\Upsilon_n)d(\Lambda_n(0),\zeta)+\Upsilon_n d(\Lambda_n(1),\zeta)\\
&\le (1-\Upsilon_n)d(\eta_n,\zeta)+\Upsilon_n d(z_n,\zeta)\\
&\le (1-\Upsilon_n)d(\eta_n,\zeta)+\Upsilon_n d(\varsigma_n,\zeta).
\end{aligned}$$

And

$$\begin{aligned}
d(\varsigma_n,\zeta) &= d(\hat\Lambda_n(\vartheta_n),\zeta)\\
&\le (1-\vartheta_n)d(\hat\Lambda_n(0),\zeta)+\vartheta_n d(\hat\Lambda_n(1),\zeta)\\
&\le (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(S(u_n^i),\zeta)\\
&\le (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(S(u_n^i),S(\zeta))\\
&\le (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(u_n^i,\zeta)
\end{aligned}$$

Similarly using above two equations, we have

$$\begin{aligned}
d(\eta_n,\zeta) &\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}d(\varsigma_{n-1},\zeta)\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}\{(1-\vartheta_{n-1})d(\eta_{n-1},\zeta)+\vartheta_{n-1}d(u^i_{n-1},\zeta)\}\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}\{(1-\vartheta_{n-1})d(\eta_{n-1},\zeta)+\vartheta_{n-1}(1-\rho)d(\eta_{n-1},\zeta)\}\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}(1-\rho\vartheta_{n-1})d(\eta_{n-1},\zeta)\\
&= (1-\rho\vartheta_{n-1}\Upsilon_{n-1})d(\eta_{n-1},\zeta).
\end{aligned}$$

Since, $\{\eta_n\}$ is a bounded sequence, so $\exists$ a constant Q in such a way that $d(\eta_n,\zeta)\leq Q\ \forall\ 0\leq n$.

$$d(\eta_n,\zeta)\leq (1-\rho\vartheta_{n-1}\Upsilon_{n-1})Q.$$

Suppose $0\leq m\leq n$, we get

$$d(\eta_n,\zeta)\leq Q\prod_{j=m}^{n-1}(1-\rho\vartheta_j\Upsilon_j).$$

Using condition (c) we can get

$$\lim_{n\to\infty}\prod_{j=m}^{n-1}(1-\rho\vartheta_j\Upsilon_j)=0,$$

and hence

$$\lim_{n\to\infty}d(\eta_n,\zeta)=0.$$

Now

$$\begin{aligned}
d(\eta_n,u^i_n) &\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(u^i_n,\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+d(\eta_n,\zeta)+d(\eta_n,\zeta)
\end{aligned}$$

applying $\lim n\to\infty$ we get $\lim\limits_{n\to\infty}d(\eta_n,u^i_n)=0$,

and

$$\begin{aligned}
d(\eta_n,S(u^i_n)) &\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(S(u^i_n),\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(u^i_n,\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+2d(\eta_n,\zeta),
\end{aligned}$$

applying $\lim n\to\infty$ we get $\lim\limits_{n\to\infty}d(\eta_n,S(u^i_n))=0$,

and

$$\begin{aligned}
d(S^m_r(\eta_n),\eta_n) &\leq d(S^m_r(\eta_n),S^m_r(w_n))+d(S^m_r(w_n),\eta_n)\\
&\leq d(\eta_n,w_n)+d(z_n,\eta_n)\\
&\leq d(u^i_n,\eta_n)+d(\eta_n,z_n),
\end{aligned}$$

applying $\lim n\to\infty$ we get $\lim\limits_{n\to\infty}d(S^m_r(\eta_n),\eta_n)=0$.

Now we show that the cluster point $\eta^\dagger$ of sequence $\{\eta_n\}$ belongs to Θ. Since we already proved that sequence $\{\eta_n\}$ is bounded. Therefore $\exists$ a subsequence $\{\eta_{n_j}\}$ of sequence $\{\eta_n\}$ which converges to the cluster point $\eta^\dagger$ of sequence $\{\eta_n\}$. Since $\lim\limits_{n\to\infty}d(\eta_n,u^i_n)=0$ it implies $\lim\limits_{j\to\infty}d(u^i_{n_j},\eta^\dagger)=0$. Using nonexpansiveness of S we get

$$\begin{aligned}
d(\eta^\dagger,S(\eta^\dagger)) &\leq d(\eta^\dagger,\eta_{n_j})+d(\eta_{n_j},S(u^i_{n_j}))+d(S(u^i_{n_j}),S(\eta^\dagger))\\
&\leq d(\eta^\dagger,\eta_{n_j})+d(\eta_{n_j},S(u^i_{n_j}))+d(u^i_{n_j},\eta^\dagger).
\end{aligned}$$

Applying $\lim\limits_{j\to\infty}$ we get

$$d(\eta^\dagger, S(\eta^\dagger)) = 0 \implies \eta^\dagger \in F(S).$$

Now we show that $\eta^\dagger \in \bigcap\limits_{i=1}^{m} EP(F_i)$. We also have for any subsequence $\{\eta_{n_j}\}$ of $\{\eta_n\}$, $\lim\limits_{j\to\infty} d(S_r^m(\eta_{n_j}), \eta_{n_j}) = 0$. We know that the mapping S_r^m is nonexpansive, it is demiclosed at 0 and hence $\eta^\dagger \in F(S_r^m)$. To prove $\eta^\dagger \in \bigcap\limits_{i=1}^{m} EP(F_i)$ we have to prove that $F(S_r^m) = \bigcap\limits_{i=1}^{m} F(T_r^{F_i})$. It is obvious that $\bigcap\limits_{i=1}^{m} F(T_r^{F_i}) \subseteq F(S_r^m)$, we only have to prove that $F(S_r^m) \subseteq \bigcap\limits_{i=1}^{m} F(T_r^{F_i})$.

Let $l \in F(S_r^m)$ and $k \in \bigcap\limits_{i=1}^{m} F(T_r^{F_i})$, and we have

$$\begin{aligned} d(l,k) &= d(S_r^m(l), k) = d(T_r^{F_m} S_r^{m-1}(l), k) \le d(S_r^{m-1}(l), k) \\ &\le d(S_r^{m-2}(l), k) \le \cdots \le d(S_r^1(l), k) = d(T_r^{F_1}(l), k) \le d(l,k). \end{aligned}$$

It implies that

$$\begin{aligned} d(l,k) &= d(S_r^m(l), k) = d(S_r^{m-1}(l), k) \\ &= d(S_r^{m-2}(l), k) = \cdots d(S_r^1(l), k) \\ &= d(T_r^{F_1}(l), k). \end{aligned}$$

Applying Lemma 2.3 we get

$$\begin{aligned} d^2(S_r^i(l), k) = d^2(S_r S_r^{i-1}(l), k) \le d^2(S_r^{i-1}(l), k) - d^2(S_r^i(l), S_r^{i-1}(l)) \\ d^2(S_r^i(l), k) + d^2(S_r^i(l), S_r^{i-1}(l)) \le d^2(S_r^{i-1}(l), k) = d^2(l,k). \end{aligned}$$

Since $d(S_r^i(l), k) = d(l,k)$, from the above equation $\forall\ i = 1, 2, \cdots, m$ we can have

$$d(S_r^i(l), S_r^{i-1}(l)) = 0 = d\left(T_r^{F_i}(l), S_r^{i-1}(l)\right) \implies S_r^{i-1}(l) \in F(T_r^{F_i}). \tag{3.3}$$

Now, if we take $i = 1$ in (3.3), we get $l \in F(T_r^{F_1}) \implies l = T_r^{F_1}(l)$ again taking $i = 2$ in (3.3), we get $l = S_r^1(l) \in F(T_r^{F_2}) \implies l = T_r^{F_2}(l)$. Similarly taking $i = 2, 3, \cdots, m$ in (3.3), we get

$$l = T_r^{F_1}(l) = T_r^{F_2}(l) = \cdots = T_r^{F_{m-1}}(l) = T_r^{F_m}(l).$$

It implies that

$$l \in \bigcap_{i=1}^{m} F(T_r^{F_i})$$

That is

$$F(S_r^m) = \bigcap_{i=1}^{m} F(T_r^{F_i}).$$

Now finally we prove that $\eta^\dagger \in (G_i + H_i)^{-1}(0)$. Since $\hat{\nu} \le \nu_n \le \tilde{\nu}$, we can choose a $\nu > 0$ in such a way that the subsequence $\{\nu_{n_j}\}$ of $\{\nu_n\}$ converges to ν. Since $u_n^i = J_{\nu_n}^{H_i}(\exp_{\eta_n}(-\nu_n G_i(\eta_n)))$. Using Lemma 2.4 and $\lim\limits_{n\to\infty} d(\eta_n, u_n^i) = 0$

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(\eta_n, u_n^i) \\ &= \lim_{j\to\infty} d(\eta_{n_j}, u_{n_j}^i) \\ &= \lim_{j\to\infty} d\left(\eta_{n_j}, J_{\nu_{n_j}}^{H_i}\left(\exp_{\eta_{n_j}}\left(-\nu_{n_j} G_i\left(\eta_{n_j}\right)\right)\right)\right) \\ &= d\left(\eta^\dagger, J_{\nu}^{H_i}\left(\exp_{\eta_{n_j}}\left(-\nu_{n_j} G_i\left(\eta_{n_j}\right)\right)\right)\right) \end{aligned}$$

Using Proposition 2.5 we get $\eta^\dagger \in (G_i + H_i)^{-1}(0)$, and hence $\eta^\dagger \in \Theta$. This completes the proof. $\square$

Corollary 3.1 *Suppose* $\mathcal{E}, \Sigma, G, H, \{F_i\}_{i=1}^{m}, \{S_r^j\}_{j=1}^{m}$ *and* S *the same as above. Suppose* $\{\eta_n\}, \{u_n\}, \{\varsigma_n\}$ *and* $\{z_n\}$ *are the sequences generated by* $\eta_0 \in \mathcal{E}$, $\forall n \geq 0$,

$$\begin{cases} u_n = J_{\nu_n}^{H}\left(\exp_{\eta_n}\left(-\nu_n G\left(\eta_n\right)\right)\right) \\ \varsigma_n = \exp_{\eta_n} \vartheta_n \exp_{\eta_n}^{-1} S u_n \\ z_n = S_r^m\left(\varsigma_n\right) \\ \eta_{n+1} = \exp_{\eta_n} \Upsilon_n \exp_{\eta_n}^{-1} z_n, \end{cases}$$

where $\forall n \in \mathbb{N}$, $\{\vartheta_n\}, \{\Upsilon_n\}$ *and* $\{\nu_n\}$ *are the sequences of positive real numbers satisfying given following assumptions:*

(i) $0 < a \leq \vartheta_n, \Upsilon_n \leq b < 1$;

(ii) $0 < \hat{\nu} \leq \nu_n \leq \tilde{\nu} < \infty$;

(iii) $\sum_{n=1}^{\infty} \vartheta_n \Upsilon_n = \infty$.

If $\Theta = \bigcap_{i=1}^{m} EP\left(F_i\right) \bigcap (G+H)^{-1}(0) \bigcap F(S)$ *is nonempty, therefore the sequence* $\{\eta_n\}$ *converges strongly to solution of the problem* (1.2).

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