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# Estimates for Solution of Regular Elliptic Systems in Besov-Type Spaces

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ABSTRACT: In this paper we show that the method used in the scalar case remains true for systems and thus gives us near-boundary a priori estimates for solutions of regular elliptic systems in Besov-type spaces because the triebel conjecture concerning the estimates for the solutions of the problems with regular elliptic limits in the scalar case in this spaces is completely solved.

Keywords: Elliptic system, Besov-type spaces, a priori-estimate.

## Contents

## 1. Introduction

The purpose of this paper is to give the regularity for the solutions of regular elliptical systems in the type-Besov space $\mathcal{B}^{s,\tau}_{p,q}$. So, this paper is the general case of [10,5] where the author obtained the regularity for the solutions of regular elliptic boundary value problems in the scalar case in the spaces $\mathcal{B}^{s,\tau}_{p,q}$.
Note also that the paper generalizes [9] where the author shows a priori estimates for solutions of regular elliptic systems in BMO spaces and its local version Companato spaces.
We first mention the well-known work for Campanato's variational systems [4] which obtained results concerning local and global regularity for solutions $v \in H^1_0(\Omega,\mathbb{R}^N)$ under Dirichlet limit conditions of second-order strongly linear elliptical systems of the following form:

$$\sum_{i,j=1}^{n}\int_{\Omega} < A_{ij}(x).D_j v \setminus D_i\Phi > dx = \sum_{i=1}^{n}\int_{\Omega} < f_i(x) \setminus D_i\Phi > dx, \qquad \forall \Phi \in \mathcal{C}^{\infty}_0(\Omega,\mathbb{R}^N).$$

In this work, they showed the following regularity result: $||Dv||_{BMO} \leq C||f||_{BMO}$. That is mean if $f \in BMO(\Omega,\mathbb{R}^{nN})$ then $Dv \in BMO(\Omega,\mathbb{R}^{nN})$.

In this paper, we deal with inhomogeneous and non-variational systems. Take for example, the following classical regular elliptic system of the second order

$$\begin{cases} Mv = f \quad in \quad \Omega, \\ v \mid \Omega = \varphi \quad in \quad \Gamma = \partial\Omega. \end{cases}$$

With;

- $M = \Sigma_{|\alpha|\leq 2} a^{ij}_{\alpha}(x) D^{\alpha}_{x}$,

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where $a_{\alpha}^{ij}$ is the $N \times N$, $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $D_x^{\alpha} = D_{x_1}^{\alpha_1} ... D_{x_n}^{\alpha_n}$ is the derivation and $D_{x_j}^{\alpha_j} = \frac{1}{i^{\alpha_j}} \times \frac{\partial^{\alpha_j}}{\partial_{x_j}^{\alpha_j}}$.

- $v,\ fand\ \ \varphi$ are the vector-valued functions in $\mathbb{R}^N$.

- $\Omega$ is an open and bounded regular domain in $\mathbb{R}^N$.

A very precise choice of the indices $s, \tau, p, q$ in Besov type space $\mathcal{B}_{p,q}^{s,\tau}$, allow us to find the BMO space and therefore the work done in [9].
We show that the method used for the scalar case in [10] to estmates the solution of degenerate elliptic problems can be adapted for the case of elliptical systems, in addition, this work generalizes the result of the above example to the elliptic systems in the sense of Douglis and Nirenberg [6] and to the general spaces defined in [7] and [9]. The result proven in this work makes the objective of several applications in geophysics more precisely in the study of Shallow waters.
The plan of the paper is the following: in the second section, we give the main definitions on Besov-type spaces and lemmas which helped us in the proof of our main result as well as we recall some results proved in old papers about derivatives, compactness and interpolation. In section 3, we give a characterization of the traces at the edge of the Besov-type space as well as the main assumptions before stating the result. Finally, section 4 deals with the main result as well as the proof of this result; we follow a method of Peetre used in the scalar case [10]. This method consists of doing a partial Fourier transform with respect to the tangential direction on the system of equations, and reducing our problem to an isomorphism theorem for a system of ordinary differential operators to estimate the tangential derivatives of the solution , then do an interpolation inequality to control the normal derivatives of the solution.

## 2. Mathematical Preliminaries

For our work to be meaningful, we must use a decomposition called Littlewood-Paley of utity: Denote $x = (t, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\xi = (\tau, \xi')$ its a dual variable and let $\varphi \in \mathcal{C}_0^{\infty}, \varphi \geq 0$ such that.

$$(1.1) \begin{cases} \varphi(\xi) = 1 \text{ if } |\xi| \leq 1, \\ \\ \varphi(\xi) = 0 \text{ if } |\xi| \geq 2, \end{cases}$$
$$\text{i.e., } \operatorname{supp}(\varphi) \subset \{\xi \in \mathbb{R}^n / |\xi| \leq 2\}.$$

For $j \in \mathbb{N}$, on pose $\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right)$.
We can verifie easily that for all $\xi \in \mathbb{N}$, we have

$$\sum_{j=0}^{\infty} \theta_j(\xi) = 1,$$

with $\theta_j = \varphi_j - \varphi_{j-1}$.
Multiply the above equality by $\hat{u}$ and apply the inverse Fourier transform, we obtain the formula of nonhomogeneous partition of $u \in \mathcal{S}'(\mathbb{R}^n)$ :

$$\sum_{j=0}^{\infty} \dot{\Delta}_j u = u = \sum_{k \geq 0} \mathcal{F}^{-1}\left[\theta_k \mathcal{F} u\right].$$

We set

$$\begin{aligned} \dot{S}_j = \varphi\left(2^{-j}\left|D_x\right|\right); \dot{S}_j{}' = \varphi\left(2^{-j}\left|D_{x'}\right|\right); \dot{S}_j{}'' = \varphi\left(2^{-j}\left|D_t\right|\right), \\ \dot{S}_{-1} = \dot{S}_{-1}{}' = \dot{S}_{-1}{}'' = 0, \\ and\ \ \dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}; \dot{\Delta}_j' = \dot{S}_j{}' - S_{j-1}'; \dot{\Delta}_j'' = \dot{S}_j{}'' - \dot{S}_{j-1}{}'' \end{aligned}$$

**Definition 2.1** *([10].) Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. We denoted by $\mathcal{B}^s_{p,q}$ the space of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\begin{cases} (i)\ \forall j \in \mathbb{N},\ \dot{\Delta}_j u \in L^p(\mathbb{R}^n), \\ \\ (ii)\ \varepsilon_j = 2^{sj}||\dot{\Delta}_j u||_{L^p} \text{ est dans } l^q. \end{cases}$$

*The space $\mathcal{B}^s_{p,q}$ is associated with the norm:*

$$\| u \|_{\mathcal{B}^s_{p,q}} = \| (\varepsilon_j)_j \|_{l^q} = \begin{cases} (\sum_{j\geq 0} \varepsilon_j^q)^{\frac{1}{q}} = (\sum_{j\geq 0} 2^{sjq}||\dot{\Delta}_j u||^q_{L^p})^{\frac{1}{q}} \quad si\ q < \infty, \\ \\ \sup_{j\leq 0} \varepsilon_j \quad si\ q = \infty. \end{cases}$$

**Definition 2.2** *([8].) Let $s, \tau \in \mathbb{R}$, $p, q \in \left]0, +\infty\right]$. The space of Besov-type is noted $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ is the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{n+1})$ such that $||u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} < \infty$, with*

$$\begin{cases} ||u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} = \sup_B \frac{1}{|B|^\tau} \left\{ \sum_{j\geq J^+} 2^{sjq}||\dot{\Delta}_j u(x)||^q_{L^p} \right\}^{\frac{1}{q}} \quad q < \infty, \\ \\ ||u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} = \sup_B \frac{1}{|B|^\tau} \sup_{j\geq J^+} 2^{sj}||\dot{\Delta}_j u(x)||_{L^p(B)} \quad q = \infty, \end{cases}$$

*where the supremum is taken over all balls $B$ of radius less than $2^{-J}$, $J \in \mathbb{N}$.*

For $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$. We denoted by $L^p(\mathbb{R}; \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}))$ and we not $L^p(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}))$ is the space of functions which of $L^p$ with respect to the first variable $t$ and of type Besov $\mathcal{B}^{s,\tau}_{p,q}$ with respect to the second variable $x'$.

**Definition 2.3** *([8].) Let $s \in \mathbb{R}, \tau \geq 0, 0 < p, q < \infty$, the space of anisotropic Besov-type noted by $L^p(\mathbb{R}; \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}))$ is the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{n+1})$ such that $||u||_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}))} < \infty$, with*

$$\begin{cases} ||u||_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} = \sup\limits_{B'_J} \dfrac{1}{|B'_J|^\tau} \left\{ \sum\limits_{j\geq J^+} 2^{sjq}||\dot{\Delta}'_j u(x)||^q \right\}^{\frac{1}{q}} \quad q < \infty, \\ ||u||_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} = \sup\limits_{B'_J} \dfrac{1}{|B'_J|^\tau} \sup\limits_{j\geq J^+} 2^{sj}||\dot{\Delta}'_j u(x)||_{L^p(B_{J'})} \quad q = \infty, \end{cases}$$

*where the supremum is taken over all balls $B$ of radius less than $2^{-J}$, $J \in \mathbb{N}$.*

**Definition 2.4** *([10].) The classical Sobolev spaces*

$$W^{t_j,p}(\mathbb{R}) = \left\{ u \in L^p(\mathbb{R});\ \ D^k_t \in L^p(\mathbb{R});\ \ \forall 1 \leq k \leq t_j \right\}.$$

**Definition 2.5** *([10].) We define the Weighted anisotropic Besov-type where the properties of differentiability in the directions $x_1, x_2, ..., x_n$ are different from those in the direction $t$ by*

$$W^{t_j,p}(\mathbb{R}; \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1})) = \left\{ u \in L^p(\mathbb{R}; \mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^{n-1})),\ \ D^k_t u \in L^p(\mathbb{R}; \mathcal{B}^{s+t_j-k,\tau}_{p,q}(\mathbb{R}^{n-1})) \right\},$$

*where $s \in \mathbb{R}, \tau \geq 0, 1 \leq p, q \leq +\infty$ and $1 \leq k \leq t_j$. The standard practice norm in these spaces is:*

$$\begin{aligned} \|u\|_{W^{2,p}_k(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} = \Big\{ &\|u\|_{L^p(\mathbb{R};\mathcal{B}^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \|tu\|_{L^p(\mathbb{R};\mathcal{B}^{s+2,\tau}_{p,q}(\mathbb{R}^n))} \\ &+ \|tD_t u\|_{L^p(\mathbb{R};\mathcal{B}^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \|D_t u\|_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &+ \|tD_t u\|_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} \Big\}^{\frac{1}{q}}. \end{aligned}$$

**Definition 2.6** *([10].) For $s \in \mathbb{R}, \tau \geq 0$, and $1 \leq p, q \leq +\infty$, we not $\mathcal{B}^{s+2,\tau}_{p,q,k}(\mathbb{R}^{n+1})$ defined by*

$$\mathcal{B}^{s+2,\tau}_{p,q,k}(\mathbb{R}^{n+1}) = \left\{u \in \mathcal{B}^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}),\ tD_t^2 u, tD_{x_j}D_{x_k}u \ \text{and} \ tD_{x_j}D_t u \in \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})\right\}.$$

*This space is endowed with the norm:*

$$||u||_{\mathcal{B}^{s+2,\tau}_{p,q,k}(\mathbb{R}^{n+1})} = ||u||_{\mathcal{B}^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1})} + ||tD_t^2 u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} + ||tD_{x_j}D_{x_k}u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} + ||tD_{x_j}D_t u||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})}.$$

Now, we recall some essential lemmas needed for the proof of the main results.

**Lemma 2.7** *([12].) Let $1 \leq p < \infty$ and let $A < 0$. If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in l^p$ for any $\nu \geq 1$, then*

$$\sum_{j\geq 1}\left(\sum_{v\geq 1} 2^{vA} a_{jv}\right)^p \lesssim \sup_{v\geq 1}\sum_{j\geq 1} a_{jv}^p.$$

**Lemma 2.8** *([8], [13].) Let $m \in \mathbb{N}\setminus\{0\}$, $s \in \mathbb{R}$ suchthat $s < m$, if $v \in L^p(\mathbb{R}; \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))$, such that $D_t^m v \in L^p(\mathbb{R}; \mathcal{B}^{s-m,\tau}_{p,q}(\mathbb{R}^{n-1}))$, then $v \in \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\exists C > 0$ such that we have the following inequalitie:*

$$\|v\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} \quad \lesssim \|D_t^m v\|_{L^p\left(\mathbb{R};\mathcal{B}^{s-m,\tau}_{p,q}(\mathbb{R}^{n-1})\right)} + \varepsilon^{-1}\|v\|_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}.$$

**Lemma 2.9** *([11].) $\exists\ C > 0$ such that for any $\varepsilon > 0$ and for any $v \in \mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^n)$, $\left[\text{resp. } W^{t_j,p}(\mathbb{R}, \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))\right]$ we have for any $k = 0, .., t_j - 1$ the following inequalitie*

$$||D_t^k v||_{\mathcal{B}^{s+t_j-j,\tau}_{p,q}(\mathbb{R}^n)} \leq C\left\{\varepsilon||D_t^{t_j}v||_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} + \varepsilon^{\frac{-k}{t_j-k}}||v||_{\mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^n)}\right\}.$$

**Lemma 2.10** *([8].) Let $C_0 > 0$ such that for any $\varphi \in \mathcal{S}(\mathbb{R}^{n+1},)$ and there exists $C_1 > 0$ satisfying for any $v \in \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})[\text{resp.}; L^p(\mathbb{R}, \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))]$ we have*

$$\begin{aligned}\|\varphi v\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} \leq & C_0\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\|v\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} + C_1\|v\|_{\mathcal{B}^{s-1}_{p,q,u}(\mathbb{R}^{n+1})}\\ & \Big[\ \text{resp.},\ \|\varphi v\|_{L^p(\mathbb{R},\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))} \leq C_0\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\|v\|_{L^p(\mathbb{R};\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ & +C_1\|v\|_{L^p\left(\mathbb{R};\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^n)\right)}\Big].\end{aligned}$$

**Lemma 2.11** *([8].) Let $s_1 \leq s_2 < s_3$ be three real numbers, and let $1 \leq p, q \leq \infty$, $u \leq p$. For any $\varepsilon > 0$ and $v \in \mathcal{B}^{s_3,\tau}_{p,q}$, [resp.; $L^p(\mathbb{R}; \mathcal{B}^{s_3,\tau}_{p,q}(\mathbb{R}^n))$,] we get:*

$$\begin{aligned}\|v\|_{\mathcal{B}^{s_2,\tau}_{p,q}(\mathbb{R}^{n+1})} \lesssim & \lesssim \varepsilon\|v\|_{\mathcal{B}^{s_3,\tau}_{p,q}(\mathbb{R}^{n+1})} + \varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|v\|_{\mathcal{B}^{s_1,\tau}_{p,q}(\mathbb{R}^{n+1})},\\ & \Big[\ \text{resp.},\ \|v\|_{L^p\left(\mathbb{R};\mathcal{B}^{s_2,\tau}_{p,q}(\mathbb{R}^n)\right)} \lesssim \varepsilon\|v\|_{L^p\left(\mathbb{R};\mathcal{B}^{s_3,\tau}_{p,q}(\mathbb{R}^n)\right)}\\ & +\varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|v\|_{L^p\left(\mathbb{R};\mathcal{B}^{s_1,\tau}_{p,q}(\mathbb{R}^n)\right)}\Big].\end{aligned}$$

## 3. Caracterisation of Traces of Besov-Type Spaces and Assumptions

For all $u \in \mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^n_+)$, we consider $t_j$ the traces of $u$ defined for $x' \in \mathbb{R}^{n-1}$ by

$$\gamma_l u(x') = \frac{\partial^l u}{\partial v^l}\mid_\Gamma = D_t^l u(0, x'); \qquad x' \in \mathbb{R}^{n-1}$$

for $l = 0, ..., t_j - 1$.

We pose $\gamma = (\gamma_0, ..., \gamma_{t_j-1})$. Notons que ce opérateur est continue de $\mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^n_+)$ vers

$\prod_{l=0}^{t_j-1} \mathcal{B}_{p,q}^{s+t_j-1-\frac{1}{p},\tau}(\mathbb{R}^n_+)$.
For $i = 1, ...m_{+,}$ let $\sigma_i$ be an integer such that $\sigma_i \leq -1$. We put

$$(B_{ij}(x'; D_{x'}).\gamma)u = \sum_{l=0}^{t_j-1} B_{ijl}(x'; D_{x'})\gamma_l u.$$

The operator $B_{ijl}(x', D_{x'})$ is a differential operator of order less than or equal to $\sigma_i + t_j - l$.
If $\sigma_i + t_j - l < 0$, we pose $B_{ijl} = 0$.
We denote

$$B_\gamma \equiv B(x', D_{x'})\gamma \equiv (B_{ij}(x', D_{x'})\gamma)_{i=1,..,m_+;j=1,..,N}.$$

This operator is continuous from $\mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}^n_+)$ to $\prod_{l=0}^{t_j-1} \mathcal{B}_{p,q}^{s+t_j-\sigma_i-\frac{1}{p},\tau}(\mathbb{R}^n_+)$.
Consider $B^0_{ij}\gamma$, the principal part of operator $B_{ij}\gamma$, given by $B^0\gamma = B^0(x', D_{x'})\gamma = (B^0_{ij}\gamma)_{i=1,...,m \ \ et \ \ j=1,...,N}$.

### 3.1. Caracterisation of Traces of Besov-type spaces

**Theorem 3.1** *([15,8]) Let* $t_j \in \mathbb{N} \setminus \{0\}$, $l \in \{0, ..., t_j - 1\}$. *For* $v \in W_{loc}^{t_j,p}(\mathbb{R}_+; \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n))$, *the series* $\sum_{j\geq 0} D_t^l \dot{\Delta}'_j v(0, .)$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and defines an element* $\gamma_l v$ *belonging to* $\mathcal{B}_{p,q}^{s+1-\frac{1}{u},\tau}(\mathbb{R}^n)$.

*In addition the mapping* $v \longmapsto \gamma_l v$ *is continuous and surjective from* $W^{t_j,p}(\mathbb{R}_+; \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n))$ *to* $\mathcal{B}_{p,q}^{s+1-\frac{1}{u},\tau}(\mathbb{R}^n)$.

*Also, there exists an extension operator* $R_0$ *from* $\mathcal{B}_{p,q}^{s+1-\frac{1}{u},\tau}(\mathbb{R}^n)$ *to* $W^{t_j,p}(\mathbb{R}_+; \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n))$ *such that*

$$\gamma_l v \circ R_0 = Id_{\mathcal{B}_{p,q}^{s+l-\frac{1}{u},\tau}(\mathbb{R}^n)}.$$

*In particular, if* $s \geq 0$, *the operator* $\gamma_l$ *is bounded and surjective from* $\mathcal{B}_{p,q,k}^{s+2,\tau}(\mathbb{R}^{n+1}_+)$ *to* $\mathcal{B}_{p,q}^{s+l-\frac{1}{u},\tau}(\mathbb{R}^n)$.

**Proof.** For the proof, it is the same thing as in the classical case, see [8], [10].

### 3.2. Basic assumptions

*Assumption (H1)* The operator $M(x; D_x)$ is elliptic for $t > 0$ in $\mathbb{R}^{n+1}_+$

$\forall(\tau, \xi) \in \mathbb{R}^n \setminus \{0\}$

$$\det(M^0_{i,j}(0; \xi', \tau))_{i,j=1,...,N} \neq 0.$$

*Assumption (H2)* For all $x' \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, the complex variable polynomial $\theta$

$$P(\theta) = \det(M^0_{i,j}(0; \xi', \theta))_{i,j=1,...,N},$$

admits $m_+(\xi)$ complex roots with positive imaginary part.

*Assumption (H3)* Suppose the following problem for all $\xi' \in \mathbb{R}^{n-1}$

$$\begin{cases} M^0\left(t, 0; \xi', D_t, \right) v(t) = 0, \\ B^0(0, \xi')\gamma v = 0, \\ v \in \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+). \end{cases}$$

admits only one solution $v \equiv 0$.
With $M^0$ is the principal part of the operator $M$.

**Corollary 3.2** *[2] Under the above assumptions, we have*

$$M : \prod_{j=1}^{N} W^{t_j,p}(\mathbb{R}_+) \longrightarrow \prod_{i=1}^{N} W^{-s_i,p}(\mathbb{R}_+),$$

*is a Fredholm operator with index equal to* $m_+$.

## 4. Main Results

**Theorem 4.1** *Let* $s, \tau$ *be two nonnegatif numbres and* $1 \leq p, q < \infty, u \leq p$. *Suppose that the hypothèses* $H(1) - H(3)$ *are holds for any compact* $K$ *in* $\overline{\mathbb{R}_+^{n+1}}$. *Then there exists a constant* $C_K$ *such that, for* $v \in \prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+^n)$ *with the support of is in* $K$, *we have:*

$$\|v\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+^n)} \leq C_M \left\{ \|Mv\|_{\prod_{i=1}^N \mathcal{B}_{p,q}^{s+s_i,\tau}(\mathbb{R}_+^n)} + \|\gamma_0 v\|_{\prod_{i=1}^m \mathcal{B}_{p,q}^{s+\sigma_i-\frac{1}{p},\tau}(\mathbb{R}_+^{n-1})} + \|v\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j-1,\tau}(\mathbb{R}_+^n)} \right\}.$$

**Proof.** To establish Theorem 4.1, we will have to go through two steps: first, we will prove the following Proposition which will allow us to give an estimate of the derivatives quasi tangential solutions, and the second step based on a Lemma that we will state it later and will make it possible to estimate the normal derivatives

**Proposition 1** *Let* $s, \tau$ *two nonnegative real numbers and let* $1 \leq p, q < +\infty$. *Under the Assumptions (H1)-(H3), for any compact set* $K$ *in* $\overline{\mathbb{R}_+^{n+1}}$, *there exists a constant* $C_K > 0$, *such that for any* $v \in \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+; \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n))$ *with supp*$(v) \subset K$, *we have:*

$$\|v\|_{\prod_{j=1}^N W^{t_j,p}\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^{n-1})\right)} \leq C_K \left\{ \|Mv\|_{\prod_{i=1}^N W^{-s_i,p}\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+^{n-1})\right)} + \|\gamma_l v\|_{\prod_{i=1}^{m_+} \mathcal{B}_{p,q}^{s-\sigma_i-\frac{1}{p},\tau}(\mathbb{R}_+^{n-1})} \right.$$
$$\left. + \|v\|_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s+t_j-1,\tau}(\mathbb{R}_+^{n-1})\right)} \right\}.$$

**Proof.** We are going to reduce our problem and show the proposition for a homogeneous system of operators with constant coefficients thanks to the Lemmas 2.8, Lemma2.10 and Lemma2.11 as well thanks to the Korn technique (See [3]).
So, we write $M^0 \equiv M^0(D_{x'}, D_t) = (M_{ij}^0(0; D_{x'}, D_t))$ and $B^0\gamma = B^0(0, D_{x'})\gamma$,
with

$$M_{ij}^0(0; D_{x'}, D_t) = \sum_{k+|\alpha'|=s_i+t_j} a_\alpha^{ij}(0) D_{x'}^{\alpha'} D_t^k.$$

Using the corollary 3.2 and like [7] and [2] , we can show that under the hypotheses H(1)-H(3) we have , $\forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ , the operator $(M^0(\xi', D_t), B^0(0, \xi')\gamma)$ is invertible from $\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)$ to $\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}$ and if $K_{\xi'}$ its inverse, then the mapping $\xi' \longmapsto K_{\xi'}$ is $\mathcal{C}^\infty$ from $\mathbb{R}^{n-1} \setminus \{0\}$ to $\mathcal{B}(\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}; \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+))$ and for all multi-index $\alpha', \exists\ C_{\alpha'} > 0$ such that for all $\xi'$; $\frac{1}{2} \leq |\xi'| \leq 2$ and for $(f, g) \in \prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}$, we have

$$||D_{\xi'}^{\alpha'} K_{\xi'}(f, g)||_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)} \leq C_{\alpha'} ||(f, g)||_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)} \tag{4.1}$$

First of all, we are going to prove that for any sufficiently large integer $L \geq 1$, there exists a constant $C > 0$ such that for any ball $B \in \mathbb{R}^{n-1}$ of radius $2^{-J}$, $J \in \mathbb{Z}$ centered at $x_0 \in \mathbb{R}^{n-1}$.

$$\begin{aligned} &\|v\|_{\prod_{j=1}^N L^p\left(B;W^{t_j,p}(\mathbb{R}_+)\right)} \\ &\leq C \Big\{ \|M^0 v\|_{\prod_{i=1}^N L^p\left(2B;W^{-s_i,p}(\mathbb{R}_+)\right)} + \|B^0\gamma v\|_{\prod_{i=1}^{m_+} L^p(2B)} \\ &\quad + |B|^{1/p} \sum_{\nu \geq -J+1} 2^{-2\nu M} |F_\nu|^{1-\frac{1}{p}} \Big( \|M^0 v\|_{\prod_{i=1}^N L^p\left(F_\nu;W^{-s_i,p}(\mathbb{R}_+)\right)} \\ &\quad + \|B^0\gamma v\|_{\prod_{i=1}^m L^p(F_\nu)} \Big) \Big\}. \end{aligned} \tag{4.2}$$

True for any $v \in \mathcal{S}(\mathbb{R}^{n-1}; W^{t_j,p}(\mathbb{R}_+))$ with support of the tangential Fourier transform of $v$ is belongs to the annulus $\frac{1}{2} < |\xi'| < 2$. With, $F_\nu = \left\{x' \in \mathbb{R}^{n-1};\ 2^\nu < |x' - x'_0| < 2^{\nu+1}\right\}$ For this, we apply the operator $(M^0(\xi', D_t), B(0, \xi')\gamma)$ to the relation:

$$\mathcal{F}v\left(.,\xi'\right) = \int e^{-iy'\cdot\xi'} v\left(.,y'\right) dy'$$

to obtain the system

$$\begin{aligned} M^0\left(\xi', D_t\right)\mathcal{F}v\left(.,\xi'\right) &= \mathcal{F}M^0 v\left(.,\xi'\right) = \int e^{-iy'\cdot\xi'} \left(M^0 v\right)\left(.,y'\right) dy' \\ B^0\left(0,\xi'\right)\gamma\mathcal{F}v\left(.,\xi'\right) &= \mathcal{F}B^0\gamma v\left(\xi'\right) = \int e^{-iy'\cdot\xi'} \left(B^0\gamma u\right)\left(y'\right) dy'. \end{aligned}$$

We apply $K_{\xi'}$ to the system to obtain

$$\mathcal{F}v\left(.,\xi'\right) = \int e^{-iy'\cdot\xi'} K_{\xi'}\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right) dy'.$$

Let $\Phi \in C_0^\infty$ equal to 1 for $\frac{1}{2} < |\xi'| < 2$ and, we integrate by parts with respect to $\xi'$, we obtain

$$v(.,x') = \iint \frac{e^{i\left(x'-y'\right)\cdot\xi'}}{\left(1+\left|x'-y'\right|^2\right)^L}\left(I - \dot{\Delta}_{\xi'}\right)^L \left\{\Phi\left(\xi'\right) K_{\xi'}\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\} \frac{dy' d\xi'}{(2\pi)^{n-1}}.$$

By (4.1) yeilds:

$$\left\|v\left(.,x'\right)\right\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)} \leq C \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+\left|x'-y'\right|^2\right)^L} \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy'.$$

Now, we integrate with respect to $x' \in B$, we obtain

$$\begin{aligned} &\|v\|_{L^p\left(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)\right)} \\ &\leq C\left\{\int_{x'\in B}\left(\int_{y'\in\mathbb{R}^{n-1}} \frac{1}{\left(1+\left|x'-y'\right|^2\right)^L} \quad \times \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy'\right)^p dx'\right\}^{1/p}. \end{aligned}$$

By the decomposition of $\mathbb{R}^{n-1}$ defined by $\mathbb{R}^{n-1} = 2B' \cup_{\nu\geq -J+1} F'_\nu$, with $F'_\nu = \left\{x' \in \mathbb{R}^n : 2^\nu \leq |x' - x'_0| \leq 2^{\nu+1}\right\}$ and $B' = \left\{x' \in \mathbb{R}^n : |x' - x'_0| < 2^{-J}, J \in \mathbb{Z}\right\}$. Then

$$\begin{aligned} &\|v\|_{L^p\left(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)\right)} \\ &\leq C\left\{\int_{x'\in B}\left(\int_{2B} \frac{1}{\left(1+\left|x'-y'\right|^2\right)^L}\chi_{2B}\left(y'\right) \times \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy'\right)^p dx'\right\}^{1/p}. \\ &+ C\left\{\int_{x'\in B}\left(\sum_{\nu\geq -J+1}\int_{F_\nu} \frac{1}{\left(1+\left|x'-y'\right|^2\right)^L} \times \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy'\right)^p dx'\right\}^{1/p} \end{aligned}$$

The first term of the second member of the above inequality is a norm $L^p$ of a convolution product of a function of $L^1(\mathbb{R}^{n-1})$ (for large L) and a function of $L^p(\mathbb{R}^{n-1})$; on the other hand, for the second term we notice that for $x' \in B, y' \in F_\nu$ and $|x'-y'| \sim |x'_0 - y'| \sim 2^\nu$. Thus we have

$$\begin{aligned}
&\|v\|_{L^p\left(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)\right)} \\
&\leq C\left\{\left\|M^0 v\right\|_{L^p\left(2B;\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\right)} + \left\|B^0\gamma v\right\|_{\prod_{i=1}^{m_+} L^p(2B)}\right\} \\
&+ C|B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu L} \int_{y'\in F_\nu} \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy' \\
&\leq C\left\{\left\|M^0 v\right\|_{L^p\left(2B;\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\right)} + \left\|B^0\gamma v\right\|_{\prod_{i=1}^{m_+} L^p(2B)}\right\} + C|B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu M} \left|F_\nu\right|^{1-\frac{1}{p}} \\
&\times \left(\int_{y'\in F_\nu} \left\|\left(M^0 v\left(.,y'\right), B^0\gamma v\left(y'\right)\right)\right\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}} dy'\right)^{1/p}.
\end{aligned}$$

So that inequality (4.2) is proved.

Now, let $v = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_N \end{pmatrix} \in \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+; \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))$ with *suppv* $\subset K$, such that $K$ is a compact set of $\mathbb{R}^n_+$. For $k \in \mathbb{N}$, we pose $v_k = \dot{\Delta}'_k v(2^{-k}x)$; $k \geq 1$, then $v$ belongs to $\prod_{j=1}^N \mathcal{S}(\mathbb{R}^{n-1}; W^{t_j,p})$

$$\left(M^0_{ij} v^j\right)_k = 2^{k(s_i+t_j)} M^0_{ij} v^j_k \quad \text{and} \quad \left(B^0_{ij}\gamma v^j\right)_k = 2^{k(\sigma_i+t_j)} B^0_{ij}\gamma v^j_k. \tag{4.3}$$

We apply inequality (4.2) for each $uk$, $k \geq 1$, we obtain

$$\begin{aligned}
&\|v_k\|_{\prod_{j=1}^N L^p\left(B;W^{t_j,p}(\mathbb{R}_+)\right)} \\
&\leq C\left\{\left\|M^0 v_k\right\|_{\prod_{i=1}^N L^p\left(2B;W^{-s_i,p}(\mathbb{R}_+)\right)} + \left\|B^0\gamma v_k\right\|_{\prod_{i=1}^{m_+} L^p(2B)}\right. \\
&\quad \left. + |B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu M} \left|F_\nu\right|^{1-\frac{1}{p}} \left(\left\|M^0 v_k\right\|_{\prod_{i=1}^N L^p\left(F_\nu;W^{-s_i,p}(\mathbb{R}_+)\right)} \quad + \left\|B^0\gamma u_k\right\|_{\prod_{i=1}^m L^p(F_\nu)}\right)\right\}
\end{aligned} \tag{4.4}$$

The operator $\dot{\Delta}_k$ commutes with the derivation then with the operator with constant coefficients $M^0$, so using (4.3) we have:

$$\begin{aligned}
\|v_k\|_{\prod_{j=1}^N L^p\left(B;W^{t_j,p}(\mathbb{R}_+)\right)} &= \sum_{j=1}^N \left\|v^j_k\right\|_{L^p\left(B;W^{t_j,p}(\mathbb{R}_+)\right)} \\
&= \sum_{j=1}^N \sum_{r=0}^{t_j} \left\|D^r_t v^j_k\right\|_{L^p(B;L^p(\mathbb{R}_+))} \\
&= \sum_{j=1}^N \sum_{r=0}^{t_j} 2^{kn/p} 2^{-kr} \left\|\dot{\Delta}'_k D^r_t v^j\right\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}.
\end{aligned}$$

$$\begin{aligned}
\left\|M^0 v_k\right\|_{\prod_{i=1}^N L^p\left(2B;W^{-s_i,p}(\mathbb{R}_+)\right)} &= \sum_{i=1}^N \left\|\left(M^0 v_k\right)^i\right\|_{L^p\left(2B;W^{-s_i,p}(\mathbb{R}_+)\right)} \\
&= \sum_{i=1}^N \left\|\sum_{j=1}^N 2^{-k(s_i+t_j)} \left(M^0_{ij} v^j\right)_k\right\|_{L^p\left(2B;W^{-s_i,p}(\mathbb{R}_+)\right)} \\
&= \sum_{i=1}^N \sum_{r=0}^{-s_i} 2^{kn/p} \left\|\sum_{j=1}^N 2^{-k(s_i+t_j+r)} \dot{\Delta}'_k D^r_t \left(M^0_{ij} v^j\right)\right\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}.
\end{aligned}$$

and

$$\begin{aligned}\left\|B^0\gamma v_k\right\|^{m_+}_{\prod_{i=1}^m L^p(2B)} &= \sum_{i=1}^{m}\left\|\left(B^0\gamma v_k\right)^i\right\|_{L^p(2B)}\\ &= \sum_{i=1}^{m_+}\left\|\sum_{j=1}^{N}2^{-k(\sigma_i+t_j)}\left(B^0_{ij}\gamma v^j\right)_k\right\|_{L^p(2B)}\\ &= \sum_{i=1}^{m_+}2^{k(n-1)/p}\left\|\sum_{j=1}^{N}2^{-k(\sigma_i+t_j)}\dot{\Delta}'_k B^0_{ij}\gamma v^j\right\|_{L^p(2^{-k+1}B)}.\end{aligned}$$

Substituting the above equalities into (4.4) yields

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{-kr}\left\|\dot{\Delta}'_k D^r_t v^j\right\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}\\ &\le C\left\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\left\|\sum_{j=1}^{N}2^{-k(s_i+t_j+r)}\dot{\Delta}'_k D^r_t L^0_{ij}v^j\right\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}+\sum_{i=1}^{m_+}\left\|\sum_{j=1}^{N}2^{-k(\sigma_i+t_j+1)}\dot{\Delta}'_k B^0_{ij}\gamma v^j\right\|_{L^p(2^{-k+1}B)}\right.\\ &\quad+|B|^{1/p}\sum_{\nu\ge -J+1}2^{-2\nu M}\left|F_\nu\right|^{1-\frac{1}{p}}\left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\left\|\sum_{j=1}^{N}2^{-k(s_i+t_j+r)}\dot{\Delta}'_k D^r_t L^0_{ij}v^j\right\|_{L^p(\mathbb{R}_+\times 2^{-k}F_\nu)}\right.\\ &\quad\left.\left.+\sum_{i=1}^{m_+}\left\|\sum_{j=1}^{N}2^{-k(\sigma_i+t_j+1)}\dot{\Delta}'_k B^0_{ij}\gamma v^j\right\|_{L^p(2^{-k}F_\nu)}\right]\right\}.\end{aligned}$$

If we now replace $u^j$ by $2^{k(s+t_j)}v^j$, we get:

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k\left(s+t_j-r\right)}\left\|\dot{\Delta}'_k D^r_t v^j\right\|_{L^p\left(\mathbb{R}_+\times 2^{-k}B\right)}\le C\left\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\left\|\sum_{j=1}^{N}\dot{\Delta}'_k D^r_t L^0_{ij}v^j\right\|_{L^p\left(\mathbb{R}_+\times 2^{-k+1}B\right)}\right.\\ &\quad+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\left\|\sum_{j=1}^{N}\dot{\Delta}'_k B^0_{ij}\gamma v^j\right\|_{L^p\left(2^{-k+1}B\right)}+|B|^{1/p}\sum_{\nu\ge -J+1}2^{-2\nu M}\left|F_\nu\right|^{1-\frac{1}{p}}\\ &\quad\left.\times\left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\left\|\sum_{j=1}^{N}\dot{\Delta}'_k D^r_t L^0_{ij}v^j\right\|_{L^p\left(\mathbb{R}_+\times 2^{-k}F_\nu\right)}+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\left\|\sum_{j=1}^{N}\dot{\Delta}'_k B^0_{ij}\gamma v^j\right\|_{L^p\left(2^{-k}F_\nu\right)}\right]\right\}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k(s+t_j-r)}\left\|\dot{\Delta}'_k D^r_t v^j\right\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}\\ &\le C\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\left\|\dot{\Delta}'_k D^r_t\left(M^0u\right)^i\right\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\left\|\dot{\Delta}'_k\left(B^0\gamma u\right)^i\right\|_{L^p(2^{-k+1}B)}\\ &\qquad+|B|^{1/p}\sum_{\nu\ge -J+1}2^{-2\nu M}\left|F_\nu\right|^{1-\frac{1}{p}}\left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\left\|\dot{\Delta}'_k D^r_t\left(M^0u\right)^i\right\|_{L^p(\mathbb{R}_+\times 2^{-k}F_\nu)}\right.\\ &\qquad\qquad\left.\left.+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\left\|\dot{\Delta}'_k\left(B^0\gamma u\right)^i\right\|_{L^p(2^{-k}F_\nu)}\right]\right\}.\end{aligned}$$

We pose $k = J + k \in \mathbb{Z}$and $\mu = \nu - k \in \mathbb{Z}$, then ball $2^{-k}$ becomes $B_k$, we deduce

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j} 2^{k(s+t_j-r)p} \left\|\dot{\Delta}'_k D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)} \\
&\leq C\left\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p}\left\|\dot{\Delta}'_k D_t^r\left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times 2B_K)} + \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p}\left\|\dot{\Delta}'_k\left(B^0\gamma u\right)^i\right\|^p_{L^p(2B_K)}\right. \\
&+\left|2^k B_K\right|\left(\sum_{\mu\geq -K+1} 2^{(\mu+K)\left(-2M+(n-1)\left(1-\frac{1}{p}\right)\right)} 2^{\mu\frac{\lambda}{p}} \times \left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\frac{2^{k(s-s_i-r)}}{|F_\mu|^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k D_t^r\left(M^0 u\right)^i\right\|_{L^p(\mathbb{R}_+\times F_\mu)}\right.\right. \\
&\left.\left.\left.+\sum_{i=1}^{m_+}\frac{2^{k(s-\sigma_i-1/p)}}{|F_\mu|^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k\left(B^0\gamma u\right)^i\right\|_{L^p(F_\mu)}\right]\right)^p\right\}.
\end{aligned}$$

A simple calculation yields

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j} 2^{k\left(s+t_j-r\right)p}\left\|\dot{\Delta}'_k D_t^r v^j\right\|^p_{L^p\left(\mathbb{R}_+\times B_K\right)} \\
&\leq C\left\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p}\left\|\dot{\Delta}'_k D_t^r\left(M^0 v\right)^i\right\|^p_{L^p\left(\mathbb{R}_+\times 2B_K\right)} + \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p}\left\|\dot{\Delta}'_k\left(B^0\gamma v\right)^i\right\|^p_{L^p(2B_K)}\right. \\
&+ 2^{(k-K)(-2N+n-1)p} 2^{-K\lambda}\left(\sum_{\mu\geq 1} 2^{\mu\left(-2M+(n-1)\left(1-\frac{1}{p}\right)+\frac{\lambda}{p}\right)}\left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\frac{2^{k(s-s_i-r)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k D_t^r\left(M^0 v\right)^i\right\|_{L^p\left(\mathbb{R}_+\times F_{\mu-K}\right)}\right.\right. \\
&\left.\left.\left.+\sum_{i=1}^{m_+}\frac{2^{k(s-\sigma_i-1/p)}}{|\ F_{\mu-K}{}^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k\left(B^0\gamma v\right)^i\right\|_{L^p\left(F_{\mu-K}\right)}\right]\right)^p\right\}.
\end{aligned}$$

We pose $A_M = -2M + (n-1)(n - \frac{1}{p}) + \frac{\lambda}{p}$ and multiply by $\frac{1}{|B_k|^\tau}$ and sum for j, $k \geq \max(K^+, 1)$,

$$\begin{aligned}
&\frac{1}{|B_K|^\tau}\sum_{k\geq\max(K^+,1)}\sum_{j=1}^{N}\sum_{r=0}^{t_j} 2^{k(s+t_j-r)p}\left\|\dot{\Delta}'_k D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)} \\
&\leq C\left\{\frac{1}{|B_K|^\tau}\sum_{k\geq K^+}\sum_{i=1}^{N}\sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p}\left\|\dot{\Delta}'_k D_t^r\left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times 2B_K)}\right. \\
&+\frac{1}{|B_K|^\tau}\sum_{k\geq K^+}\sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p}\left\|\dot{\Delta}'_k\left(B^0\gamma v\right)^i\right\|^p_{L^p(2B_K)} \\
&+\sum_{k\geq K^+}\left(\sum_{\mu\geq 1} 2^{\mu A_M}\left[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\frac{2^{k(s-s_i-r)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k D_t^r\left(M^0 v\right)^i\right\|_{L^p(\mathbb{R}_+\times F_{\mu-K})}\right.\right. \\
&\left.\left.\left.+\sum_{i=1}^{m_+}\frac{2^{k(s-\sigma_i-1/p)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}}\left\|\dot{\Delta}'_k\left(B^0\gamma v\right)^i\right\|_{L^p(F_{\mu-K})}\right]\right)^p\right\}.
\end{aligned}$$

Using Lemma 2.7 for obtain

$$\begin{aligned}
&\frac{1}{|B_K|^{\tau}} \sum_{k\geq \max(K^+,1)} \sum_{j=1}^{N}\sum_{r=0}^{t_i} 2^{k(s+t_j-r)p} \left\|\dot{\Delta}'_k D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)} \\
\leq& C\left\{ \frac{1}{|B_K|^{\tau}} \sum_{k\geq K^+} \sum_{i=1}^{N}\sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p} \left\|\dot{\Delta}'_k D_t^r \left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times 2B_K)} \right.\\
&+ \frac{1}{|B_K|^{\tau}} \sum_{k\geq K^+} \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p} \left\|\dot{\Delta}'_k \left(B^0\gamma v\right)^i\right\|^p_{L^p(2B_K)} \\
&+ \sup_{\mu\geq 1} \sum_{k\geq K} \left[ \sum_{i=1}^{N}\sum_{r=0}^{-s_i} \frac{2^{k(s-s_i-r)p}}{|F_{\mu-K}|^{\tau}} \left\|\dot{\Delta}'_k D_t^r \left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times F_{\mu-K})} \right.\\
&\left.\left. + \sum_{i=1}^{m_+} \frac{2^{k(s-\sigma_i-1/p)p}}{|F_{\mu-K}|^{\tau}} \left\|\dot{\Delta}'_k \left(B^0\gamma v\right)^i\right\|^p_{L^p(F_{\mu-K})} \right]\right\}.
\end{aligned}$$

We add the terms associated with $k = 0$ to the left of the above inequality, and like $F_{\mu-K} \subset B_{K-\mu-1}$, we deduce

$$\begin{aligned}
&\frac{1}{|B_K|^{\tau}} \sum_{k\geq K^+} \sum_{j=1}^{N}\sum_{r=0}^{t_j} 2^{k(s+t_j-r)p} \left\|\dot{\Delta}'_k D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)} \\
\leq& C\left\{ \frac{1}{|B_K|^{\tau}} \sum_{k\geq K^+} \sum_{i=1}^{N}\sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p} \left\|\dot{\Delta}'_k D_t^r \left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times 2B_K)} \right.\\
&+ \frac{1}{|B_K|^{\tau}} \sum_{k\geq K^+} \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p} \left\|\dot{\Delta}'_k \left(B^0\gamma v\right)^i\right\|^p_{L^p(2B_K)} \\
&+ \sup_{\mu\geq 1} \sum_{k\geq (K-\mu-1)+i=1} \sum_{r=0}^{N} \left[ \sum_{r=0}^{-s_i} \frac{2^{k(s-s_i-r)p}}{|F_{\mu-K}|^{\tau}} \left\|\dot{\Delta}'_k D_t^r \left(M^0 v\right)^i\right\|^p_{L^p(\mathbb{R}_+\times B_{K-\mu-1})} \right.\\
&\left.\left. + \sum_{i=1}^{m_+} \frac{2^{k(s-\sigma_i-1/p)p}}{|F_{\mu-K}|^{\tau}} \left\|\dot{\Delta}'_k \left(B^0\gamma v\right)^i\right\|^p_{L^p(B_{K-\mu-1})} \right]\right\} + R_0^K,
\end{aligned}$$

where

$$R_0^K = \frac{1}{|B_K|^{\tau}} \sum_{j=1}^{N}\sum_{r=0}^{t_j} \left\|\dot{\Delta}'_0 D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)}.$$

Taking the supremum over $K$ and $B_K$ yields

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j} \left\|D_t^r v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-r,\tau}_{p,q}(\mathbb{R}^{n-1})\right)} \\
\leq& C\left\{ \sum_{i=1}^{N}\sum_{r=0}^{-s_i} \left\|D_t^r \left(M^0 v\right)^i\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s-s_i-r,\tau}_{p,q}(\mathbb{R}^{n-1})\right)} + \sum_{i=1}^{m_+} \left\|\left(B^0\gamma v\right)^i\right\|^p_{\mathcal{B}^{s-\sigma_i-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^{n-1})} \right\} + R_0,
\end{aligned}$$

where

$$R_0 = \sup_{K,B_K} \frac{1}{|B_K|^{\tau}} \sum_{j=1}^{N}\sum_{r=0}^{t_j} \left\|\dot{\Delta}'_0 D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)}.$$

Finally, we have

$$\|v\|^p_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))} \leq C\left\{\left\|M^0v\right\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))} + \left\|B^0\gamma v\right\|^p_{\prod_{i=1}^m \mathcal{B}^{s-\sigma_i-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^{n-1})}\right\} + R_0. \tag{4.5}$$

To estimate the terms of $R_0$, we write

$$R_0^K = \frac{1}{|B_K|^\tau}\sum_{j=1}^N\sum_{r=0}^{t_j}\left\|\dot{\Delta}'_0 D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)} + \frac{1}{|B_K|^\tau}\sum_{j=1}^N\left\|\dot{\Delta}'_0 D_t^{t_j} v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)}.$$

To estimate the first term of $R_0^K$ , we use Lemma 2.8, we get

$$\begin{aligned}
&\frac{1}{|B_K|^\tau}\sum_{j=1}^N\sum_{r=0}^{t_j-1}\left\|\dot{\Delta}'_0 D_t^r v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)}\\
&\leq \sum_{j=1}^N\sum_{r=0}^{t_j-1}\left\|D_t^r v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-r-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\\
&\leq C\sum_{j=1}^N\left\{\varepsilon^p\left\|D_t^{t_j}v^j\right\|^p_{L^p(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))} + \sum_{r=0}^{t_j-1}\varepsilon^{-\frac{rp}{t_j-r}}\left\|v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}\\
&\leq C\sum_{j=1}^N\left\{\varepsilon^p\left\|v^j\right\|^p_{W^{t_j,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))} + C'_\varepsilon\left\|v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}.
\end{aligned}$$

To estimate the second term of $R_0^K$ we return to the equation $M^0v = f$.
$\forall i = 1,\dots,N,\quad \sum_{j=1}^N L^0_{ij}v^j = f^i$, so $\sum_{j=1}^N\sum_{r+|\alpha'|=s_i+t_j} a^{ij}_{r,\alpha'}(0)D^{\alpha'}_{x'}D^r_t v^j = f^i$, here $f^i = \left(M^0u\right)^i$.
Thus

$$\sum_{j=1}^N a^{ij}_{s_i+t_j,\alpha'}(0)D_t^{s_i+t_j}v^j = f^i - \sum_{j=1}^N\sum_{\substack{r+|\alpha'|=s_i+t_j\\ 0\leq r\leq s_i+t_j-1}} a^{ij}_{r,\alpha'}(0)D^{\alpha'}_{x'}D^r_t v^j$$

We apply $D_t^{-s_i}$ to both sides of this inequality, we get

$$\sum_{j=1}^N a^{ij}_{s_i+t_j,\alpha'}(0)D_t^{t_j}v^j = D_t^{-s_i}f^i - \sum_{\substack{j=1\\ N}}^N\sum_{\substack{r'+|\alpha'|=t_j\\ -s_i\leq r'\leq t_j-1}} a^{ij}_{r'+s_i,\alpha'}(0)D^{\alpha'}_{x'}D^{r'}_t v^j. \tag{4.6}$$

The ellipticity hypothesis gives that the matrix $A = (a^{ij}_{s_i+t_j,\alpha'}(0))_{i,j}$ is invertible. Let us put
$D^T_t v = (D_t^{t_j}v^j)_j,\ D_t^{-s}f = (D_t^{-s_i}f^i)_i,$
$V = (v^i)_i = \sum_{j=1}^N\sum_{\substack{r'+|\alpha'|=t_j\\ -s_i\leq r'\leq t_j-1}} a^{ij}_{r'+s_i,\alpha'}(0)D^{\alpha'}_{x'}D^{r'}_t v^j,$
and using (4.6), we obtain

$$D_t^T v = A^{-1}D_t^{-S}f - A^{-1}V$$

and then

$$\dot{\Delta}'_0 D_t^T v = A^{-1}\dot{\Delta}'_0 D_t^{-S}f - A^{-1}\dot{\Delta}'_0 V.$$

To estimate the second term of $R_0^K$, we write

$$\begin{aligned}
&\frac{1}{|B_K|^\tau}\sum_{j=1}^N \left\|\Delta_0' D_t^{t_j} v^j\right\|^p_{L^p(\mathbb{R}_+\times B_K)}\\
&=\frac{1}{|B_K|^\tau}\left\|\Delta_0' D_t^{T} v\right\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+\times B_K)}\\
&\leq C\left\{\frac{1}{|B_K|^\tau}\left\|\Delta_0' D_t^{-S} f\right\|^p_{\prod_{i=1}^N L^p(\mathbb{R}_+\times B_K)}+\frac{1}{|B_K|^\tau}\left\|\Delta_0' V\right\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+\times B_K)}\right\}\\
&\leq C\left\{\left\|D_t^{-S} f\right\|^p_{\prod_{i=1}^N L^p(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}+\left\|V\right\|^p_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}\\
&\leq C\left\{\left\|f\right\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}+\left\|V\right\|^p_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}.
\end{aligned}\tag{4.7}$$

Now, we have

$$\begin{aligned}
\left\|V\right\|^p_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)} &\leq C\sum_{j=1}^N\sum_{r'+|\alpha'|=t_j}\left\|D_{x'}^{\alpha'}D_t^{r'}v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\\
&\leq C\sum_{j=10\leq r'\leq t_j-1}^N\sum_t\left\|D_t^{r'}v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-r'-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}.
\end{aligned}$$

In the same way as for (4.5) there is a constant $C>0$ such that $\forall\varepsilon>0$, we have

$$\begin{aligned}
&\left\|V\right\|^p_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}^{s-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\\
&\leq C\sum_{j=1}^N\left\{\varepsilon^p\left\|D_t^{t_j}v^j\right\|^p_{L^p(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}+\sum_{r'=0}^{t_j-1}\varepsilon^{-\frac{r'p}{t_j-r}}\left\|v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}\\
&\leq C\sum_{j=1}^N\left\{\varepsilon^p\left\|v^j\right\|^p_{W^{t_j,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}+C'_\varepsilon\left\|v^j\right\|^p_{L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}.
\end{aligned}\tag{4.8}$$

Finally, by using (4.4) and (4.8)

$$\begin{aligned}
R_0 \leq C&\left\{\|f\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}+\varepsilon^p\|v\|^p_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^{n-1}))}\right.\\
&\left.+C'_\varepsilon\|v\|^p_{\prod_{j=1}^N L^p\left(\mathbb{R}_+;\mathcal{B}^{s+t_j-1,\tau}_{p,q}(\mathbb{R}^{n-1})\right)}\right\}.
\end{aligned}\tag{4.9}$$

Putting together the inequality (4.3) and choosing arbitrarily $\varepsilon>0$ small, we get the Proposition 1 for the system $(M^0, B^0\gamma)$.
To complete the proof of Theorem 4.1, we have the following lemma to estimate the normal derivative of solution.

**Lemma 4.2** *Let $s,\tau$ be two nonegative real numbers and $1\leq p,q<+\infty$. For any compact set $K$ in $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant $C_K>0$, such that for any $v\in\prod_{j=1}^N\mathcal{B}^{s+t_j,\tau}_{p,q}\left(\mathbb{R}^n_+\right)$ avec supp$(v)\subset K$, we have:*

$$\|v\|_{\prod_{j=1}^N\mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^n_+)}\leq C_K\left\{\|Mv\|_{\prod_{i=1}^N\mathcal{B}^{s+s_i,\tau}_{p,q}(\mathbb{R}^n_+)}+\|v\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{B}^{s+t_j,\tau}_{p,q}(\mathbb{R}^{n-1}_+))}\right\}.$$

**Proof.** Same idea of the proof of Proposition 1 we can restrict ourselves to the operator $M^0$. First take $0 \le s < 1$. We have

$$\left\|v^j\right\|_{\mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+^n)} = \left\|D_t^{t_j} v^j\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)} + \sum_{\substack{k+\left|\alpha'\right|\le t_j \\ 0\le k\le \le_j -1}} \left\|D_{x'}^{\alpha'} D_t^k v^j\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)}.$$

The interpolation Lemma 2.9 gives, for $k+|\alpha'| < t_j,\ 0\le k\le t_j-1$

$$\begin{aligned}
&\left\|D_{x'}^{\alpha'} D_t^k v^j\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)} \\
&\le C\left\{\left\|D_{x'}^{\alpha'} D_t^k v^j\right\|_{L^p(\mathbb{R}_+\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^{n-1}))} + \left\|D_{x'}^{\alpha'} D_t^{k+1} v^j\right\|_{L^p\left(\mathbb{R}_+\mathcal{B}_{p,q}^{s-1,\tau}(\mathbb{R}^{n-1})\right)}\right\} \\
&\le C\left\{\left\|D_t^k v^j\right\|_{L^p\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s+t_j-k,\tau}(\mathbb{R}^{n-1})\right)} + \left\|D_t^{k+1} v^j\right\|_{L^p\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s+t_j-k-1,\tau}(\mathbb{R}^{n-1})\right)}\right\} \\
&\le C\left\|v^j\right\|_{W^{t_j,p}(\mathbb{R}_+\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^{n-1}))}.
\end{aligned} \tag{4.10}$$

To estimate $||D_t^k v^j||_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)}$, we return again to the equation $M^0 v = f$, to get

$$D_t^T v = A^{-1} D_t^{-s} f - A^{-1} V,$$

and with aid of (4.10), we have

$$\begin{aligned}
\left\|D_t^T v\right\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)} &= \sum_{j=1}^N \left\|D_t^{t_j} v^j\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)} \\
&\le C\left\{\sum_{i=1}^N \left\|D_t^{-s_i}\left(M^0 v\right)^i\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)} + \sum_{\substack{k+\left|\alpha'\right|=t_j \\ 0\le k\le t_j-1}} \left\|D_{x'}^{\alpha'} D_t^k v^j\right\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}_+^n)}\right\} \\
&\le C\left\{\sum_{i=1}^N \left\|\left(M^0 v\right)^i\right\|_{\mathcal{B}_{p,q}^{s-s_i,\tau}(\mathbb{R}_+^n)} + \sum_{j=1}^N \left\|v^j\right\|_{W^{t,p}(\mathbb{R}_+;\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^{n-1}))}\right\}.
\end{aligned} \tag{4.11}$$

Then the Lemma 4.2 is proved for $0 \le s < 1$.
Now, for $s \ge 0$, we write $s = q + r,\ q \in \mathbb{N},\ 0 \le r < 1$.
We do a proof by induction on $q$. This is true for the case $q = 0$. Assuming that the estimate of Lemma 4.2 is true for all $q$, we show that it holds for $q+1$. Let $K$ be a compact set of $\overline{\mathbb{R}_+}$ and $u \in \prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j+1,\tau}(\mathbb{R}_+)$ with *suppu* $\subset K$ We notice that $v^j \in \mathcal{B}_{p,q}^{s+t_j+1,\tau}(\mathbb{R}_+)$ if and only if $v^j \in \mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+)$ and $D_{x_k} v^j \in \mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+)$ and $D_t^{t_j} v^j \in \mathcal{B}_{p,q}^{s+1,\tau}(\mathbb{R}_+)$. Thus

$$\|v\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+1+t_j}(\mathbb{R}_+^n)} \le C\left\{\|v\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j,\tau}(\mathbb{R}_+^n)} + \sum_{k=1}^{n-1} \|D_{x_k} u\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j}(\mathbb{R}_+^n)} + \left\|D_t^T v\right\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+1,\tau}(\mathbb{R}_+^n)}\right\}.$$

By the induction hypothesis, we have

$$\|D_{x_k} v\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+t_j}(\mathbb{R}_+^n)} \le C_K\left\{\left\|M^0 v\right\|_{\prod_{i=1}^N \mathcal{B}_{p,q}^{s-s_i+1,\tau}(\mathbb{R}_+^n)} + \|v\|_{\prod_{j=1}^N W^{t_j,p}\left(\mathbb{R}_+;\mathcal{B}_{p,q}^{s+1,\tau}(\mathbb{R}^{n-1})\right)}\right\}. \tag{4.12}$$

We substitute in (4.12) $v$ by $D_{x_k} v$; $1 \le k \le n-1$ and the coefficients of $M^0$ are constant, so we obtain

$$\left\|D_t^T v\right\|_{\prod_{j=1}^N \mathcal{B}_{p,q}^{s+1,\tau}(\mathbb{R}_+^n)} \le C\left\{\sum_{i=1}^N \left\|\left(M^0 v\right)^i\right\|_{\mathcal{B}_{p,q}^{s-s_i+1,\tau}(\mathbb{R}_+^n)} + \sum_{j=1}^N \left\|v^j\right\|_{W^{t_j,p}\left(\mathbb{R}_+\mathcal{B}_{p,q}^{s+1,\tau}(\mathbb{R}^{n-1})\right)}\right\}. \tag{4.13}$$

Finally, we combine the inequalities (4.12) and (4.13) in (4.11) to obtain the inequality of Lemma 4.2 for $s+1=q+1+r$. The Theorem 4.1 is a consequence of Proposition 1 and Lemma 4.2.

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