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f-Wijsman Deferred Statistical Convergence and Some Asymptotic Results in Metric Spaces

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ABSTRACT: In this paper, Wijsman deferred statistical convergence of sequences of sets in any metric spaces is generalized by the help of modulus function named f-Wijsman deferred statistical convergence. Also some new results about this new concept is given.

Key Words: Wijsman convergence, deferred density, modulus function, statistically equivalent sequences.

Contents

1. Introduction and Background

The concept of statistical convergence was first defined by Fast [16] and Steinhaus [35] then reintroduced by Schoenberg [33]. Its popularity in summability theory has increased after the initiator works of Fridy [17] and Šalát [32]. Some authors studied this concept as a nonmatrix summability method [8,9,13,14,15,17,18,33,34].

The asymptotic density of $M \subseteq \mathbb{N}$ is defined by $\delta(M) = \lim_{m\to\infty} \frac{|M(m)|}{m}$, where $|M(m)|$ represent the number of elements of $M(m)$ and express $M(m) = \{k \le m : k \in M\}$ for $m \in \mathbb{N}$.

$\xi = (\xi_m)_{m\in\mathbb{N}}$ is statistical convergent to ξ_0 if for every $\varepsilon > 0$,

$$\delta(\{m : |\xi_m - \xi_0| \ge \varepsilon\}) = 0$$

holds (denoted by $st - \lim_{m\to\infty} \xi_m = \xi_0$).

The concept of convergence of sequences has been extended by severel authors such as Aizuru *et al.* [2], Bhardwaj and Dhawan [5,6], Cakalli [7], Connor [8], Et et al. [10,11,12], Kucukaslan et al. [22], Kucukaslan and Yilmazturk [23], Mursaleen [27] and many others.

In 1932, R. P. Agnew in [1] defined the deferred Cesàro mean $D_{\varsigma,\vartheta}$ of a sequence $\xi = (\xi_m)$ by

$$(D_{\varsigma,\vartheta}\xi)_m := \frac{1}{\vartheta - \varsigma} \sum_{k=\varsigma+1}^{\vartheta} \xi_k,$$

where $\varsigma = (\varsigma_m)$ and $\vartheta = (\vartheta_m)$ are sequences in $\mathbb{N}^+$ under which

$$\varsigma_m < \vartheta_m \quad \text{and} \quad \lim_{m\to\infty} \vartheta_m = \infty. \tag{1.1}$$

For brevity, ς and ϑ will be used instead of (ς_m) and (ϑ_m), respectively.

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[21] Deferred density of $M \subseteq \mathbb{N}$ is defined as follows:

$$\delta_D(M) := \lim_{m\to\infty} \frac{1}{\vartheta - \varsigma} \big|\{\varsigma < k \leq \vartheta : k \in M\}\big|.$$

In this paper we take account one of them named Wijsman convergence. Statistical convergence of the sequence of sets was investigated by Nuray and Rhoades [29]. They introduced Wijsman statistical convergences of sequences of sets.

Let (X, κ) be an arbitrary metric space. The symbol $d_x(T)$ denotes the distance of the point $x \in X$ to the set T.i.e.,

$$d_x(T) := \inf\{\kappa(x,t) \ : \ t \in T\}.$$

Definition 1.1 *Let (X, κ) be a metric space. For any closed (nonempty) subsets $A_k, T \subseteq X$, $k \in \mathbb{N}$ we say that the sequence $A = (A_k)_{k\in\mathbb{N}}$ is*

- *Wijsman convergent to the set T ($W - \lim A_k = T$) if*

  $$\lim_{k\to\infty} d_x(A_k) = d_x(T),$$

  exists for each $x \in X$ [37].

- *Wijsman statistically convergent to T ($WS - \lim A_k = T$) if the sequence $(d_x(A_k))$ is statistically convergent to $d_x(T)$ for each $x \in X$.*

- *Wijsman strongly deferred Cesàro summable to the set T ($WD - \lim A_k = T$) if for each $x \in X$,*

  $$\lim_{m\to\infty} \frac{1}{\vartheta - \varsigma} \sum_{k=\varsigma+1}^{\vartheta} |d_x(A_k) - d_x(T)| = 0$$

  holds [3].

- *Wijsman deferred statistically convergent to the set T ($WDS - \lim A_k = T$) if for every $\varepsilon > 0$ and $x \in X$*

  $$\lim_{m\to\infty} \frac{1}{\vartheta - \varsigma} |\{\varsigma < k \leq \vartheta \ : |d_x(A_k) - d_x(T)| \geq \varepsilon\}| = 0$$

  holds [3].

In this paper, by using the concept of f-density which was defined by Aizpuru et. al. [2], we give a generalization of Wijsman deferred density. Then we will define f-Wijsman deferred statistical convergence for sequences of closed subsets of any metric spaces. For this purpose let us recall the definition of modulus function.

Density by moduli was defined in [2] by using modulus function. They also obtained a generalization of statistical convergence by using this new concept.

A modulus is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(a) = 0 \Leftrightarrow a = 0$,

(ii) $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$,

(iii) f is increasing,

(iv) f is continuous.

If f, g are modulus functions and $\alpha, \beta \in \mathbb{R}^+$, then $f \circ g$, $\alpha f + \beta g$ and $f \vee g$ are also modulus functions. $f(a) = a^p$ where $0 < p \leq 1$ is an example of unbounded, $g(a) = \frac{a}{1+a}$ is an example of bounded modulus.

Modulus function was first defined by Nakano [28]. Many redefined and investigated sequence spaces by the help of modulus function have been introduced by Ruckle [31] and Maddox [24].

Definition 1.2 *([2]) Let f be a modulus from $[0,\infty)$ to $[0,\infty)$. f-density of $M \subseteq \mathbb{N}$ is defined by*

$$\delta^f(M) := \lim_{m\to\infty} \frac{f(|M(m)|)}{f(m)}$$

if the limit exists (note that f is unbounded).

Note that, if M is finite, then $\delta^f(M) = 0$. Also, if $\delta^f(M) = 0$, then $\delta^f(M^c) = 1$ where M^c is complement of M.

A sequence (ξ_m) is f-statistical convergent to $\xi_0 \in \mathbb{R}$ if, $\delta^f(\{m \in \mathbb{N} : |\xi_m - \xi_0| \geq \varepsilon\}) = 0$ for $\varepsilon > 0$.

Also, we will give some asymptotic results about f-Wijsman deferred statistical convergence of sequences of sets in the third section. Now let us recall some basic definitions about asymptotically equivalent.

Definition 1.3 *([26]) Let $\alpha = (\alpha_m)$ and $\beta = (\beta_m)$ be non-negative sequences. α and β are asymptotically equivalent if*

$$\lim_{m\to\infty} \frac{\alpha_m}{\beta_m} = 1. \tag{1.2}$$

It is denoted by $\alpha \sim \beta$.

By combination the definition of statistical convergence and Definition 1.3 asymtotically statistical equivalent with multiple L of two non-negative sequences is defined by Patterson in [30] as follows:

Definition 1.4 *([30])Let $\alpha = (\alpha_m)$ and $\beta = (\beta_m)$ be non-negative sequences. α and β are asymptotically equivalent with multiple L if for every $\varepsilon > 0$*

$$\lim_{m\to\infty} \frac{1}{m} \left| \left\{ k \leq m : \quad \left| \frac{\alpha_k}{\beta_k} - L \right| \geqslant \varepsilon \right\} \right| = 0, \tag{1.3}$$

exists (denoted by $\alpha \overset{S_L}{\sim} \beta$).

Also, if $L = 1$ in (1.3), the sequences α and β are called asymptotically statistical equivalent (denoted by $\alpha \overset{S}{\sim} \beta$).

Asymptotically equivalent and asymptotically statistical equivalent of sequences of sets is defined by Ulusu and Nuray in [36] as follow:

Definition 1.5 *Let (X, κ) be a metric space. For any closed (nonempty) subsets $A = (A_k)$, $B = (B_k) \subseteq X$ such that $d_x(A_k) > 0$ and $d_x(B_k) > 0$ for each $x \in X$. The sequences $A = (A_k)$ and $B = (B_k)$ are*

- *asymptotically equivalent(in the Wijsman sense) with mutiple L if for each $x \in X$,*

$$\lim_{m\to\infty} \frac{d_x(A_k)}{d_x(B_k)} = L, \tag{1.4}$$

  (denoted by $A \overset{W_L}{\sim} B$) [36].

- *asymptotically statistical equivalent(in the Wijsman sense) with mutiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{m\to\infty} \frac{1}{m} \left| \left\{ k \leq m : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| = 0, \tag{1.5}$$

  (denoted by $A \overset{WS_L}{\sim} B$) [36].

2. f-Wijsman Deferred Statistical Convergence

In this section, we will build a new concept named f-Wijsman deferred statisitically convergence by the help of modulus functions. Then we will give some results about this new concept.

Throughout the paper, f will be taken as unbounded modulus function. Also (X,κ) be a metric space, the sets A_k, $k\in\mathbb{N}$ and T be nonempty closed subset of X.

Definition 2.1 *f-deferred density of $M\subseteq\mathbb{N}$ is defined as follows:*

$$\delta_D^f(M) := \lim_{m\to\infty}\frac{1}{f(\vartheta-\varsigma)}f\left(|\{\varsigma<k\le\vartheta : k\in M\}|\right)$$

if the limit exists [20].

Definition 2.2 *The sequence $(A_k)_{k\in\mathbb{N}}$ is f-Wijsman statistical convergent to T ($WS^f-\lim A_k=T$) if the sequence $(d_x(A_k))$ is f-statistical convergent to $d_x(T)$ for each $x\in X$.*

Definition 2.3 *The sequence $(A_k)_{k\in\mathbb{N}}$ is f-Wijsman strongly deferred Cesàro summable to the set T ($WD^f-\lim A_k=T$) if for each $x\in X$,*

$$\lim_{m\to\infty}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta}f(|d_x(A_k)-d_x(T)|)=0 \tag{2.1}$$

hold.

Proposition 2.1 *Let f be a modulus and $0<\delta<1$. Then, we have $f(x)\le 2f(1)x/\delta$ for each $x\ge\delta$ [6].*

Theorem 2.1 *If $WD-\lim A_k=T$, then $WD^f-\lim A_k=T$.*

Proof: Let us assume that $WD-\lim A_k=T$. Then for each $x\in X$

$$D(m) := \frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta}|d_x(A_k)-d_x(T)|\to 0, \quad (m\to\infty)$$

Let $\varepsilon>0$, choose $0<\delta<1$ such that $f(x)<\varepsilon$ for every x with $0\le x\le\delta$. So, by using Proposition 2.1

$$\begin{aligned}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta}f(|d_x(A_k)-d_x(T)|) &= \frac{1}{\vartheta-\varsigma}\left(\sum_{\substack{k=\varsigma+1\\ |d_x(A_k)-d_x(T)|\le\delta}}^{\vartheta}f(|d_x(A_k)-d_x(T)|)\right)\\ &+ \frac{1}{\vartheta-\varsigma}\left(\sum_{\substack{k=\varsigma+1\\ |d_x(A_k)-d_x(T)|>\delta}}^{\vartheta}f(|d_x(A_k)-d_x(T)|)\right)\\ &\le \varepsilon+\frac{2f(1)D(m)}{\delta(\vartheta-\varsigma)}\end{aligned}$$

Hence, $WD^f-\lim A_k=T$. □

For the converse of Theorem 2.1 let us examine following example:

Example 2.1 Let X be the set of real numbers, $\kappa(x,y)$ be the usual metric on $\mathbb{R}$ and $f(x)=\log(x+1)$. Let us define a sequence (A_k) as follows:

$$A_k := \begin{cases}\{\vartheta-\varsigma\}, & k\in(\varsigma,\vartheta] \text{ such that } k=\varsigma+1\\ \{0\}, & \text{otherwise}\end{cases} \tag{2.2}$$

For each $x \in \mathbb{R}$

$$\begin{aligned}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta} f(|d_x(A_k)-d_x(\{0\})|) &= \frac{1}{\vartheta-\varsigma} f(|d_x(A_{\varsigma+1})-d_x(\{0\})|)\\ &= \frac{1}{\vartheta-\varsigma} f(|d_x(\{\vartheta-\varsigma\})-d_x(\{0\})|)\\ &\leq \frac{1}{\vartheta-\varsigma} f(|(x-(\vartheta-\varsigma))-(x-0)|)\\ &= \frac{1}{\vartheta-\varsigma} f(\vartheta-\varsigma)=\frac{\log(\vartheta-\varsigma+1)}{\vartheta-\varsigma}\to 0\end{aligned}$$

when $m\to\infty$. So, $WD^f-\lim A_k=\{0\}$. But for $x=0$

$$\begin{aligned}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta} |d_x(A_k)-d_x(\{0\})| &= \frac{1}{\vartheta-\varsigma}|d_x(A_{\varsigma+1})-d_x(\{0\})|\\ &= \frac{1}{\vartheta-\varsigma}||x-(\vartheta-\varsigma)|-|x-0||=\frac{(\vartheta-\varsigma)}{\vartheta-\varsigma}\to 1\end{aligned}$$

when $m\to\infty$. So, $WD-\lim A_k\neq\{0\}$.

In [25], Maddox proved that there exists $\lim_{a\to\infty}\frac{f(a)}{a}$ for any modulus function f. By this condition converse of Theorem 2.1 holds.

Theorem 2.2 *Let us assume* $\lim_{a\to\infty}\frac{f(a)}{a}>0$ *holds. If* $WD^f-\lim A_k=T$*, then* $WD-\lim A_k=T$*.*

Proof: As in the proof of Proposition 1 in [25], we have $\alpha=\lim_{a\to\infty}\frac{f(a)}{a}=\inf\{\frac{f(a)}{a}: a>0\}$. From the description of α, we have $f(a)\geq\alpha a$ for all $a>0$. $0<\alpha\leq f(1)$, so we have $a\leq\alpha^{-1}f(a)$ for all $a\geq 0$. Thus,

$$\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta}|d_x(A_k)-d_x(T)|\leq\alpha^{-1}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta} f(|d_x(A_k)-d_x(T)|)$$

holds. If we take limit for $m\to\infty$ we obtain $WD-\lim A_k=T$. □

Theorem 2.3 *If* $WD^f-\lim A_k=T$*, then* $WDS-\lim A_k=T$*.*

Proof: Let us assume that $WD^f-\lim A_k=T$. For every $x\in X$ and $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta} f(|d_x(A_k)-d_x(T)|) &\geq \frac{1}{\vartheta-\varsigma}\left(\sum_{\substack{k=\varsigma+1\\ |d_x(A_k)-d_x(T)|\geq\varepsilon}}^{\vartheta} f(|d_x(A_k)-d_x(T)|)\right)\\ &\geq \frac{1}{\vartheta-\varsigma}|\{\varsigma<k\leq\vartheta: |d_x(A_k)-d_x(T)|\geq\varepsilon\}|f(\varepsilon).\end{aligned}$$

If we take limit when $m\to\infty$, we obtain $WDS-\lim A_k=T$. □

For the converse of Theorem 2.3 let us examine following example:

Example 2.2 Let $X=\mathbb{R}$, $\kappa(x,y)$ be the usual metric on $\mathbb{R}$ and $f(x)=2x$. Let (A_k) defined as in Example 2.1. (A_k) is Wijsman Deferred statistical convergent to $\{0\}$. Actually, for every $x\in\mathbb{R}$ and $\varepsilon>0$

$$\lim_{m\to\infty}\frac{1}{\vartheta-\varsigma}|\{\varsigma<k\leq\vartheta: |d_x(A_k)-d_x(\{0\})|\geq\varepsilon\}|=\lim_{m\to\infty}\frac{1}{\vartheta-\varsigma}=0.$$

But (A_k) is not f-Wijsman Deferred strongly convergent.

$$\begin{aligned}\frac{1}{\vartheta-\varsigma}\sum_{k=\varsigma+1}^{\vartheta} f(|d_x(A_k)-d_x(\{0\})|) &= \frac{1}{\vartheta-\varsigma} f(|d_x(A_{\varsigma+1})-d_x(\{0\})|)\\ &= \frac{1}{\vartheta-\varsigma} f(|d_x(\{\vartheta-\varsigma\})-d_x(\{0\})|)\\ &\le \frac{1}{\vartheta-\varsigma} f(|(x-(\vartheta-\varsigma))-(x-0)|)\\ &= \frac{1}{\vartheta-\varsigma} f(\vartheta-\varsigma)=\frac{2(\vartheta-\varsigma)}{\vartheta-\varsigma}\nrightarrow 0.\end{aligned}$$

Definition 2.4 *A sequence $(A_k)_{k\in\mathbb{N}}$ is f-Wijsman deferred statistical convergent to T ($WDS^f-\lim A_k = T$) if for every $x \in X$,*

$$\lim_{m\to\infty}\frac{1}{f(\vartheta-\varsigma)} f\left(|\{\varsigma<k\le\vartheta\ : |d_x(A_k)-d_x(T)|\ge\varepsilon|\}\right)=0 \tag{2.3}$$

hold.

Theorem 2.4 *Let (X,κ) be a metric space and the inclusions $A_k \subseteq B_k \subseteq C_k$ hold for all $k \in \mathbb{N}$ for $A=(A_k)$, $B=(B_k)$ and $C=(C_k)$. If $WDS^f-\lim A_k = WDS^f-\lim C_k = T$, then $WDS^f-\lim B_k = T$.*

Proof: Let $x \in X$ be an arbitrary fixed point and consider the saequences $(d_x(A_k))$, $(d_x(B_k))$ and $(d_x(C_k))$. It is clear from the inclusion $A_k \subseteq B_k \subseteq C_k$ tha the inequality

$$d_x(C_k)\le d_x(B_k)\le d_x(A_k)$$

holds for all $k \in \mathbb{N}$. From this inequality, we have

$$\begin{aligned}\{\varsigma<k\le\vartheta : |d_x(B_k)-d_x(T)|\ge\varepsilon\} &= \{\varsigma<k\le\vartheta : d_x(B_k)\ge d_x(T)+\varepsilon\}\\ &\cup \{\varsigma<k\le\vartheta : d_x(B_k)\le d_x(T)-\varepsilon\}\\ &\subset \{\varsigma<k\le\vartheta : d_x(A_k)\ge d_x(T)+\varepsilon\}\\ &\cup \{\varsigma<k\le\vartheta : d_x(C_k)\le d_x(T)-\varepsilon\}\end{aligned}$$

for $\varepsilon>0$. It is also clear that

$$\{\varsigma<k\le\vartheta : |d_x(A_k)-d_x(T)|\ge\varepsilon\}\supset\{\varsigma<k\le\vartheta : d_x(A_k)\ge d_x(T)+\varepsilon\}$$

and

$$\{\varsigma<k\le\vartheta : |d_x(C_k)-d_x(T)|\ge\varepsilon\}\supset\{\varsigma<k\le\vartheta : d_x(C_k)\le d_x(T)-\varepsilon\}$$

are true. Also we have

$$\delta_D^f(\{\varsigma<k\le\vartheta : d_x(A_k)\ge d_x(T)+\varepsilon\})=0,$$
$$\delta_D^f(\{\varsigma<k\le\vartheta : d_x(C_k)\ge d_x(T)-\varepsilon\})=0.$$

So,

$$\delta_D^f(\{\varsigma<k\le\vartheta : |d_x(B_k)-d_x(T)|\ge\varepsilon\})=0.$$

□

Definition 2.5 *Let (A_k) and (B_k) be sequences of sets.*

- *If the set $A=(A_k)$ have a property $\mathcal{P}$ for all $k\in\mathbb{N}$ except a set which has zero f-deferred density. In this case, we say the sequence $A=(A_k)$ has the property $\mathcal{P}$ f-deferred almost all $k\in\mathbb{N}$ (denoted by "$f-D-a.a.k$").*

- *If f-deferred density of $\{k \in \mathbb{N} : A_k \neq B_k\}$ is zero, then it is said that the sequence (A_k) is f-deferred almost all equal to the sequence (B_k) (denoted by $(A_k) \equiv (B_k)(f - D - a.a.k)$).*

Theorem 2.5 *Let $(A_k) \equiv (B_k)(f - D - a.a.k)$. Then, f-Wijsman deferred statistical convergency of the sequence (A_k) implies f-Wijsman deferred statistical convergency of the sequence (B_k), vice versa.*

Proof: Assume that $WDS^f - \lim A_k = T$. Namely,

$$\lim_{m\to\infty} \frac{1}{f(\vartheta - \varsigma)} f\left(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) = 0 \tag{2.4}$$

holds for $x \in X$.

Since $A_k \neq B_k(f - D - a.a.k)$, then we have

$$\lim_{m\to\infty} \frac{1}{f(\vartheta - \varsigma)} f\left(|\{\varsigma < k \leq \vartheta : A_k \neq B_k\}|\right) = 0. \tag{2.5}$$

Also, the set

$$\{\varsigma < k \leq \vartheta : |d_x(B_k) - d_x(T)| \geq \varepsilon\}$$

can be represent as

$$\{\varsigma < k \leq \vartheta : A_k = B_k\} \cup \{\varsigma < k \leq \vartheta : A_k \neq B_k\} \tag{2.6}$$

for k when $|d_x(B_k) - d_x(T)| \geq \varepsilon$.

From (2.4), (2.5) and (2.6) we have

$$\lim_{m\to\infty} \frac{1}{f(\vartheta - \varsigma)} f\left(|\{\varsigma < k \leq \vartheta : |d_x(B_k) - d_x(T)| \geq \varepsilon\}|\right) = 0$$

and this gives the proof. The converse can be proved by the same way. □

Corollary 2.1 *Let (A_k), (B_k) and (C_k) be sequences of sets such that $A_k \subset B_k \subset C_k(f - D - a.a.k)$. If $WDS^f - \lim A_k = WDS^f - \lim C_k = T$, then $WDS^f - \lim B_k = T$.*

Theorem 2.6 *If $W - \lim A_k = T$, then $WDS^f - \lim A_k = T$. But the converse need not to be true.*

Every finite set have zero f-density. So, it is clear that Wijsman convergent sequences are also f-Wijsman deferred statistical convergent with same limit.

For the converse of Theorem 2.6, let X be the set of real numbers, $f(x) = x^p$, $0 < p \leq 1$ and (A_k) defined as follows:

$$A_k := \begin{cases} [2, \vartheta - \varsigma], & k \geq 2 \text{ and } k \in (\varsigma, \vartheta] \text{ is a square} \\ \{1\}, & \text{otherwise} \end{cases}$$

This sequence is not Wijsman convergent but

$$\frac{1}{f(\vartheta - \varsigma)} f(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(\{1\})| \geq \varepsilon\}|) \leq \frac{f(\sqrt{\vartheta - \varsigma})}{f(\vartheta - \varsigma)} = \frac{(\sqrt{\vartheta - \varsigma})^p}{(\vartheta - \varsigma)^p} \to 0$$

when $m \to \infty$, for each $x \in \mathbb{R}$ and $\varepsilon > 0$. So, $WDS^f - \lim A_k = \{1\}$.

Theorem 2.7 *If $WDS^f - \lim A_k = T$, then $WDS - \lim A_k = T$ holds.*

Proof: Let $WDS^f - \lim A_k = T$. Suppose that (A_k) is not Wijsman deferred statistically convergent to T. Then there exist $x \in X$ and $\varepsilon > 0$ such that

$$\limsup_{m\to\infty} \frac{|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|}{\vartheta - \varsigma} > 0.$$

So, there exist $s \in \mathbb{N}$ and a sequence $(m_t) \subset \mathbb{N}$ such that

$$\lim_{t\to\infty} m_t = \infty \tag{2.7}$$

and

$$\frac{|\{\varsigma(m_t) < k \leq \vartheta(m_t) : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|}{\vartheta(m_t) - \varsigma(m_t)} \geq \frac{1}{s}$$

for every $t \in \mathbb{N}$. Last inequality can be written as follows:

$$\vartheta(m_t) - \varsigma(m_t) \leq s|\{\varsigma(m_t) < k \leq \vartheta(m_t) : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|. \tag{2.8}$$

From the third property (f is increasing) of modulus f and (2.8) we have

$$f(\vartheta(m_t) - \varsigma(m_t)) \leq sf(|\{\varsigma(m_t) < k \leq \vartheta(m_t) : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|).$$

So

$$\frac{f(|\{\varsigma(m_t) < k \leq \vartheta(m_t) : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|)}{f(\vartheta(m_t) - \varsigma(m_t))} \geq \frac{1}{s} \tag{2.9}$$

holds for every $t \in \mathbb{N}$. (2.7) and (2.9) imply

$$\limsup_{m\to\infty} \frac{f(|\{\varsigma(m) < k \leq \vartheta(m) : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|)}{f(\vartheta(m) - \varsigma(m))} \geq \frac{1}{s},$$

contrary to hypothesis of theorem. □

From Theorem 2.7 following result obtained:

Theorem 2.8 *Let f_1, f_2 be unbounded modulus functions. If*

$$WDS^{f_1} - \lim A_k = T \quad \textit{and} \quad WDS^{f_2} - \lim A_k = K \tag{2.10}$$

hold for nonempty closed subsets A_k, T, K of X for $k \in \mathbb{N}$, then $T = K$.

Proof: Let us assume that (2.10) hold. By Theorem 2.7 the sequence (A_k) is Wijsman Deferred statistical convergent to T and K. from the uniqueness of this limit we obtain that $d_x(T) = d_x(K)$ for every $x \in X$. It implies that $T = K$ because T and K are closed subsets of X.

So, we can say that f-Wijsman deferred statistical limit is unique. □

Let us assume that following inequqlity holds for the sequences $\varsigma = \varsigma_m$, $\vartheta = \vartheta_m$, $\varsigma* = \varsigma*_m$, and $\vartheta* = \vartheta*_m$:

$$\varsigma \leq \varsigma* < \vartheta* \leq \vartheta \tag{2.11}$$

for all $m \in \mathbb{N}$. In the folowing theorems by considering (2.11), we obtain some comparison results.

Theorem 2.9 *If $\{k : \varsigma < k \leq \varsigma*\}$ and $\{k : \vartheta* < k \leq \vartheta\}$ are finite sets for all $k \in \mathbb{N}$, then*

$$WDS^f - \lim A_k = T \ w.r.t.\ (\varsigma* \ \textit{and}\ \vartheta*)$$

implies

$$WDS^f - \lim A_k = T \ w.r.t.\ (\varsigma \ \textit{and}\ \vartheta).$$

Proof: Let us assume that $WDS^f - \lim A_k = T$ $w.r.t.$ $(\varsigma*$ and $\vartheta*)$. For an arbitrary $\varepsilon > 0$ we have

$$\begin{aligned}\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\} &= \{\varsigma < k \leq \varsigma* : |d_x(A_k) - d_x(T)| \geq \varepsilon\} \cup \\ &\cup \{\varsigma* < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\} \\ &\cup \{\vartheta* < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}\end{aligned}$$

It is also clear that following inequality

$$\begin{aligned} |\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}| &\leq |\{\varsigma < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\}| + \\ &+ |\{\varsigma* < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\}| \\ &+ |\{\vartheta* < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}| \end{aligned}$$

holds. From the second and third properties of f we have

$$\begin{aligned} &\frac{1}{f(\vartheta - \varsigma)} f\left(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) \leq \\ \leq\; &\frac{1}{f(\vartheta * - \varsigma *)} f\left(|\{\varsigma < k \leq \varsigma * : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) \\ +\; &\frac{1}{f(\vartheta * - \varsigma *)} f\left(|\{\varsigma * < k \leq \vartheta * : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) \\ +\; &\frac{1}{f(\vartheta * - \varsigma *)} f\left(|\{\vartheta * < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) \end{aligned}$$

holds. If we take limit when $m \to \infty$, it is obtain that

$$\frac{1}{f(\vartheta - \varsigma)} f\left(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|\right) = 0.$$

□

Theorem 2.10 *Under the condition (2.11), if $\frac{f(\vartheta-\varsigma)}{f(\vartheta*-\varsigma*)}$ is bounded $WDS^f - \lim A_k = T$ w.r.t. (ς and ϑ) implies $WDS^f - \lim A_k = T$ w.r.t. ($\varsigma*$ and $\vartheta*$).*

Proof: From the following inclusion and the third property of f

$$\{\varsigma* < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\} \subset \{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}$$

we have

$$f(|\{\varsigma* < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|) \leq f(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|).$$

So,

$$\begin{aligned} &\frac{1}{f(\vartheta * - \varsigma *)} f(|\{\varsigma* < k \leq \vartheta* : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|) \\ &\leq \frac{f(\vartheta - \varsigma)}{f(\vartheta * - \varsigma *)} \frac{1}{f(\vartheta - \varsigma)} f(|\{\varsigma < k \leq \vartheta : |d_x(A_k) - d_x(T)| \geq \varepsilon\}|) \end{aligned}$$

holds. For $m \to \infty$, desired result obtained. □

3. WDS_L^f-Equivalence of Sequences of Sets

In this section, our aim is to give a generalization of Definition 1.4 by considering f-deferred statistical density which is defined in [20]. Then we will give some general results about this new concept.

$A = (A_k)$, $B = (B_k)$ be nonempty closed subsets of X such that $d_x(A_k) > 0$ and $d_x(B_k) > 0$ hold for each $x \in X$ and $k \in \mathbb{N}$. For brevity, let the set of all such subsets be denoted by $\mathcal{CL}(X)$.

If $A_k \subseteq B_k$ holds for all $k \in \mathbb{N}$, then it is shown by $A \prec B$.

Definition 3.1 *Let (X, κ) be a metric space. $A, B \in \mathcal{CL}(X)$. It is said that the sequences A and B are*

- *asymptotically f-statistical equivalent (in the Wijsman sense) with multiple L if for each $\varepsilon > 0$ and $x \in X$,*

$$\lim_{m\to\infty} \frac{1}{f(m)} f\left(\left|\{k \leq m : \mid \frac{d_x(A_k)}{d_x(B_k)} - L \mid \geq \varepsilon\}\right|\right) = 0$$

(denoted by $A \overset{WS_L^f}{\sim} B$)

- *asymptotically f-deferred equivalent (in the Wijsman sense) with mutiple L if for each $x \in X$,*

$$\lim_{m\to\infty} \frac{1}{\vartheta - \varsigma} \sum_{k=\varsigma+1}^{\vartheta} f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) = 0 \tag{3.1}$$

(denoted by $A \overset{WD_L^f}{\sim} B$).

- *asymptotically f-deferred statistical equivalent (in the Wijsman sense) with mutiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{m\to\infty} \frac{1}{f(\vartheta - \varsigma)} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) = 0 \tag{3.2}$$

(denoted by $A \overset{WDS_L^f}{\sim} B$).

Theorem 3.1 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ and $A \prec C$, then $C \overset{WDS_L^f}{\sim} B$.*

Proof: Let us assume that $A \overset{WDS_L^f}{\sim} B$ and $A \prec C$. Let $x \in X$ be an arbitrary fixed point. Since $A \prec C$, then

$$d_x(C_k) \leq d_x(A_k)$$

hold for all $k \in \mathbb{N}$. Therefore, the inequality

$$\left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \leq \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|$$

holds for all sufficiently large $n \in \mathbb{N}$, then the inclusion

$$\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \subseteq \left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}$$

holds. So, for any $\varepsilon > 0$, following inequality

$$\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \leq \left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|$$

holds. From the third property of modulus function f

$$f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) \leq f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right)$$

holds. If we divide the inequalty by $f(\vartheta - \varsigma)$ and take limit when $m \to \infty$, it is obtained that $C \overset{WDS_L^f}{\sim} B$. □

Theorem 3.2 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ and $C \prec B$, then $A \overset{WDS_L^f}{\sim} C$.*

Proof: Assume that $A \overset{WDS_L^f}{\sim} B$ and $C \prec B$. Let $x \in X$ be an arbitrary fixed point. Since $C \prec B$, then

$$d_x(B_k) \leq d_x(C_k)$$

hold for all $k \in \mathbb{N}$. therefore, the inequality

$$\left|\frac{d_x(A_k)}{d_x(C_k)} - L\right| \leq \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|$$

holds for sufficiently large $m \in \mathbb{N}$. Hence, by the same way above theorem it is obtained that $A \overset{WDS_L^f}{\sim} C$. □

Corollary 3.1 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ then $A \cup C \overset{WDS_L^f}{\sim} B$ and $A \overset{WDS_L^f}{\sim} B \cap C$ hold.*

For any sequence of sets $C = (C_k)$ we have $A_k \subset A_k \cup C_k$ and $B_k \cap C_k \subset B_k$ for all $k \in \mathbb{N}$. It means that $A \prec A \cup C$ and $B \cap C \prec B$. Hence, the proof of Corollary 3.1 is obtained from Theorem 3.1 and Theorem 3.2. So it is omitted.

Following theorems are generalizations of Theorem 3.1 and Theorem 3.2.

Theorem 3.3 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ and $A \prec C$ $(f - D - a.a.k)$ then $C \overset{WDS_L^f}{\sim} B$.*

Proof: Let us take account $M = \{k : \ C_k \subset A_k\}$. From the assumption, $\delta_D^f(M) = 0$ holds. So, following inequality

$$\left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \leq \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|$$

holds $f - D - a.a.k$. Then, we have

$$\begin{aligned} & \frac{1}{f(\vartheta - \varsigma)} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) \\ \leq \quad & \frac{1}{f(\vartheta - \varsigma)} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) + \frac{1}{f(\vartheta - \varsigma)} f\left(|M|\right). \end{aligned}$$

By taking limit when $m \to \infty$, it is obtained that $C \overset{WDS_L^f}{\sim} B$. □

Theorem 3.4 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ and $B \prec C$ $(f - D - a.a.k)$, then $A \overset{WDS_L^f}{\sim} C$.*

Theorem 3.4 can be proved by following the proof of Theorem 3.3.

Theorem 3.5 *Let $A, B, C \in \mathcal{CL}(X)$. If $A \overset{WDS_L^f}{\sim} B$ and $A = C$ $(f - D - a.a.k)$, then $C \overset{WDS_L^f}{\sim} B$.*

Proof: Let us take acoount the set $M := \{k : \ A_k \neq C_k\}$. From the assumption of this theorem we have $\delta_D^f(M) = 0$. Thus, for any $\varepsilon > 0$, the following inclusion

$$\begin{aligned} \left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} &= \left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \cap \left(M^C \cup M\right) \\ &\subseteq \left(\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \cap M^C\right) \\ &\cup \left(\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \cap M\right) \\ &\subseteq \left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \cup M \end{aligned}$$

holds. Therefore,

$$\begin{aligned}&\frac{1}{f(\vartheta-\varsigma)}f\left(\left|\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\right)\\ \leq\ &\frac{1}{f(\vartheta-\varsigma)}f\left(\left|\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\right)+\frac{1}{f(\vartheta-\varsigma)}f\left(|M|\right)\end{aligned}$$

holds. By taking limit when $m\to\infty$, it is obtained that $C\overset{WDS_L^f}{\sim}B$. □

Theorem 3.6 *Let $A,B,C\in\mathcal{CL}(X)$. If $A\overset{WDS_L^f}{\sim}B$ and $B=C$ $(f-D-a.a.k)$, then $A\overset{WDS_L^f}{\sim}C$.*

Proof: Let us take account the set $M:=\{k:\ B_k\neq C_k\}$. From the assumption $\delta_D^f(M)=0$. That is, $d_x(B_k)=d_x(C_k)$ $(f-D-a.a.k)$ satisfied for any $x\in X$. So, following inclusion

$$\begin{aligned}\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}&=\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap\left(M^C\cup M\right)\\&\subseteq\left(\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M^C\right)\\&\cup\left(\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap M\right)\end{aligned}$$

holds. Since

$$\left[\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M^C\right]\subseteq\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}$$

and

$$\left[\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap M\right]\subseteq M,$$

then we have

$$\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\subseteq\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup M.$$

Therefore,

$$\begin{aligned}&\frac{1}{f(\vartheta-\varsigma)}f\left(\left|\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\right|\right)\\ \leq\ &\frac{1}{f(\vartheta-\varsigma)}f\left(\left\{\varsigma<k\leq\vartheta:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\right)+\frac{1}{f(\vartheta-\varsigma)}f\left(|M|\right)\end{aligned}$$

holds. By taking limit when $m\to\infty$, it is obtained that $A\overset{WDS_L^f}{\sim}C$. □

4. Comparison of WD_L^f and WDS_L^f-Equivalence

In this section, WD_L^f-equivalence and WDS_L^f-equivalence will be compared. Also, it will be shown that WD_L^f-equivalence is equal WDS_L^f-equivalence under some conditions. This results are generalized versions of some results in [4].

Theorem 4.1 *Let $A,B\in\mathcal{CL}(X)$. Let us assume that $\lim_{a\to\infty}\frac{f(a)}{a}>0$. Then, $A\overset{WD_L^f}{\sim}B$ implies $A\overset{WDS_L^f}{\sim}B$.*

Proof: Assume that $A \overset{WD_L^f}{\sim} B$ i.e.,

$$\lim_{m\to\infty} \frac{1}{\vartheta-\varsigma} \sum_{k=\varsigma+1}^{\vartheta} f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) = 0.$$

For an arbitrary $\varepsilon > 0$, the following inequality

$$\begin{aligned}
\frac{1}{\vartheta-\varsigma} \sum_{k=\varsigma+1}^{\vartheta} f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) &= \frac{1}{\vartheta-\varsigma}\left(\sum_{\substack{k=\varsigma+1 \\ f\left(\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\right)\geq\varepsilon}}^{\vartheta} + \sum_{\substack{k=\varsigma+1 \\ f\left(\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\right)<\varepsilon}}^{\vartheta}\right) f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) \\
&\geq \frac{1}{\vartheta-\varsigma} \sum_{\substack{k=\varsigma+1 \\ f\left(\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right)}}^{\vartheta} f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) \\
&\geq \varepsilon \frac{1}{\vartheta-\varsigma} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right)
\end{aligned}$$

holds. So we have

$$\frac{1}{\vartheta-\varsigma} \sum_{k=\varsigma+1}^{\vartheta} f\left(\left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|\right) \geq \frac{f(\vartheta-\varsigma)}{f(\vartheta-\varsigma)} \frac{1}{\vartheta-\varsigma} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right)$$

If we take limit when $m \to \infty$, from the hypothesis we obtain

$$\lim_{m\to\infty} \frac{1}{f\,(\vartheta-\varsigma)} f\left(\left|\left\{\varsigma < k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) = 0.$$

This gives the proof. □

Following definition is an f-generalization of properly deferred method which defined in [1].

Definition 4.1 *A method $D^f_{\varsigma,\vartheta}$ is called properly f-deferred when $\varsigma = \{\varsigma(m)\}$ and $\vartheta = \{\vartheta(m)\}$ satisfy in addition to (1.1), the condition*

$$\left\{\frac{f(\varsigma(m))}{f(\vartheta(m)-\varsigma(m))}\right\}_{m\in\mathbb{N}}$$

is bounded.

In the following theorem, it is shown that WS_L^f-equivalence implies WDS_L^f- equivalence.

Theorem 4.2 *In order that $A \overset{WS_L^f}{\sim} B$ implies $A \overset{WDS_L^f}{\sim} B$ if and only if the method $D^f_{\varsigma,\vartheta}$ is properly f-deferred.*

Proof: Since $A \overset{WS_L^f}{\sim} B$, then we have

$$\lim_{m\to\infty} \frac{1}{f(m)} f\left(\left|\left\{k \leq m : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) = 0.$$

Therefore, following limit

$$\lim_{m\to\infty} \frac{1}{f(\vartheta)} f\left(\left|\left\{k \leq \vartheta : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|\right) = 0$$

exists because $\vartheta(m)\to\infty$, $m\to\infty$. It is clear from set comparison that the following inequality

$$\left|\left\{\varsigma<k\le\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|\le\left|\left\{k\le\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|$$

holds for every $\varepsilon>0$. Hence,

$$\frac{1}{f(\vartheta-\varsigma)}f\left(\left|\left\{\varsigma<k\le\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|\right)$$
$$\le\frac{f(\vartheta)}{f(\vartheta-\varsigma)}\frac{1}{f(\vartheta)}f\left(\left|\left\{k\le\vartheta:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|\right)$$

After taking limit when $m\to\infty$, we obtain desired result if and only if $D^f_{\varsigma,\vartheta}$ is properly f-deferred. □

Theorem 4.3 *If* $A\overset{WDS_L^f}{\sim}B$ *w.r.t an arbitrary* ς *and* $\vartheta=m$*, then* $A\overset{WS_L^f}{\sim}B$ *hold.*

Proof: Let $A\overset{WDS_L^f}{\sim}B$ for $\vartheta=m$ and arbitrary ς. For any $m\in\mathbb{N}$, there is a $q\in\mathbb{N}$ such that $m^{q+1}=0$ and the inequality

$$\varsigma(m)=m^{(1)}>\vartheta\left(m^{(1)}\right)=m^{(2)}>\varsigma\left(m^{(2)}\right)=m^{(3)}>...>\varsigma\left(m^{(q-1)}\right)=m^{(q)}\ge 1$$

holds. So, the set $\left\{k\le m:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}$ may be represent as

$$\left\{k\le m^{(1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\cup\left\{m^{(1)}<k\le m:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}.$$

Similarly the left hand set in the union can be represent as

$$\left\{k\le m^{(2)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\cup\left\{m^{(2)}<k\le m^{(1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}.$$

After some steps (at most h steps)

$$\begin{aligned}&\left\{k\le m^{(q-1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\\=\;&\left\{k\le m^{(q)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\cup\left\{m^{(q)}<k\le m^{(q-1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\end{aligned}$$

is obtained. Therefore,

$$\frac{1}{f(m)}f\left(\left|\left\{k\le m:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|\right)=\sum_{a=0}^{q}\frac{f(m^{(a)}-m^{(a+1)})}{f(m)}U_a,$$

where

$$U_a:=\frac{1}{f(m^{(a)}-m^{(a+1)})}f\left(\left|\left\{m^{(a+1)}<k\le m^{(a)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\ge\varepsilon\right\}\right|\right).$$

If we consider a matrix $S:=(s_{m,a})$ as

$$s_{m,a}:=\begin{cases}\frac{m^{(a)}-m^{(a+1)}}{m}, & a=0,1,2,...,q,\\ 0, & otherwise,\end{cases}$$

then the sequence

$$\left\{ \frac{1}{m} \left| \left\{ k \leq m : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| \right\}_{m \in \mathbb{N}}$$

is $(s_{m,a})$ transformation of the sequence (U_a).

Since the matrix $S = (s_{m,a})$ satisfies Silverman-Toeplitz Theorem (see in [19]) and from assumption on $A = (A_k)$ and $B = (B_k)$. So we have desired result. □

Combining Theorem 4.2 and Theorem 4.3 we can give following theorem without proof:

Theorem 4.4 *WDS_L^f-asymptotically equivalence w.r.t. any ς and $\vartheta = m$ is equivalent to WS_L^f-equivalence if and only if $\left\{ \frac{f(\varsigma)}{f(m-\varsigma)} \right\}$ is bounded for all $m \in \mathbb{N}$.*

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