Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.67138

Independent differential in graphs

Zeynep Nihan Berberler

ABSTRACT: Let $G = (V, E)$ be a graph of order n and let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D. The independent differential of an independent vertex set D is defined as $\partial_i(D) = |B(D)| - |D|$ and the maximum value of $\partial_i(D)$ for any independent subset D of V is the independent differential of G. An independent set S of vertices of a graph G is said to be an independent dominating set if every vertex in $V \setminus S$ is adjacent to a vertex in S. G is an independent dominant differential graph if it contains a ∂_i-set which is also an independent dominating set. In this paper, the study on properties of $\partial_i(G)$ is initiated and some upper and lower bounds on $\partial_i(G)$ are presented. This paper is devoted to the computation of independent differential of wheel, cycle, path-related graphs, and graph operations. Furthermore, independent dominant differential graph types are recognized.

Key Words: Differential, domination, independence, graph invariant.

Contents

1 Introduction **1**

2 Elementary Results **2**

3 Independent differential in wheel related graphs **3**

4 Independent differential in cycle related graphs **5**

5 Independent differential in path related graphs **8**

6 Final remarks **11**

1. Introduction

In this paper, simple, finite and undirected graphs without loops and multiple edges are considered. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *order* of G is given by $|V(G)| = n$ and the *size* is defined as $|E(G)| = m$ where $|*|$ denotes the number of elements in the set (i.e. the cardinality). The *neighborhood* of a vertex $v \in V(G)$ is the set of vertices adjacent to v, denoted $N_G(v)$ or just $N(v)$, and the *closed neighborhood* of v is given by $N[v] = N(v) \cup \{v\}$. Thus, $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $N(v)$ is referred to as the *open neighborhood* of v. The *degree* of a vertex $v \in V$ is defined as $d(v) = |N(v)|$. For a set $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. An *end-vertex* or a *pendant* or *pendent vertex* is a vertex of degree one and its neighbor is called a *support vertex*. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. Let G_1 and G_2 be two disjoint graphs. The *union* of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The join of graphs G and H, denoted by $G \vee H$, is obtained from the disjoint union G and H by adding the edges $\{xy | x \in V(G), y \in V(H)\}$ [1].

For any real number x we define the *ceiling function* $\lceil x \rceil$ as the smallest integer greater than or equal to x and similarly we define the *floor function* $\lfloor x \rfloor$ as the largest integer smallest than or equal to x .

The differential in graphs is a subject of increasing interest, both in pure and applied mathematics. The research and application of the $\partial(G)$ appears mainly in computational mathematics. The differential of a graph was introduced in [7] in 2006, and studied by several authors [5,8,9,10,13,16,17,18,19,20,21,22,23], motivated by its applications to information diffusion in social networks. The study of the mathematical properties of the differential in graphs together with a variety of other kinds of differentials

2010 *Mathematics Subject Classification:* 05C69.

Submitted February 15, 2023. Published September 24, 2025

of a set, stated in [5,7,8,9,10,13,16,17,18,19,20,21,22,23]. This parameter has been studied by many authors, both from the viewpoint of combinatorics and from the viewpoint of the algorithmic complexity. We refer to the papers [5,7,8,9,10,13,16,17,18,19,20,21,22,23] and the literature quoted therein.

Let $G = (V, E)$ be a graph of order n, for every set $D \subseteq V$ let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D . The *differential* of D is defined as $\partial(D) = |B(D)| - |D|$ and the *differential of a graph* G , written $\partial(G)$, is equal to $max\{\partial(D) : D \subseteq V\}$. The *independent differential of an independent vertex set* D is defined as $\partial_i(D) = |B(D)| - |D|$ and the maximum value of $\partial_i(D)$ for any independent subset D of V is the *independent differential of* G, that is $\partial_i(G) = max\{\partial_i(D) : D \subseteq V, D \text{ is independent}\}$ [25]. We will say that $D \subseteq V$ is an *independent differential set* or ∂_i-set if $\partial_i(D) = \partial_i(G)$. Note that the connectivity of G is not an important restriction, since if G has connected components $G_1, ..., G_k$, then $\partial_i(G) = \partial_i(G_1) + ... + \partial_i(G_k)$. Therefore, we will only consider connected graphs.

A set is *independent (or stable)* if no two vertices in it are adjacent. An independent set S of vertices in a graph G is said to be an *independent dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. The *independent domination number* of G, denoted by $\gamma_i(G)$, is the minimum size of an independent dominating set. An independent dominating set of G of size $\gamma_i(G)$ is called an γ_i-set. For more information on independent domination in graphs see [26]. Finally, we will say that G is an *independent dominant differential* graph if it contains a ∂_i-set which is also an independent dominating set.

Independent differential is a new concept only defined in [25] and as far as we are aware no work has been done on this topic. The main aim of this paper is to initiate the research on determining the upper and lower bounds for $\partial_i(G)$. In section 2, we will present some results of $\partial_i(G)$ which are similar to that of $\partial(G)$. In the following sections, the independent differential of wheel, cycle, path-related types of graphs and graph operations are computed and exact formulae are derived.

2. Elementary Results

Notice that for a graph G of order n, $0 \leq \partial_i(G) \leq n - 2$.

It is easy to calculate the exact values of the independent differential in the following families of graphs.

Remark 2.1 *For paths P_n $(n \geq 2)$ and cycles C_n $(n \geq 3)$, $\partial_i(P_n) = \partial_i(C_n) = \lfloor \frac{n}{3} \rfloor$. For a complete, star and wheel graph of n vertices, we have $\partial_i(K_n) = \partial_i(S_n) = \partial_i(W_n) = n - 2$. If $\Gamma_{r,t} = (V_r, V_t, E)$ is a bipartite graph, where $|V_r| = r$ and $|V_t| = t$, then $\partial_i(\Gamma_{r,t}) \leq max\{r - 1, t - 1\}$. Moreover, if $K_{r,t}$ is a complete bipartite graph, then $\partial_i(K_{r,t}) = max\{r - 1, t - 1\}$.*
Complete graphs, star graphs, wheel graphs, and path graphs P_n and cycle graphs C_n with $n = 3k$ or $n = 3k + 2$ are the examples of independent dominant differential graphs.

Proposition 2.1 *For any graph G of order n without isolated vertices, $\partial_i(G) \geq n - 2\gamma_i(G)$.*

Proof: Let X be a γ_i-set. Notice that,

$$\partial_i(G) \geq \partial_i(X) = |B(X)| - |X| = (n - \gamma_i(G)) - \gamma_i(G) = n - 2\gamma_i(G).$$

□

Proposition 2.2 *For any graph G with maximum degree $\Delta(G)$, $\Delta(G) - 1 \leq \partial_i(G)$.*

Proof: Let $X = \{v\}$, where v is a vertex of maximum degree $\Delta(G)$. We have

$$\partial_i(G) \geq \partial_i(X) = |B(X)| - |X| = \Delta(G) - 1.$$

□

Lemma 2.1 *If D is a ∂_i-set of a graph G, then $|D| \leq \gamma_i(G)$.*

Proof: Let A be a minimum independent dominating set. If $|A| < |D|$, then we have

$$\partial_i(A) = n - 2|A| > n - |D| - |D| \geq |B(D)| - |D| = \partial_i(G),$$

a contradiction. Therefore, $|D| \leq \gamma_i(G)$.

□

Theorem 2.1 *A graph G of order n is independent dominant differential if and only if $\partial_i(G) = n - 2\gamma_i(G)$.*

Proof: If D is a ∂_i-set of G which is an independent dominating set, by Lemma 2.1, we have $|D| \leq \gamma_i(G)$, so $|D| = \gamma_i(G)$ and $\partial_i(G) = n - 2\gamma_i(G)$.

If A is a minimum independent dominating set and $\partial_i(G) = n - 2|A| = |B(A)| - |A|$, then A is a γ_i -set and an independent dominating set. □

3. Independent differential in wheel related graphs

In this section, the independent differential of wheel-related graphs including gear, helm and friendship graphs are calculated.

Gear graphs

Gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. G_n has $2n+1$ vertices and $3n$ edges [4]. G_n includes an even cycle C_{2n}. There are two types of vertices of C_{2n} in G_n as vertices of degree two and three, respectively. The vertices of degree two are referred to as minor vertices and vertices of degree three to as major vertices [3]. The central vertex c of G_n has degree of n. Label the major and minor vertices, respectively, as $v_0, \ldots, v_{n-1}$ and $w_0, \ldots, w_{n-1}$ and let w_i be adjacent to the vertices v_i and v_{i+1} for $0 \leq i \leq n-1$, where $i+1$ is taken modulo n.

Theorem 3.1 *The independent differential of the gear graph G_n of order $2n+1$ is*

$$\partial_i(G_n) = \begin{cases} 2 + \lfloor \frac{2n-3}{3} \rfloor, & if \quad n = 3; \\ n-1, & if \quad n \geq 4. \end{cases}$$

Proof:

If we take the central vertex c and so $D_1 = \{c\}$, then we have that $B(D_1) = \{v_0, \ldots, v_{n-1}\}$ and so $\partial_i(D_1) = n-1$, and taking any other subset of $V(G_n)$ to the set D_1 yields $\partial_i(D_1) < n-1$.
If we take a major vertex v_i $(0 \leq i \leq n-1)$ of G_n to the set D_2, that is $D_2 = \{v_i\}$, then we have $B(D_2) = \{c, w_{i-1}, w_i\}$, where $i-1$ is taken modulo n, yielding $\partial_i(D_2) = 2$.
Let $S_1 = V(G_n) \backslash N_{G_n}[v_i]$ and so we have that $G_n[S_1] = P_{2n-3}$. If we take the maximal ∂_i -set of $G_n[S_1]$ to the set D_2 having the set D_3, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we receive $\partial_i(D_3) = \partial_i(D_1) + \partial_i(P_{2n-3}) = 2 + \lfloor \frac{2n-3}{3} \rfloor$.
If we take a minor vertex w_i $(0 \leq i \leq n-1)$ of G_n to the set D_4, that is $D_4 = \{w_i\}$, then we have $B(D_4) = \{v_i, v_{i+1}\}$, where $i+1$ is taken modulo n, yielding $\partial_i(D_4) = 1$.
Let $S_2 = V(G_n) \backslash \{c, N_{G_n}[w_i]\}$ and so we have that $G_n[S_2] = P_{2n-3}$. If we take the maximal ∂_i-set of $G_n[S_2]$ to the set D_4 having the set D_5, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$ and the maximal ∂_i-set of $G_n[S_2]$ includes at least one major vertex, we receive $\partial_i(D_5) = \partial_i(D_4) + (\partial_i(P_{2n-3}) + 1) = 2 + \lfloor \frac{2n-3}{3} \rfloor$.
By the definition of graph independent differential, among all the differential sets, we get
$\partial_i(G_n) = max\{\partial_i(D_k)\}$ $(1 \leq k \leq 5)$

$$\partial_i(G_n) = \begin{cases} 2 + \lfloor \frac{2n-3}{3} \rfloor, & \text{if} \quad n = 3; \\ n-1, & \text{if} \quad n \geq 4. \end{cases}$$

Thus, the proof holds. □

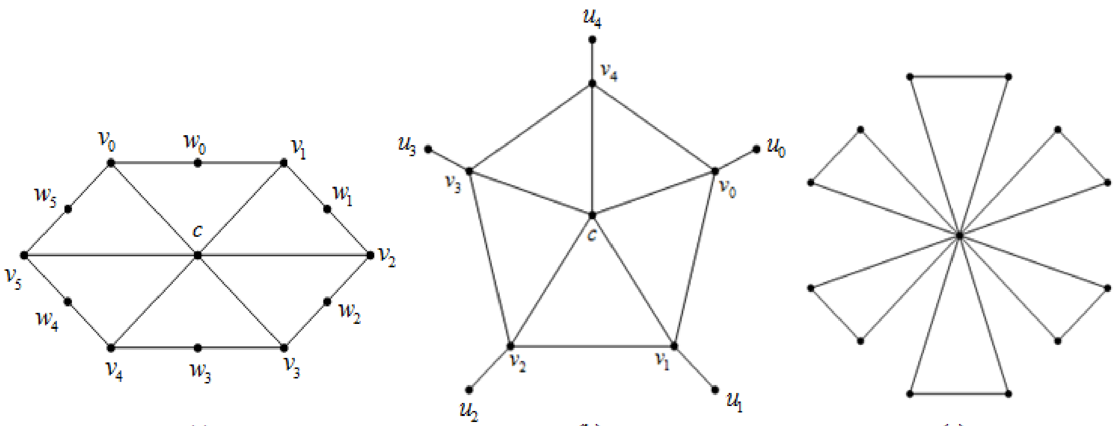

Figure 1: (a) Gear graph G_n for $n = 6$ (b) Helm graph H_n for $n = 5$ (c) Friendship graph f_n for $n = 6$

Remark 3.1 *For $n = 3, 4, 6$, the ∂_i-set of G_n is also an independent dominating set. Hence, we conclude that gear graphs are independent dominant differential for $n = 3, 4, 6$.*

Helm graphs

Helm H_n is a graph of order $2n + 1$ obtained from a wheel W_n with cycle C_n having a pendant edge attached to each vertex of the cycle. H_n consists of the vertex set $V(H_n) = \{v_i | 0 \leq i \leq n-1\} \cup \{u_i | 0 \leq i \leq n-1\} \cup \{c\}$ and edge set $E(H_n) = \{v_i v_{i+1} | 0 \leq i \leq n-1\} \cup \{v_i u_i | 0 \leq i \leq n-1\} \cup \{v_i c | 0 \leq i \leq n-1\}$, where $i+1$ is taken modulo n [4]. The central vertex c of H_n has a vertex degree of n. There are two types of vertices in $H_n \setminus \{c\}$ as the vertices of degree four and one, respectively. The vertices of degree one and four are referred to as minor and major vertices, respectively [3].

Theorem 3.2 *The independent differential of the helm graph H_n of order $2n + 1$ is*

$$\partial_i(H_n) = \begin{cases} 3 + 2\lfloor \frac{n-3}{3} \rfloor, & if \quad n = 3; \\ n-1, & if \quad n \geq 4. \end{cases}$$

Proof: If we take the central vertex c of H_n to the set D_1, then we have $B(D_1) = \{v_0, \ldots, v_n - 1\}$ yielding $\partial_i(D_1) = n - 1$, and taking any other subset of $V(H_n)$ to the set D_1 yields $\partial_i(D_1) < n - 1$.
If we take the major vertex v_i $(0 \leq i \leq n-1)$ of H_n to the set D_2, then we have $B(D_2) = \{c, u_i, v_{i+1}, v_{i-1}\}$, where $i + 1$ and $i - 1$ are taken modulo n, yielding $\partial_i(D_2) = 3$.
Let $S_1 = V(H_n) \backslash N_{H_n}[v_i]$ and so we have that $H_n[S_1] = P^*_{n-3} \cup \{u_i, u_{i+1}\}$ where P^*_n is the path graph of order n with a pendant vertex attached to each vertex of the path. If we take the maximal ∂_i-set of P_{n-3} to the set D_2 having the set D_3, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$ and every vertex of P_{n-3} is adjacent to a pendant vertex, we receive

$$\partial_i(D_3) = \begin{cases} \partial_i(D_2) + \partial_i(P_{n-3}) + \lfloor \frac{n-3}{3} \rfloor, & \text{if} \quad n = 3k \text{ or } 3k+1; \\ \partial_i(D_2) + \partial_i(P_{n-3}) + \lceil \frac{n-3}{3} \rceil, & \text{if} \quad n = 3k+2. \end{cases}$$

$$\partial_i(D_3) = \begin{cases} 3 + 2\lfloor \frac{n-3}{3} \rfloor, & \text{if} \quad n = 3k \text{ or } 3k+1; \\ 3 + \lfloor \frac{n-3}{3} \rfloor + \lceil \frac{n-3}{3} \rceil, & \text{if} \quad n = 3k+2. \end{cases}$$

, where $k \in \mathbb{Z}^+$ and taking any other subset of $V(H_n)$ to the set D_3, yields

$$\partial_i(D_3) < \begin{cases} 3 + 2\lfloor \frac{n-3}{3} \rfloor, & \text{if} \quad n = 3k\ ; \\ 3 + \lfloor \frac{n-3}{3} \rfloor + \lceil \frac{n-3}{3} \rceil, & \text{if} \quad n = 3k+2. \end{cases},$$

and if $n = 3k + 1$, then $\partial_i(D_3) \leq 3 + 2\lfloor \frac{n-3}{3} \rfloor$.
If we take the minor vertex u_i $(0 \leq i \leq n-1)$ of H_n to the set D_4, then we have $B(D_4) = \{v_i\}$ yielding $\partial_i(D_4) = 0$.

Let $S_2 = V(H_n)\backslash N_{H_n}[u_i]$ and so we have the graph $H_n[S_2]$ including the central vertex c, $n-1$ major and $n-1$ minor vertices of H_n, and also $n-1$ major vertices induce the subgraph P_{n-1} in $H_n[S_2]$. If we take the maximal ∂_i-set of P_{n-1} to the set D_4 having the set D_5, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$ and every major vertex is adjacent to a pendant vertex and the central vertex c, we receive

$$\partial_i(D_5) = \begin{cases} \partial_i(D_4) + \partial_i(P_{n-1}) + 1 + \lfloor \frac{n-1}{3} \rfloor, & \text{if} \quad n = 3k+1 \text{ or } 3k+2; \\ \partial_i(D_4) + \partial_i(P_{n-1}) + 1 + \lceil \frac{n-1}{3} \rceil, & \text{if} \quad n = 3k. \end{cases}$$

$\partial_i(D_5) = \begin{cases} 2\lfloor \frac{n-1}{3} \rfloor + 1, & \text{if} \quad n = 3k+1 \text{ or } 3k+2; \\ \lfloor \frac{n-1}{3} \rfloor + \lceil \frac{n-1}{3} \rceil + 1, & \text{if} \quad n = 3k. \end{cases}$, where $k \in \mathbb{Z}^+$ and taking any other subset of $V(H_n)$ to the set D_5 yields $\partial_i(D_5) < \begin{cases} 2\lfloor \frac{n-1}{3} \rfloor + 1, & \text{if} \quad n = 3k+1; \\ \lfloor \frac{n-1}{3} \rfloor + \lceil \frac{n-1}{3} \rceil + 1, & \text{if} \quad n = 3k. \end{cases}$, and if $n = 3k+2$, then $\partial_i(D_5) \leq 2\lfloor \frac{n-1}{3} \rfloor + 1$.

By the definition of graph independent differential, we have

$\partial_i(H_n) = max\{\partial_i(D_l)\}$ $(1 \leq l \leq 5)$

$$\partial_i(H_n) = \begin{cases} 3 + 2\lfloor \frac{n-3}{3} \rfloor, & \text{if} \quad n = 3; \\ n-1, & \text{if} \quad n \geq 4. \end{cases}$$

Thus, the proof holds.

□

Friendship graphs

Friendship graph f_n is collection of n triangles with a common point. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternate edges of the cycle. Another way of obtaining friendship graph is joining of K_1 and n copies of K_2 [3,4]. The central vertex c of f_n has a vertex degree of $2n$, where other vertices of the triangles are of degree two.

Theorem 3.3 *The independent differential of the friendship graph f_n of order $2n+1$ is $\partial_i(f_n) = 2n-1$.*

Proof: If we take the central vertex c of f_n to the set D_1, that is $D_1 = \{c\}$, then we have $B(D_1) = V(f_n) \setminus \{c\}$ yielding $\partial_i(D_1) = 2n-1$.

If we take a vertex v of degree two of f_n to the set D_2, that is $D_2 = \{v\}$, then we have $\partial_i(D_2) = 1$, and taking any other subset of $V(f_n)$ to the set D_2 yields $\partial_i(D_2) = 1$.

By the definition of graph independent differential, we get

$\partial_i(f_n) = max\{\partial_i(D_j)\}$ $(j = 1, 2)$

$\partial_i(f_n) = 2n-1$.

The theorem is thus proved.

□

Remark 3.2 *We can easily observe that $\gamma_i(f_n) = 1$ and the ∂_i-set of f_n is also an independent dominating set. Hence, we conclude that friendships graphs are independent dominant differential.*

4. Independent differential in cycle related graphs

In this section, the independent differential of cycle-related graphs including fan, k-pyramid, n-gon book of k-pages graphs are calculated.

Fans

If we join a vertex of C_n to all other vertices, then the resulting graph is called Fan, also known as *shell*, and is denoted by F_n. For $n = 3$, we notice that $F_3 \equiv C_3$. Fans can also be described by the join operation $F_n = P_{n-1} \vee K_1$, where $n \geq 3$. The degree set of F_n is {2,3,n-1} [4]. Let $c, v_0, \ldots, n_{n-2}$ be the vertices of F_n, where v_0 and v_{n-2} are the vertices of degree two and let c be the vertex that is adjacent to all other vertices. Then, c is the central vertex of F_n with degree $n-1$. The vertices of $F_n \setminus \{c\}$ are of two kinds: vertices of degree two and three. The vertices of degree two are referred to as minor vertices and vertices of degree three to as major vertices.

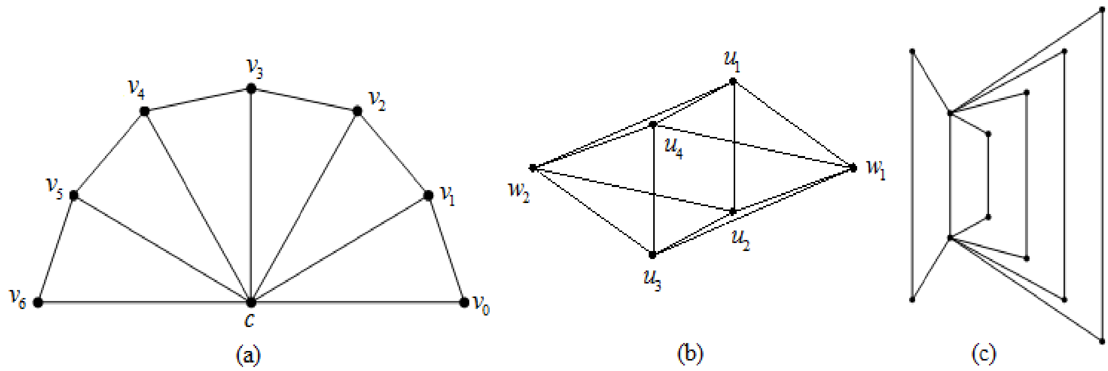

Figure 2: (a) Fan graph F_n for $n = 8$ (b) Bipyramid graph $BP(n)$ (c) n-gon book of k-pages graph $B(n,k)$ for $n = 4$, $k = 5$

Theorem 4.1 *The independent differential of the fan graph F_n of order n is $\partial_i(F_n) = n - 2$.*

Proof: If we take the central vertex c of F_n to the set D_1, that is $D_1 = \{c\}$, then we get $\partial_i(D_1) = n-2$. Let $S_1 = V(F_n)\backslash\{c\}$ and so we have that $F_n[S_1] = P_{n-1}$. If we take the maximal ∂_i-set of $F_n[S_1]$ to the set D_2, then since $\partial_i(P_n) = \lfloor\frac{n}{3}\rfloor$ and every vertex of $F_n[S_1]$ is adjacent to the central vertex c of F_n, we receive $\partial_i(D_2) = \lfloor\frac{n-1}{3}\rfloor + 1$, and for $n = 3k+2$ ($k \in \mathbb{Z}^+$) taking any other subset of $V(F_n)$ yields $\partial_i(D_2) < \lfloor\frac{n-1}{3}\rfloor + 1$.
By the definition of graph independent differential, we get
$\partial_i(F_n) = max\{\partial_i(D_j)\}$ $(j = 1, 2)$
$\partial_i(F_n) = \begin{cases} \lfloor\frac{n-1}{3}\rfloor + 1, & \text{if} \quad n = 3; \\ n-2, & \text{if} \quad n \geq 3. \end{cases}$
The theorem is thus proved.

□

Remark 4.1 *We can easily observe that $\gamma_i(F_n) = 1$ and the ∂_i-set of F_n is also an independent dominating set. Hence, we conclude that fans are independent dominant differential graphs.*

k-pyramids

The join graph $C_n \vee N_k$ $(n \geq 3, k \geq 1)$, where N_k is the null graph of order k, is called *k -pyramid* and is denoted by $kP(n)$. The 2-pyramid graph $C_n \vee N_2$ is called *bipyramid graph* and is denoted by $BP(n)$. The 1-pyramid graph $C_n \vee N_1$ is the wheel graph W_n [4].
Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of C_n and $w_1, w_2, w_3, \ldots, w_k$ be the vertices of N_k. Then, we have $deg(u_i) = k+2$ $(1 \leq i \leq n)$ and $deg(w_j) = n$ $(1 \leq j \leq k)$.

Theorem 4.2 *The independent differential of the k-pyramid graph $kP(n)$ of order $n+k$ is*

$$\partial_i(kP(n)) = \begin{cases} n-1, & \begin{cases} if\, n = 3t\, and\, n \geq \frac{3k+3}{2}; \\ if\, n = 3t+1\, and\, n \geq \frac{3k+2}{2}; \quad where\ t \in \mathbb{Z}^+. \\ if\, n = 3t+2\, and\, n \geq \frac{3k+1}{2}. \end{cases} \\ \lfloor\frac{n}{3}\rfloor + k, & otherwise. \end{cases}$$

Proof: If we take a vertex w_j $(1 \leq j \leq k)$ of N_k in $kP(n)$ to the set D_1, that is $D_1 = \{w_j\}$, then we have $B(D_1) = \{u_1, \ldots, u_n\}$ and so $\partial_i(D_1) = n-1$, and taking any other subset of $V(kP(n))$ to the set D_1 yields $\partial_i(D_1) < n-1$.
If we take the maximal ∂_i-set of C_n in $kP(n)$ to the set D_2, then since $\partial_i(C_n) = \lfloor\frac{n}{3}\rfloor$ and every vertex u_i

$(1 \le i \le n)$ of C_n in $kP(n)$ is adjacent to every vertex w_j $(1 \le j \le k)$ of N_k, we have $\partial_i(D_2) = \lfloor \frac{n}{3} \rfloor + k$, and for $n = 3t+1$ $(t \in \mathbb{Z}^+)$ taking any other subset of $V(kP(n))$ to the set D_2 yields $\partial_i(D_2) < \lfloor \frac{n}{3} \rfloor + k$. By the definition of graph differential, we have
$\partial_i(kP(n)) = max\{\partial_i(D_l)\}$ $(l = 1, 2)$

$$\partial_i(kP(n)) = \begin{cases} n-1, & \begin{cases} if\, n = 3t \text{ and } n \ge \frac{3k+3}{2}; \\ if\, n = 3t+1 \text{ and } n \ge \frac{3k+2}{2}; \\ if\, n = 3t+2 \text{ and } n \ge \frac{3k+1}{2}. \end{cases} \text{ where } t \in \mathbb{Z}^+. \\ \lfloor \frac{n}{3} \rfloor + k, & \text{otherwise.} \end{cases}$$

Thus, the proof holds.

□

Corollary 4.1 *The independent differential of the bipyramid graph $BP(n)$ of order $n+2$ is*

$$\partial_i(BP(n)) = \begin{cases} \lfloor \frac{n}{3} \rfloor + 2, & if \quad n = 3; \\ n-1, & if \quad n > 3. \end{cases}$$

Remark 4.2 *Since the wheel graph W_n is 1-pyramid graph $C_n \vee N_1$, by taking $k = 1$, the value of $\partial_i(W_n)$ holds.*

Remark 4.3 *We can easily observe that $\gamma_i(kP(n)) = min\{k, \lceil \frac{n}{3} \rceil\}$ and by Theorem 4.2 we conclude that k-pyramids are independent dominant differential graphs for* $\begin{cases} if\, n = 3t \, and\, n = \frac{3k+3}{2}; \\ if\, n = 3t+1 \, and\, n = \frac{3k-1}{2}; \\ if\, n = 3t+2 \, and\, n = \frac{3k+1}{2}. \end{cases}$ *, where $t \in \mathbb{Z}^+$.*

n-gon books

When k copies of C_n $(n \ge 3)$ share a common edge, it will form an $n-gon\ book\ of\ k\ pages$ and is denoted by $B(n,k)$. The degree set of $B(n,k)$ is $\{2, k+1\}$ [4]. Therefore, the vertices of $B(n,k)$ are of two kinds: vertices of degree 2 are referred to as minor vertices and vertices of degree $k+1$ to as major vertices. For $k = 1$, we notice that $B(n,k) \cong C_n$.

Theorem 4.3 *The independent differential of the n-gon book of k pages graph $B(n,k)$ of order $(n-2)k+2$ is $\partial_i(B(n,k)) = k(\lfloor \frac{n-3}{3} \rfloor + 1)$.*

Proof: If we take one of the major vertices of $B(n,k)$ -say vertex u , to the set D_1, that is $D_1 = \{u\}$, then since $|N_{B(n,k)}(u)| = k+1$, we have $\partial_i(D_1) = k$.
Let $S_1 = V(B(n,k)) \backslash N_{B(n,k)}[u]$ and so we have that $B(n,k)[S_1] = \cup_{i=1}^{k} P_{n-3}$. If we take the maximal ∂_i-set of $B(n,k)[S_1]$ to the set D_1 having the set D_2, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_2) = k\lfloor \frac{n-3}{3} \rfloor + k$, and for $n = 3t+1$ $(t \in \mathbb{Z}^+)$, taking any other subset of $V(B(n,k))$ to the set D_2 yields $\partial_i(D_2) < k\lfloor \frac{n-3}{3} \rfloor + k$. By the definition of graph independent differential, we receive
$\partial_i(B(n,k)) = k(\lfloor \frac{n-3}{3} \rfloor + 1)$.
The theorem is thus proved. □

Remark 4.4 *For $n = 3t$ and $n = 3t+2$ $(t \in \mathbb{Z}^+)$, the ∂_i-set of $B(n,k)$ is also an independent dominating set. Hence, we conclude that n -gon book of k pages graphs $B(n,k)$ are independent dominant differential graphs for $n = 3t$ and $n = 3t+2$ $(t \in \mathbb{Z}^+)$.*

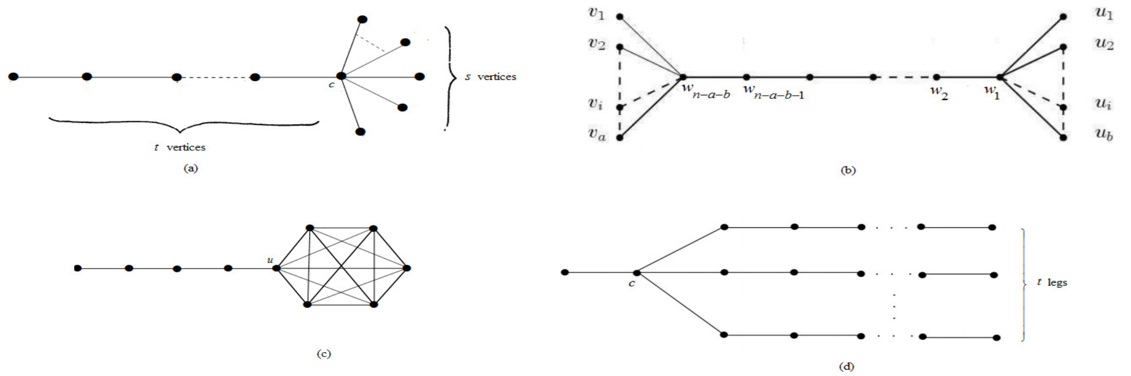

Figure 3: (a) Comet graph $C_{s,t}$ (b) Double comet graph $DC(n,a,b)$ (c) Lollipop graph $L_{n,d}$ for $n=10$ and $d=5$ (d) E_p^t graph

5. Independent differential in path related graphs

In this section, the independent differential of path-related graphs including comet, double comet, lollipop, and E_p^t graphs are calculated.

Comet graphs

The comet $C_{s,t}$ where s and t are positive integers, denotes the tree obtained by identifying the center of the star $K_{1,s}$ with an end-vertex of P_t, the path of order t. So $C_{s,t} \cong K_{1,s}$ and $C_{1,p-1} \cong P_p$ [6]. Let the center of the star $K_{1,s}$ -that is one end-vertex of P_t be the vertex c.

Theorem 5.1 *The independent differential of the comet graph $C_{s,t}$ of order $s+t$ is $\partial_i(C_{s,t}) = s + \lfloor \frac{t-2}{3} \rfloor$.*

Proof: If we take the center vertex c of $K_{1,s}$ in $C_{s,t}$ to the set D_1, that is $D_1 = \{c\}$, then we receive $\partial_i(D_1) = s$.

Let $S_1 = V(C_{s,t}) \backslash N_{C_{s,t}}[c]$ and so we have that $C_{s,t}[S_1] = P_{t-2}$. If we take the maximal ∂_i-set of $C_{s,t}[S_1]$ to the set D_1 having the set D_2, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_2) = s + \lfloor \frac{t-2}{3} \rfloor$, and for $t = 3k+2$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(C_{s,t})$ to the set D_2 yields $\partial_i(D_2) < s + \lfloor \frac{t-2}{3} \rfloor$.
If we take one of the vertices of $K_{1,s}$ except the center vertex c in $C_{s,t}$ to the set D_3, then we have $\partial_i(D_3) = 0$.
Let $S_2 = V(C_{s,t}) \backslash N_{C_{s,t}}[D_3]$ and so we have that $C_{s,t}[S_2] = P_{t-1} \cup \bar{K}_{s-1}$. If we take the maximal ∂_i-set of $C_{s,t}[S_2]$ to the set D_3 having the set D_4, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_4) = \lfloor \frac{t-1}{3} \rfloor$, and taking any other subset of $V(C_{s,t})$ to the set D_4 yields $\partial_i(D_4) < \lfloor \frac{t-1}{3} \rfloor$.
If we take the maximal ∂_i -set of P_t to the set D_5 such that $c \in B(D_5)$, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_5) = \lfloor \frac{t}{3} \rfloor$, and taking any other subset of $V(C_{s,t})$ to the set D_5 yields $\partial_i(D_5) < \lfloor \frac{t}{3} \rfloor$.
By the definition of graph independent differential, we get
$\partial_i(C_{s,t}) = max\{\partial_i(D_j)\}$ $(1 \le j \le 5)$
$\partial_i(C_{s,t}) = s + \lfloor \frac{t-2}{3} \rfloor$
Thus the proof holds. □

Remark 5.1 *For $t = 3k+1$ and $t = 3k+2$ $(k \in \mathbb{Z}^+)$, the ∂_i-set of $C_{s,t}$ is also an independent dominating set. Hence, we conclude that comets $C_{s,t}$ are independent dominant differential graphs for $t = 3k+1$ and $t = 3k+2$ $(k \in \mathbb{Z}^+)$.*

Double comet graphs

For $a, b \geq 1$, $n \geq a+b+2$ by $DC(n,a,b)$ we denote a double comet, which is a tree composed of a path containing $n-a-b$ vertices with a pendent vertices attached to one of the ends of the path and b pendent vertices attached to the other end of the path. Thus, $DC(n,a,b)$ has n vertices and $a+b$ leaves [11]. Let $v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_b, w_1, w_2, \ldots, w_{n-a-b}$ be the vertex set of the double comet $DC(n,a,b)$, which is obtained from a path P_{n-a-b} of vertices $w_1, w_2, \ldots, w_{n-a-b}$ by attaching the pendant vertices $u_1, u_2, \ldots, u_b$ to the one end vertex w_1 of P_{n-a-b} and attaching the pendant vertices $v_1, v_2, \ldots, v_a$ to the other end vertex w_n of P_{n-a-b}.

Theorem 5.2 *The independent differential of the double comet graph $DC(n,a,b)$ of order n $(a, b > 1, n-a-b > 2)$ is $\partial_i(DC(n,a,b)) = a+b+\lfloor \frac{n-a-b-4}{3} \rfloor$.*

Proof: If we take the vertex w_1 of $DC(n,a,b)$ to the set D_1, then we have $\partial_i(D_1) = b$.
Let $S_1 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[w_1]$ and so we have $DC(n,a,b)[S_1] = C_{a,n-a-b-2}$, where $C_{a,n-a-b-2}$ is a comet graph of order $n-b-2$. If we take the maximal ∂_i -set of $DC(n,a,b)[S_1]$ to the set D_1 having the set D_2, then since $\partial_i(C_{s,t}) = s+\lfloor \frac{t-2}{3} \rfloor$, we receive $\partial_i(D_2) = b+a+\lfloor \frac{n-a-b-4}{3} \rfloor$, and for $n-a-b = 3k+2$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(DC(n,a,b))$ to the set D_2 yields $\partial_i(D_2) < b+a+\lfloor \frac{n-a-b-4}{3} \rfloor$.
If we take the vertex w_{n-a-b} of $DC(n,a,b)$ to the set D_3, then we have $\partial_i(D_3) = a$.
Let $S_2 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[w_{n-a-b}]$ and so we have $DC(n,a,b)[S_2] = C_{b,n-a-b-2}$. If we take the maximal ∂_i-set of $DC(n,a,b)[S_2]$ to the set D_3 having the set D_4, then since $\partial_i(C_{s,t}) = s+\lfloor \frac{t-2}{3} \rfloor$, we have $\partial_i(D_4) = a+b+\lfloor \frac{n-a-b-4}{3} \rfloor$, and for $n-a-b = 3k+2$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(DC(n,a,b))$ to the set D_4 yields $\partial_i(D_4) < a+b+\lfloor \frac{n-a-b-4}{3} \rfloor$.
If we take a vertex v_i $(1 \leq i \leq a)$ of $DC(n,a,b)$ to the set D_5, then we get $\partial_i(D_5) = 0$.
Let $S_3 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[v_i]$ and so we have that $DC(n,a,b)[S_3] = \bar{K}_{a-1} \cup C_{b,n-a-b-1}$. If we take the maximal ∂_i-set of $DC(n,a,b)[S_3]$ to the set D_5 having the set D_6, then since $\partial_i(C_{s,t}) = s+\lfloor \frac{t-2}{3} \rfloor$, we have $\partial_i(D_6) = \partial_i(C_{b,n-a-b-1}) = b+\lfloor \frac{n-a-b-3}{3} \rfloor$, and taking any other subset of $V(DC(n,a,b))$ to the set D_6 yields $\partial_i(D_6) < b+\lfloor \frac{n-a-b-3}{3} \rfloor$.
If we take a vertex u_j $(1 \leq j \leq b)$ of $DC(n,a,b)$ to the set D_7, then we have $\partial_i(D_7) = 0$.
Let $S_4 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[u_j]$ and so we have that $DC(n,a,b)[S_4] = \bar{K}_{b-1} \cup C_{a,n-a-b-1}$. If we take the maximal ∂_i-set of $DC(n,a,b)[S_4]$ to the set D_7 having the set D_8, then since $\partial_i(C_{s,t}) = s+\lfloor \frac{t-2}{3} \rfloor$, we have $\partial_i(D_8) = \partial_i(C_{a,n-a-b-1}) = a+\lfloor \frac{n-a-b-3}{3} \rfloor$, and taking any other subset of $V(DC(n,a,b))$ to the set D_8 yields $\partial_i(D_8) < a+\lfloor \frac{n-a-b-3}{3} \rfloor$.
For $n-a-b = 3k+1$ $(k \in \mathbb{Z})$, if we take the maximal ∂_i-set of P_{n-a-b} to the set D_9 such that $w_1 \in B(D_9)$ or $w_{n-a-b} \in B(D_9)$, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_9) = \lfloor \frac{n-a-b}{3} \rfloor$. If $w_1 \in B(D_9)$, then let $S_5 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[D_9]$ and so we have $DC(n,a,b)[S_5] = \bar{K}_b \cup K_{1,a}$. If we take the maximal ∂_i-set of $DC(n,a,b)[S_5]$ to the set D_9 having the set D_{10}, then since $\partial_i(K_{1,n}) = n-1$, we have $\partial_i(D_{10}) = \lfloor \frac{n-a-b}{3} \rfloor + (a-1)$, and taking any other subset of $V(DC(n,a,b))$ to the set D_{10} yields $\partial_i(D_{10}) < \lfloor \frac{n-a-b}{3} \rfloor + (a-1)$. If $w_{n-a-b} \in B(D_9)$, then let $S_6 = V(DC(n,a,b)) \backslash N_{DC(n,a,b)}[D_9]$ and so we have $DC(n,a,b)[S_6] = \bar{K}_a \cup K_{1,b}$. If we take the maximal ∂_i-set of $DC(n,a,b)[S_6]$ to the set D_9 having the set D_{10}, then since $\partial_i(K_{1,n}) = n-1$, we have $\partial_i(D_{10}) = \lfloor \frac{n-a-b}{3} \rfloor + (b-1)$, and taking any other subset of $V(DC(n,a,b))$ to the set D_{10} yields $\partial_i(D_{10}) < \lfloor \frac{n-a-b}{3} \rfloor + (b-1)$.
For $n-a-b = 3k$ or $n-a-b = 3k+2$ $(k \in \mathbb{Z}^+)$, if we take the maximal ∂_i-set of P_{n-a-b} to the set D_{11} such that $\{w_1, w_{n-a-b}\} \in B(D_{11})$, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_{11}) = \lfloor \frac{n-a-b}{3} \rfloor$, and taking any other subset of $V(DC(n,a,b))$ to the set D_{11} yields $\partial_i(D_{11}) < \lfloor \frac{n-a-b}{3} \rfloor$.
By the definition of graph independent differential, we get
$\partial_i(DC(n,a,b)) = max\{\partial_i(D_l)\}$ $(1 \leq l \leq 11)$
$\partial_i(DC(n,a,b)) = a+b+\lfloor \frac{n-a-b-4}{3} \rfloor$ for $a, b > 1$ and $n-a-b > 2$.
Thus the proof holds. □

Remark 5.2 *The ∂_i-set of $DC(n,a,b)$ is also an independent dominating set. Hence, we conclude that double comets are independent dominant differential graphs.*

Lollipop graphs

The lollipop graph $L_{n,d}$ is a graph obtained from a complete graph K_{n-d} and a path P_d, by joining one of the end vertices of P_d [12], let this vertex be the vertex u, to all the vertices of K_{n-d}.

Theorem 5.3 *The independent differential of the lollipop graph $L_{n,d}$ $(d > 1)$ of order n is $\partial_i(L_{n,d}) = n - d + \lfloor \frac{d-2}{3} \rfloor$.*

Proof: If we take the vertex u of $L_{n,d}$ to the set D_1, that is $D_1 = \{u\}$, then we have $\partial_i(D_1) = n - d$.
Let $S_1 = V(L_{n,d}) \backslash N_{L_{n,d}}[u]$ and so we have that $L_{n,d}[S_1] = P_{d-2}$. If we take the maximal ∂_i-set of $L_{n,d}[S_1]$ to the set D_1 having the set D_2, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_2) = n - d + \lfloor \frac{d-2}{3} \rfloor$, and for $n = 3k$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(L_{n,d})$ to the set D_2 yields $\partial_i(D_2) < n - d + \lfloor \frac{d-2}{3} \rfloor$.
If we take one of the vertices of K_{n-d} to the set D_3, then we have $\partial_i(D_3) = n - d - 1$.
Let $S_2 = V(L_{n,d}) \backslash N_{L_{n,d}}[D_3]$ and so we have that $L_{n,d}[S_2] = P_{d-1}$. If we take the maximal ∂_i-set of $L_{n,d}[S_2]$ to the set D_3 having the set D_4, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_4) = n-d-1+\lfloor \frac{d-1}{3} \rfloor$, and for $n = 3k+2$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(L_{n,d})$ to the set D_4 yields $\partial_i(D_4) < n-d-1+\lfloor \frac{d-1}{3} \rfloor$.
If we take the maximal ∂_i-set of P_d to the set D_5, such that $u \in B(D_5)$, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_5) = \lfloor \frac{d}{3} \rfloor$.
If we take the ∂_i-set of K_{n-d} to the set D_5 having the set D_6, then since $\partial_i(K_n) = n - 2$, we get $\partial_i(D_6) = \lfloor \frac{d}{3} \rfloor + n - d - 2$, and for $n = 3k + 1$ $(k \in \mathbb{Z}^+)$ taking any other subset of $V(L_{n,d})$ to the set D_6 yields $\partial_i(D_6) < \lfloor \frac{d}{3} \rfloor + n - d - 2$.
By the definition of graph independent differential, we get
$\partial_i(L_{n,d}) = max\{\partial_i(D_j)\}$ $(1 \leq j \leq 6)$
$\partial_i(L_{n,d}) = n - d + \lfloor \frac{d-2}{3} \rfloor$ for $d > 1$.
Thus the proof holds. □

Remark 5.3 *For $d = 3k+1$ and $d = 3k+2$ $(k \in \mathbb{Z}^+)$, the ∂_i-set of $L_{n,d}$ is also an independent dominating set. Hence, we conclude that lollipop graphs are independent dominant differential for $d = 3k + 1$ and $d = 3k + 2$ $(k \in \mathbb{Z}^+)$.*

E_p^t graphs

The graph E_p^t is a tree of order $pt + 2$ obtained from a path with two vertices having one of the end-vertices attached to t legs and each leg has p vertices [24]. Let the end-vertex attached to t legs be the vertex c, and the vertex degree of the vertex c is $deg(c) = t + 1$.

Theorem 5.4 *The independent differential of the E_p^t graph of order $pt + 2$ is $\partial_i(E_p^t) = t(\lfloor \frac{p-1}{3} \rfloor + 1)$.*

Proof: If we take the vertex c of E_p^t to the set D_1, then we have $\partial_i(D_1) = t$.
Let $S_1 = V(E_p^t) \backslash N_{E_p^t}[c]$ and so we have that $E_p^t[S_1] = P_{p-1}$. If we take the maximal ∂_i-set of $E_p^t[S_1]$ to the set D_1 having the set D_2, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_2) = t + t\lfloor \frac{p-1}{3} \rfloor$, and for $p = 3k + 2$ $(k \in \mathbb{Z})$ taking any other subset of $V(E_p^t)$ to the set D_2 yields $\partial_i(D_2) < t + t\lfloor \frac{p-1}{3} \rfloor$.
If we take the other end-vertex that is adjacent to the vertex c in P_2 to the set D_3, then we have $\partial_i(D_3) = 0$.
Let $S_2 = V(E_p^t) \backslash N_{E_p^t}[D_3]$ and so we have that $E_p^t[S_2] = P_p$. If we take the maximal ∂_i-set of $E_p^t[S_2]$ to the set D_3 having the set D_4, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we receive $\partial_i(D_4) = t\lfloor \frac{p}{3} \rfloor$, and for $p = 3k + 1$ $(k \in \mathbb{Z})$ taking any other subset of $V(E_p^t)$ to the set D_4 yields $\partial_i(D_4) < t\lfloor \frac{p}{3} \rfloor$.
If we take the maximal ∂_i-set of one of the legs to the set D_5 such that $c \in B(D_5)$, then since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we have $\partial_i(D_5) = \lfloor \frac{p+1}{3} \rfloor$.
If we take the maximal ∂_i-sets of $t - 1$ legs of E_p^t to the set D_5 having the set D_6, since $\partial_i(P_n) = \lfloor \frac{n}{3} \rfloor$, we receive $\partial_i(D_6) = \lfloor \frac{p+1}{3} \rfloor + (t - 1)\lfloor \frac{p}{3} \rfloor$, and taking any other subset of $V(E_p^t)$ to the set D_6 yields $\partial_i(D_6) < t + t\lfloor \frac{p-1}{3} \rfloor$.
By the definition of graph independent differential, we get
$\partial_i(E_p^t) = max\{\partial_i(D_l)\}$ $(1 \leq l \leq 6)$

$\partial_i(E_p^t) = t(\lfloor \frac{p-1}{3} \rfloor + 1)$.
Thus the proof holds. □

Remark 5.4 *For $p = 3k$ and $p = 3k+1$ ($k \in \mathbb{Z}$), the ∂_i-set of E_p^t is also an independent dominating set. Hence, we conclude that E_p^t graphs are independent dominant differential for $p = 3k$ and $p = 3k+1$ ($k \in \mathbb{Z}$).*

6. Final remarks

To finish this work we must say that we have also studied the differential in other types of graphs obtained by some operation on two graphs, namely the join and the corona product of two graphs. The obtained results give explicit formulas for the independent differential depending on parameters in both graphs. We present here these results because they might be interesting for the reader.

The join of two graphs G and H, denoted by $G + H$, is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. We can give the exact value for the independent differential of a join of two graphs depending on the orders and the independent differentials of these graphs.

Proposition 6.1 *Let G and H be two graphs of order n_1 and n_2, respectively. Then,*
(a) If $n_1 > 1$ and $n_2 > 1$, then $\partial_i(G + H) = max\{\partial_i(G) + n_2, \partial_i(H) + n_1\}$.
(b) If $n_1 = 1$, then $\partial_i(G + H) = n_2 - 1$.
(c) If $n_2 = 1$, then $\partial_i(G + H) = n_1 - 1$.

The corona product of graphs was introduced in [14] and some applications of this type of product can be seen in [2,15]. Let G and H be two graphs of order n_1 and n_2, respectively, the corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the ith-copy of H with the ith-vertex of G.

Proposition 6.2 *Let G and H be two graphs of order n_1 and n_2, respectively. Then,*
(a) If $n_2 > 1$, then $\partial_i(G \odot H) = n_1(\partial_i(H) + 1) + \gamma_i(G)(n_2 - 2 - \partial_i(H))$.
(b) If $n_2 = 1$, then $\partial_i(G \odot H) = n_1 - \gamma_i(G)$.

References

1. D. B. West, *Introduction to Graph Theory, Prentice Hall*, NJ, (2001).
2. I. G. Yero, J. A. Rodríguez-Velázquez, *On the randić index of corona product graphs*, ISRN Discrete Math. 2011, Article ID 262183, (2011).
3. I. Javaid, S. Shokat, *On the partition dimension of some wheel related graphs*, Journal of Prime Research in Mathematics 4, 154-164, (2008).
4. J. A. Gallian, *A dynamic survey of graph labeling*, Elect. Jour. Combin. 15, DS6, (2008).
5. J. C. Hernández-Gómez, *Differential and operations on graphs*, Int. J. Math. Anal. 9, 341-349, (2015).
6. J. H. Hattingh, E. Ungerer, *The signed and minus k-subdomination numbers of comets*, Discrete Mathematics 183, 141-152, (1998).
7. J. L. Mashburn, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, P. J. Slater, *Differentials in graphs*, Utilitas Mathematica 69, 43-54, (2006).
8. J. M. Sigarreta, *Differential in Cartesian Product Graphs*, Ars Combinatoria CXXVI, 259-267, (2016).
9. J. R. Lewis, *Differentials of graphs*, Master's Thesis, East Tennessee State University, (2004).
10. L. A. Basilio, S. Bermudo, J. M. Sigarreta, *Bounds on the differential of a graph*, Util. Math. 103, 319-334, (2017).
11. M. Cygan, M. Pilipczuk, R. Škrekovski, *Relation between Randić index and average distance of trees*, MATCH Commun. Math. Comput. Chem. 66, 605-612, (2011).
12. M. J. Morgan, S. Mukwembi, H. C. Swart, *On the eccentric connectivity index of a graph*, Discrete Mathematics 311(13), 1229-1234, (2011).
13. P. Roushini Leely Pushpam, D. Yokesh, *Differential in certain classes of graphs*, Tamkang Journal of Mathematics 41, 129-138, (2010).

14. R. Frucht, F. Harary, *On the corona of two graphs*, Aequationes Math. 4, 322-325, (1970).

15. R. M. Casablanca, A. Diánez, P. García-Vázquez, *Toughness of the corona of two graphs*, Int. J. Comput. Math. 88(13), 2697-2706, (2011).

16. S. Bermudo, H. Fernau, *Combinatorics for smaller kernels: the differential of a graph*, Theoret. Comput. Sci. 562, 330-345, (2015).

17. S. Bermudo, H. Fernau, *Computing the differential of a graph: hardness, approximability and exact algorithms*, Discrete Applied Mathematics 165, 69-82, (2014).

18. S. Bermudo, H. Fernau, J. M. Sigarreta, *The differential and the Roman domination number of a graph*, Appl. Anal. Discrete Math. 8, 155-171, (2014).

19. S. Bermudo, H. Fernau, *Lower bounds on the differential of a graph*, Discrete Math. 312, 3236-3250, (2012).

20. S. Bermudo, J. C. Hernández-Gómez, J. M. Rodríguez, J. M. Sigarreta, *Relations between the differential and parameters in graphs*, Electronic Notes in Discrete Mathematics 46, 281-288, (2014).

21. S. Bermudo, J. M. Rodríguez, J. M. Sigarreta, *On the differential in graphs*, Utilitas Mathematica 97, 257-270, (2015).

22. S. Bermudo, L. De la Torre, A. M. Martín-Caraballo, J. M. Sigarreta, *The differential of the strong product graphs*, Int. J. Comput. Math. 92(6), 1124-1134, (2015).

23. S. Bermudo, *On the differential and Roman domination number of a graph with minimum degree two*, Discrete Applied Mathematics 232, 64-72, (2017).

24. T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein, *Introduction to Algorithms*, The MIT Press. (Third edition), (2009).

25. W. Goddard, M. A. Henning, *Generalised domination and independence in graphs*, Congr. Numer., 123, 161-171, (1997).

26. W. Goddard, M. A. Henning, *Independent domination in graphs: A survey and recent results*, Discrete Math. 313(7), 839-854, (2013).

Zeynep Nihan Berberler,

Department of Computer Science,

Dokuz Eylul University,

Turkey.

E-mail address: zeynep.berberler@deu.edu.tr