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A New Structure of Random approach Vector Space

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ABSTRACT: This paper is explaining the fundamental goal of A-Random approach on nonempty set if it meets the condition. A duo (Ω, δ_R) is dubbed A-Random approach space, the relationship between approach space and A-Random approach space is clarified. We define the δ_R-contraction function and debate some of its properties. We introduce the definition of A-Random approach semi group, A-Random approach group, A-Random approach vector space and we will also discuss solve various problems.

Key Words: Random space, A-Random approach group, A-Random approach vector space.

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1. Introduction

The concept of A-Random approach (appr.) space is central to modern functional analysis, and in recent years, applications in various other fields of mathematics have been studied in order to find and compare their properties. Random normed space and approach space theory is important in quantitative domain theory; there are many examples of approach structure in functional analysis, measure theory, probability space, and approximation theory. As in the metric case. The topological space generates the A-Random appr. space, it is called "topological," and the metric space generates the topological, it is said to be "metric." " The part of the numerical data that exists carrying from the ARP-product," if topological product compatibility with the family of underlying metric topologies, it can be retained." The fundamental difference in existence, There is a difference between A-Random appr. and appr. spaces. " in the reality that in an A-Random appr. space.

The A-Random appr. defines all the distances between the points," where such a point-set distance doesn't command to gain the two coincidentally infimum over the accounted set of all the point distances "As in the metric case, an A-Random appr. space is defined.

Šerstnev in 1962 [20] defined standard random spaces closely along the lines of standard (classical) space theory, so he used them to study the best methods of rounding in statistics. Accordingly, we will adopt the usual terminology and codification of the theory of random appr. spaces . In the sequel, the theory of random normed spaces will adopt usual terminology, notation and conventions, accordingly [2,3,8,19]. Lowen [14] found definition approach spaces were introduced in 1987. Lowen's monographs [10] can be used to set up an overall realization of appr. spaces. Lowen & Sioen introduced the definitions of separation axioms in approach spaces and determined their relation to each other in 2000 and 2003 [17,15]. The distance between points and sets in a metric space were studied sue Lowen in 1989 [12]. The relationship between Functional ideas and Topological Theories are found via Lowen, Van Olmen, and Vroegrijk in 2004 [16].

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The theory of appr. spaces, a generalization of metric and topological spaces, is based on point-to-set distances rather than point-to-point distances. The most important motivation was to solve the problem of a product of metric spaces infinite. Another reason for introducing approach spaces is to unify metric, uniformity, topological, and convergence theories. Barn and Qasim [5,6] the local distance-approach spaces is characterized, Appr. spaces, A-Random appr. spaces and compared them with usual , appr. spaces. Colebuders, Sion,... etc [7] show that some considerable consequences on real valued contractions. Martinez-Moreno1, Rpld'an2,...etc [18] found definition the concept of fuzzy A-Random appr. spaces as spaces popularization of fuzzy metric spaces and demonstrate a few Properties of fuzzy A-Random appr. space.

Gutierres, Hofmann [9] calculated the concept of completeness for appr. spaces and calculated a few properties in completeness appr. spaces. Van Opdenbosch [21] set up new isomorphic descriptions of A-Random appr. spaces, A-Random pre-appr. spaces, convergence A-Random appr. spaces, topological spaces, and convergence spaces, topological spaces, metric spaces, and spaces that are consistent.

Baekeland and Lowen [4] institute the measures of Lindelof and separability in A-Random appr. spaces. Lowen and Verwulgen [10] institute define A-Random appr. vector spaces. Lowen and Windels [11] defined an A-Random appr. groups spaces, semi-group spaces, and uniformly convergent. Lowen [13] detail of this book A-Random appr. theory completely disband this by" presentation properly the new two" kinds of numerically" form spaces that are" wanted: A-Random appr. spaces on the domestic level" and united gauge spaces" on the united" level. And Hussein and Abbas [1] through which you can find out Normed approach space. In Hussein and Saeed [18] defined the distance between two different sets in approach normed space, topological approach Banach space. The goal of this paper is two - fold: first, we want to put random approach group checking space in the proper perspective when random approach vector spaces.

This paper is splitted into four divisions: In division 1, introducing the research and Preliminaries with basic definitions. In Section 2, we introduce new definition which is called random and explains the relationship between random space and - approach space. In Section 3, we demonstrated some properties of δ_R- contractions. Section 4. We discuss convergent sequence in random appr. space with new results. Section 5 introduced the definitions of random appr. group, random appr. semi-group, random appr. sub-group, and solved a few examples in random appr. group, as well as introduced the definition of random appr. vector space and proved some examples in random appr. vector space.

2. Structure of A-Random Space

Definition 2.1 *A function $\psi : [0,1] \times [0,1] \to [0,1]$ is said to a triangular norm (shortly , ψ-norm) if the conditions that follow hold, $\forall\ h, l, p \in [0,1]$*

$(\psi 1)\, \psi(h,l) = \psi(l,h)$ *(commutativity)*
$(\psi 2)\ \psi(h,1) = h$ *(boundary condition)*
$(\psi 3)\ \psi(h,p) \geq\ \psi(h,l)$ *wherenever* $p \geq l$ *(monotonicity)*
$(\psi 1)\ \psi(h,\ \psi(l,p)) =\ \psi(\ \psi(h,l), p)$ *(associativity)*

Basic examples are the

1. *Lukasiewicz ψ-norm ψ_L , $\psi_L(h,l) = max\{h+l-1, 0\}$*
2. $\psi_p(h,l) = hl$
3. $\psi_M(h,l) = min\{h,l\}$
4. $\psi_D(h,l) = min\{h,l\}\,,\ if\ max\{h,l\} = 1$

Definition 2.2 *The space of each mappings $H : \mathcal{R}^* \to\ [0,\ 1]$ is said to distribution functions space and denote ∇^+*

1. $H(0) = 0 and H(+\infty) = 1,.$

2. *The set* ∇^{+} *is partially ordered by the usual point wise ordering of functions ,that is ,* $H \leq G$ *if and only if* $H(t) \leq G(t)$ *for all* $r \in \mathcal{R}$. *The maximal element for* ∇^{+} *in this order is the distribution function* K_0 *given by*

$$K_0(r) = \begin{cases} 0, & if\ r \leq 0 \\ 1, & if\ r > 0 \end{cases} \quad , \ K_{\infty}(r) = \begin{cases} 0, & if\ 0 \leq r < \infty \\ 1, & if\ r = \infty \end{cases}$$

Definition 2.3 Ω *is non-empty set,* ψ *is continuous* ψ*-norm and* F *is a mapping from* $\Omega \times \Omega$ *in to* M^{+} *such that a triple* (Ω, F, ψ), *if* $\mathcal{F}_{x,y}$ *denotes the value of* $\mathcal{F}$ *at a point* $(x, y) \in \Omega \times \Omega$, *the following condition hold*

$RM1)\ \mathcal{F}_{x,y}(tr) = K_0(r)\ if\ and\ only\ if\ x = y,\ for\ all\ t > 0$
$RM2)\ \mathcal{F}_{x,y}(r) = \mathcal{F}_{y,x}(r)$
$RM3)\ \mathcal{F}_{x,z}(r) \geq \psi_p\ (\mathcal{F}_{x,y}\ (r),\ \mathcal{F}_{y,z}\ (e)),\ for\ all\ x, y \in \Omega,\quad r, e \geq 0$

The a triple (Ω, F, ψ) *is Random metric space*

Definition 2.4 *Let* Ω *be a non-empty vector space* ψ *is continuous* ψ*-norm and* σ *is mapping from* Ω *into* M^{+} *in which, that the conditions is holding*

1. $\sigma_x(r) = K_0(r)\ if\ and\ only\ if x = 0,\ \forall\ r > 0$
2. $\sigma_{\lambda x}(r) = \sigma_x\left(\frac{r}{|\lambda|}\right),\ \lambda \neq 0, \forall\ x \in \Omega$
3. $\sigma_{x+y}(r + e) \geq \psi(\sigma_x(r), \sigma_y(e)), \forall\ x, y \in \Omega, r, e \geq 0$

The triple (Ω, σ, ψ) *is named a random normed space RN-space*

Definition 2.5 *Let* Ω *be a non-empty vector space,* ψ *is continuous t-norm and* σ *is mapping from* Ω *into* ∇^{+} *in which, that the conditions is holding.*

$AR1)\sigma_x(r) = K_0(r)\ if\ and\ only\ if\ x = 0\quad,\ \forall\ t > 0$
$AR2)\quad \sigma_{\lambda x}(r) = \sigma_x(r),\quad where\ |\lambda| = 1, \forall\ x \in X$
$AR3)\ \lim_{\lambda \longrightarrow 0} \sigma_{\lambda x}(r) = K_0(r)$
$AR4)\sigma_{x+y}(r + e) \geq \psi(\sigma_x(r), \sigma_y(e)),\quad for\ all\quad x, y \in X,\quad r, e \geq 0$

Then a triple (Ω, σ, ψ) *is said to be A-Random normed space*

Example 2.1 Let $(\Omega, \|.\|_R)$ be a L. normed spaces. Define function.

$$\sigma_x(r) = \begin{cases} K_0(0), & if\ r \leq 0 \\ e^{-\frac{\|m\|}{r}}, & if\ r > 0 \end{cases}$$

Hence (Ω, σ, ψ_p) is A-Random normed space

Proof:

1. $\sigma_m(r) = 1$ then, $e^{-\frac{\|m\|}{r}} = 1$ therefor, $e^{-\frac{\|m\|}{r}} = e^0$

$$hence,\ m = 0$$

   the conversely, it is clear.

2. $\sigma_{\lambda m}(r) = e^{-\frac{\|\lambda m\|}{r}} = e^{-\frac{|\lambda|\|m\|}{r}} = e^{-\frac{\|m\|}{r}} = \sigma_m(r)$

3. $\lim_{\lambda\longrightarrow 0}\sigma_{\lambda m}(r) = \lim_{\lambda\longrightarrow 0} e^{-\frac{\|\lambda m\|}{r}} = \lim_{\lambda\longrightarrow 0} e^{-\frac{|\lambda|\|m\|}{r}} = e^{0} = 1,\ t > 0,$ then $\lim_{\lambda\longrightarrow 0}\sigma_{\lambda m}(r) = K_0(r)$

4. $\psi_p(\sigma_m(r), \sigma_n(p)) = e^{-\frac{\|m\|}{r}} e^{-\frac{\|n\|}{p}} = e^{-\frac{\|m\|}{r}-\frac{\|n\|}{p}} \le e^{-\frac{\|m\|}{r+p}} \cdot e^{-\frac{\|n\|}{r+p}} = e^{-\left(\frac{\|m\|+\|n\|}{r+p}\right)} \le e^{-\left(\frac{\|m+n\|}{r+p}\right)} = \sigma_{m+n}(r+p)$

□

Example 2.2 Let $(\Omega, \|.\|_R)$ be a L. normed spaces. Define function.

$$\sigma_x(t) = \begin{cases} K_0(0),\ if\ t \le 0 \\ \frac{t}{t+\|w\|},\ if\ t > 0 \end{cases}$$

Then (Ω, σ, ψ_p) is A-Random normed space

Proof:

1. $\sigma_w(t) = 1\ then, \frac{t}{t+\|w\|} = 1$ therefor, $\|w\| = 0$ hence, $w = 0$ the conversely , it is clear.

2. $\sigma_{\lambda w}(t) = \frac{t}{t+\|\lambda w\|} = \frac{t}{t+|\lambda|\|w\|} = \frac{t}{t+\|w\|} = \sigma_w(t)$

3. $\lim_{\lambda\longrightarrow 0}\sigma_{\lambda w}(t) = \lim_{\lambda\longrightarrow 0}\frac{t}{t+\|\lambda w\|} = \lim_{\lambda\longrightarrow 0}\frac{t}{t+|\lambda|\|w\|} = \frac{t}{t} = 1,\ t > 0 \Longrightarrow \lim_{\lambda\longrightarrow 0}\sigma_{\lambda w}(t) = K_0(t)$

4. $T_p(\sigma_w(t), \sigma_l(s)) = \frac{t}{t+\|w\|} \cdot \frac{s}{s+\|l\|} \cdot = \frac{1}{1+\frac{\|w\|}{t}} \cdot \frac{1}{1+\frac{\|l\|}{s}} \frac{1}{1+\frac{\|w\|+\|l\|}{t+s}} = \frac{t+s}{t+s+\|w+l\|} = \sigma_{w+l}(t+s)$

□

Example 2.3 Let $(\Omega, \|.\|_R)$ be a L. normed spaces. Define function.

$$\sigma_x(t) = \begin{cases} K_0(0),\ if\ t \le 0 \\ \max\left\{1 - \frac{\|x\|}{t}, 0\right\},\ if\ t > 0 \end{cases}$$

Then(Ω, σ, ψ_L) is A-Random normed space

Proof:

1. $\sigma_x(t) = 1\ then, \left\{1 - \frac{\|x\|}{t}, 0\right\} = 1$ therefor, $x = 0$.
   the conversely , it is clear.

2. $\sigma_{\lambda x}(t) = \max\left\{1 - \frac{\|\lambda x\|}{t}, 0\right\} = \max\left\{1 - \frac{|\lambda|\|x\|}{t}, 0\right\} = \max\left\{1 - \frac{\|x\|}{t}, 0\right\} = \sigma_x(t)$

3. $\lim_{\lambda\longrightarrow 0}\sigma_{\lambda x}(t) = \lim_{\lambda\longrightarrow 0}\max\left\{1 - \frac{\|\lambda x\|}{t}, 0\right\} = \lim_{\lambda\longrightarrow 0}\max\left\{1 - \frac{|\lambda|\|x\|}{t}, 0\right\} = 1,$
   $t > 0 \Longrightarrow \lim_{\lambda\longrightarrow 0}\sigma_{\lambda x}(t) = K_0(t)$

4. $\sigma_{x+y}(t+s) = \max\left\{1 - \frac{\|x+y\|}{t+s}, 0\right\} = \max\left\{1 - \left\|\frac{x}{t+s} + \frac{y}{t+s}\right\|, 0\right\}$
   $\ge max\left\{1 - \left\|\frac{x}{t}\right\| - \left\|\frac{y}{s}\right\|, 0\right\} = \ \psi_L(\sigma_x(t), \sigma_y(s))$

□

3. Structure of A-Random Approach Space

Definition 3.1 *Let* Ω *be non -empty set . A function* $\delta_R : \Omega \times 2^{\Omega} \longrightarrow \nabla^{+}$ *is called distance on* Ω *if satisfy the following:*

$R1)\ \delta_R(n, \{n\}) = K_0(r),\ for\ any\ n \in 2^{\Omega}$
$R2)\ \delta_R(n, \emptyset) = K_{\infty}(r),\ \forall\ n \in \Omega$
$R3)\ \delta_R(n, E \cup D) = \min\{\delta_R(n, E), \delta_R(n, D)\},\quad for\ any\ n \in \Omega,\quad A, B \in 2^{\Omega}$
$R4)\ \delta_R(n, E) \geq\ \delta_R\left(n, E^{k(t)}\right)\ + k(t),\ for\ any\ \ n \in \Omega,\ k(t) \in \nabla^{+},$

where $E^{k(t)} = \{n \in \Omega : \delta_R(n, E) \geq\ k(t)\}$ *.*
A pair $(\Omega,\ \ \delta_R)$ *where* δ_R *is distance is said to be Random appr. space .*

Example 3.1 Let (Ω, F, ψ) be RM-space. normed spaces. Define function.

$$\sigma_x(t) = \begin{cases} K_0(0),\ if\ p \leq 0 \\ \frac{p}{p+\|w-n\|},\ if\ p > 0 \end{cases}$$

Then $(\Omega, \mathcal{F}, \psi_p)$ is random metric space

Proof:

1. $\mathcal{F}_{w,n}\ (p) = K_0(p)$ then $\frac{p}{p+\|w-n\|} = 1$ therefore, $w - n = 0$ hence $w = n$
2. $\mathcal{F}_{w,n}\ (p) =\ \frac{p}{p+\|w-n\|} = \frac{p}{p+\|n-w\|} = \mathcal{F}_{n,w}(p)$
3. $\psi_p(\mathcal{F}_{w,n}(p),\ \mathcal{F}_{n,q}(t)) = \frac{p}{p+\|w-n\|} \cdot \frac{t}{t+\|n-q\|} = \frac{1}{1+\frac{\|n-m\|}{r}} \cdot \frac{1}{1+\frac{\|m-q\|}{s}}$
   $\leq \frac{1}{1+\frac{\|w-n\|}{p+t}} \cdot \frac{1}{1+\frac{\|n-q\|}{p+t}} = \frac{p+t}{p+t+\|w-q\|} = \mathcal{F}_{w,q}\ (p+t)$

Then $(\Omega, \mathcal{F}, \psi_p)$ is random metric space.
Given a Random metric $(\Omega, \mathcal{F}, T_p)$, define $\delta_{\mathcal{F}} : \Omega \times 2^{\Omega} \longrightarrow \nabla^{+}$ by □

$$\sigma_x(t) = \begin{cases} K_0(\infty), if\ A = \emptyset \\ \inf \mathcal{F}_{x,y}(t),\ if\ A \neq \emptyset \end{cases}$$

Proof: If $N \neq \emptyset$, for all $x \in \Omega : \delta_{\mathcal{F}}(x, \{x\}) = K_0(0)$.

1. If $N = \emptyset$, $\delta_{\mathcal{F}}(x, \emptyset) = K_0(\infty)$
2. If $N \neq \emptyset,\ for\ all\ x \in \Omega\ , \delta_{\mathcal{F}}(x, \emptyset) = inf_{\emptyset \in A}\mathcal{F}_{x,\emptyset}(t) = K_0(\infty)$
3. If $N \neq \emptyset,\ for\ all\ x \in \Omega, N, E \in 2^{\Omega}$
   $\delta_{\mathcal{F}}(x, \emptyset) = inf_{\emptyset \in A}\mathcal{F}_{x,\emptyset}(t) = K_0(\infty)\ \delta_{\mathcal{F}}(x, N \cup E) = inf_{a \in N \cup E}\mathcal{F}_{x,N \cup E}(t)$
   $= \min(inf_{a \in A}\mathcal{F}_{x,a}(t), inf_{a \in B}\mathcal{F}_{x,a}(t)) = \min(\ \delta_{\mathcal{F}}(x, N), \delta_{\mathcal{F}}(x, E))$
4. If $N = \emptyset$
   $\delta_{\mathcal{F}}(x, \emptyset) \geq \delta_{\mathcal{F}}(x, \emptyset) + k(t)$
   $If N \neq \emptyset,$
   $\delta_{\mathcal{F}}(x, N) \inf_{a \in N}\mathcal{F}_{x,a}(t) \geq inf_{a \in N}\mathcal{F}_{x,a}(t) + k(t) = \delta_{\mathcal{F}}\left(x, N^{k(t)}\right) + k(t),\ \ k(t) \in \nabla^{+}$

□

4. New Results of δ_R-Contractions on A-Random Approach Spaces

Definition 4.1 *Let* (Ω, δ_R) *and* $\left(\mho, \acute{\delta_R}\right)$ *are A-Random appr. spaces. The function* $\vartheta : \Omega \rightarrow \mho$ *is said to be* δ_R *–contraction if* $\acute{\delta_R}(\vartheta\ (x)\ , \vartheta(A)) \geq \delta_R\,(x, A)\,,\ forall x \in \Omega$ *and for any* $A \in 2^{\Omega}$.

Proposition 4.1 *Let* $(\Omega,\ \delta_R)$ *be A-Random appr. spaces and* $\vartheta : (\Omega, \delta_R) \rightarrow (\Omega, \delta_R)$ *then for all* $M,\ N \in 2^{\Omega}$.

1. *I:* $(\Omega, \delta_R) \rightarrow (\Omega, \delta_R)$ *is* δ_R *– contraction.*
2. *The constant map is* δ_R*- contraction.*

Proof: It is clear. □

Proposition 4.2 *Let* $(\Omega, \delta_R), (\acute{\Omega}, \acute{\delta_R})$ *and* $(\acute{\acute{\Omega}}, \acute{\acute{\delta_R}})$ *be A-Random approach spaces . The function* $\vartheta : (\Omega, \delta_R) \rightarrow (\acute{\Omega}, \acute{\delta_R})$
$h : (\acute{\Omega}, \acute{\delta_R}) \ \rightarrow \ (\acute{\acute{\Omega}}, \acute{\acute{\delta_R}}) are \delta_R - contraction. Then h\ o\vartheta : (\Omega,\ \delta_R) \rightarrow (\acute{\acute{\Omega}}, \acute{\acute{\delta_R}}) is \delta_R\breve{}\ contraction.$

Proof: Let $M,\ N \in 2^{\Omega}$ then $\acute{\acute{\delta_R}}(h\ o\ \vartheta\ (N), h\ o\vartheta(M) \geq \acute{\delta_R}(\ \vartheta(N), \vartheta(M))$ since ϑ is β- contraction ,so $\acute{\delta_R}(\vartheta(N), \vartheta(M)) \geq \delta_R(N, M)$.
Thus $\acute{\acute{\delta_R}}(h\ o\ \vartheta(N),\ h\ o\ \vartheta(M)) = \acute{\acute{\delta_R}}\ (h\ (\vartheta(N), h(\vartheta(M) \geq \acute{\delta_R}((N), (M)) \geq \delta_R(N, M) h\ o\ \vartheta$ is δ_R-contraction. □

Proposition 4.3 *Let* (Ω, δ_R) *and* $(\acute{\Omega},\ \acute{\delta_R})$ *be A-Random appr. spaces and* $\vartheta : (\Omega,\ \delta_R) \ \rightarrow \ (\acute{\Omega},\ \acute{\delta_R})$ *is* δ_R*- contraction. Then the restriction* $\vartheta|_{\mathcal{B}}$ *is the* δ_R*- contraction for* $\mathcal{B} \subset \Omega$.

Proof: Suppose $\vartheta : (\Omega,\ \delta_R) \ \rightarrow \ (\acute{\Omega},\ \acute{\delta_R})$ is δ_R- contraction and $\mathcal{B} \subset \Omega$. Define $f :\ \mathcal{B} \rightarrow \acute{\Omega}$ by $f(n) = \vartheta(n)$ for all $n \in \mathcal{B}$. $\acute{\delta_R}(fn), f(A)) = \acute{\delta_R}(\vartheta\ (n), \vartheta(A)) \geq \delta_R(n, A)$. □

Proposition 4.4 *Let* (Ω_i, δ_{Ri}) *be a family of A-Random appr. spaces that any* $\varsigma \in I$*. Then, the projection pr :Л* $\Omega i \rightarrow \ \Omega i$ *is* δ_R*– contraction.*

Proof: Let $x_i \in \ \Omega_i, M \in P($, $pr :$ Л $\Omega i \ \rightarrow \ \Omega i$ projection function .
$\acute{\delta_R} i(Pr(x_i),\ Pr(M)) = \acute{\delta_R} i(Pr(x1, \ldots, xi)\ ,\ Pr(Mi))$ for $k \in \ I$
$\acute{\delta_R} i((x_i), (M) \leq (\delta_R 1((x1), M1) \times \delta_R 2((x2), M2) \times \ldots \times \delta_R i((xi), (Mi) = \prod_{i\in I} \delta_R i(\prod_{i\in I} xi, \prod_{i\in I} Mi)) = \delta_R i(\prod_{i\in I} xi, \prod_{i\in I} Mi)$. Hence $\acute{\delta_R} i(Pr(xi), Pr(M)) \leq \acute{\delta_R}(\prod_{i\in I} xi, M)$. Then $Pr(x)$ is δ_R - contraction. □

Proposition 4.5 *Let* $\vartheta : \Omega \rightarrow \Omega$ *be* δ_R*– contraction. Then, the map contraction* $\vartheta \times I_N : \Omega \times N \rightarrow \Omega \times N$ *is* δ_R*– contraction*

Proof: For all $s \in \Omega, n \in N$ and $M \in 2^{\Omega}$
$\acute{\delta_R}(\vartheta\ (\Omega,\ N), \vartheta\ (M,\ \Omega))) = \acute{\delta_R}\ (\vartheta s) \text{x}\ I_N,\ \vartheta\ (m) \text{x}\ I_N\ = \acute{\delta_R}\ ((\vartheta s\ \text{x} N,\ \vartheta\ (M)\ \text{x}\ N)$
$= \min \acute{\delta_R}((\vartheta s), \vartheta(M))), \acute{\acute{\delta_R}}\ (N, N) \geq \min \delta_R(w, M), \acute{\acute{\delta_R}}(N, N)$
$= \beta((s, \delta_R N), (M, N))$. So $'((\vartheta(w, \mathcal{V})\pounds(M, \mathcal{V})) contraction$ □

5. New Structure of A-Random Approach Vector Space

Definition 5.1 *We say* $(\Omega\ , \delta_R,\ *)$ *is an A-Random appr. semi-group if and only if:*

1. (Ω, δ_R) *is A-Random appr. space.*
2. $(\Omega, *)$ *is a semi – group.*
3. $* : \Omega \times \Omega \longrightarrow \Omega, (x, y) = x * y$ *is* δ_R*-contraction.*

Definition 5.2 *We say* $(\Omega\ , \delta_R, *)$ *is A-Random appr. group if it satisfies:*

1. (Ω, δ_R) *is A-Random appr. space.*
2. $(\Omega, *)$ *is group.*
3. $* : \Omega \times \Omega \longrightarrow \Omega, (x, y) = x * y$ *is* δ_R*-contraction.*
4. $\aleph : \Omega \to \Omega, x \to -x$ *is* δ_R *– contraction.*

Definition 5.3 *(A-Random approach sub – space): A subset B of A-Random approach vector space over the field* Γ *is called A-Random approach subspace if satisfy the following*

1. *B subspace of vector space* $(\Omega, +, .)$.
2. (B, δ_R) *A-Random approach space*

Proposition 5.1 *Let* $\mathbb{R}$ *be set of real number and* $(R^n,\ \delta_R)$ *is A-Random approach space with usual distance*

Proof: $(R^n, \delta_R, +)$ is A-Random approach group with usual distance and addition for $i = 1, \ldots, n$ For all $x \in R^n\ for\ all\ M \in 2^{R^n}$
$\delta_R :\ 2^{R^n}\ \times\ 2^{R^n}\ \to \nabla^+$ define as:

$$\delta_R\left(x, M\right) = \begin{cases} \inf_{x_i\ \in M} \mathcal{F}_{x_i, y_i}\ (t), & M \neq \emptyset \\ K_0\left(\infty\right),\ M = \emptyset \end{cases}$$

1. We will prove $(R^n,\ \delta_R)$ A-Random approach space
   (a) $M \neq \emptyset$, for all $(x_1, x_2, \ldots.., x_n) \in R^n$
   $\delta_R\left(x, \{x\}\right) = inf_{a\in A}\mathcal{F}_{x_i,a}\left(t\right) = inf_{x_i\in\{x_i\}}\mathcal{F}_{x_i,\ x_i}\left(t\right) = K_0\left(t\right),\ for\ all,\ i = 1, 2, \ldots, n.$
   $If M = \emptyset,\ \delta_R\left(x, M\right) = inf_{\emptyset\in M}\mathcal{F}_{x_i,\emptyset}\left(t\right) = K_0(t)$
   (b) $\delta_R\left(x_i, \emptyset\right) = inf_{\emptyset\in M}\mathcal{F}_{x_i,\emptyset}\left(t\right) = inf\left(\left(\mathcal{F}_{x_1,\emptyset}\left(t\right), \mathcal{F}_{x_2,\emptyset}\left(t\right), \ldots, \mathcal{F}_{x_n,\emptyset}\left(t\right)\right) = K_0(t)\right.$
   (c) $\delta_R\left(x_i, M \cup B\right) = inf_{a_i\in M\cup B}\mathcal{F}_{x_i,a_i}\left(t\right) \leq \min\left(inf_{a_i\in M}\left(\mathcal{F}_{x_i,a}\left(t\right), \mathcal{F}_{x_i,a}\left(t\right)\right)\right)$
   $= \min\left(inf_{a_i\in M}\mathcal{F}_{x_i,\ a}\left(t\right), inf_{a_i\in B}\mathcal{F}_{x_i,\ a}\left(t\right)\right) = \min\left(\delta_R\left(x_i, M\right), \delta_R\left(x_i, B\right)\right)$
   (d) $\delta_R\left(x_i, M\right) = inf_{a_i\in M}\mathcal{F}_{x_i,a}\left(t\right) \geq inf_{a_i\in M}\mathcal{F}_{x_i,a}\left(t\right) + g\left(t\right) = \delta_R\left(x_i, M^{g(t)}\right) + g(t)$

   Then $(R^n,\ \delta_R\ )$ is A-Random approach space
2. It is clear $(R^n, *)$ is group with usull addition.
   (a) $\acute{\delta_R}\left(x + y, M + B\right) = inf_{a\in M,b\in B}\mathcal{F}_{x_i+y_i,a+b}\left(t\right) \geq inf_{a\in M}\mathcal{F}_{x_i,a}\left(t\right) +, inf_{a_i\in B}\mathcal{F}_{y_i,b}\left(t\right)$
   $= \delta_R\left(x_i, M\right) + \delta_R\left(y_i, B\right)$
   $\acute{\delta_R}\left(f\left(x\right), f\left(M\right)\right) = \acute{\delta_R}\left(-x, \{-m\}\right) = inf_{m\in M}\mathcal{F}_{-x,-m}\left(t\right) = inf_{-m\in M}\mathcal{F}_{x,m}\left(t\right) = \acute{\delta_R}\left(x, M\right)$

   Therefore, the inverse function is $\acute{\delta_R}$-contraction that is $(R^n,\ \delta_R,\ +)$ is A-Random approach group
   .

□

Proposition 5.2 *$\mathbb{Z}$ be a set of all integer numbers, then $(\mathbb{Z}, \delta_R,\ *)$ is A-Random appr. group with the usual addition*

$$\delta_R(x, B) = \begin{cases} \inf_{b\ \in B} \mathcal{F}_{x,b}(t),\ B \neq \emptyset \\ K_0(\infty),\ B = \emptyset \end{cases}$$

Such that $\delta_R : 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \to \nabla^+$

Proof:

1. We will prove $(\mathbb{Z}, \delta_R\ )$ A-Random appr. space
   (a) $B \neq \emptyset$, for all $x \in \mathbb{Z}$
   $\delta_R(x, \{x\}) = inf_{b\in B}\mathcal{F}_{x,b}(t) = inf_{x\in\{x\}}\mathcal{F}_{x,\ b}(t) = K_0(t)$
   If $B = \emptyset, \delta_R(x, M) = inf_{\emptyset\in M}\mathcal{F}_{x_i,\emptyset}(t) = K_0(t)$.
   (b) $\delta_R(x, \emptyset) = inf_{\emptyset\in B}\mathcal{F}_{x,\emptyset}(t) = inf\left((\mathcal{F}_{x,\emptyset}(t)\right) = K_0(t)$.
   (c) $\delta_R(x, M \cup B) = inf_{b\in M\cup B}\mathcal{F}_{x,b}(t) \leq \min(inf_{b\in M}(\mathcal{F}_{x,b}(t), \mathcal{F}_{x,b}(t)))$
   $= \min(inf_{b\in M}\mathcal{F}_{x,b}(t), inf_{b\in B}\mathcal{F}_{x,b}(t)) = \min(\delta_R(x, M), \delta_R(x, B))$.
   (d) $\delta_R(x, B) = inf_{b\in B}\mathcal{F}_{x,\ b}(t) \geq inf_{b\in B}\mathcal{F}_{x,b}(t) + g(t) = \delta_R\left(x, B^{g(t)}\right) + g(t)$

   Then $(\mathbb{Z}, \delta_R)$ is A-Random approach space.

2. It is clear $(\mathbb{Z}, *)$ is group with usual addition.
   (a) $\acute{\delta_R}(f(x, y), f(M, B)) = \acute{\delta_R}(x + y, M + B) = inf_{a\in M, b\in B}\mathcal{F}_{x+y,\ a+b}(t)$
   $\geq inf_{a\in M}\mathcal{F}_{x,\ a}(t) +, inf_{b\in B}\mathcal{F}_{y,\ b}(t) = \delta_R(x, M) + \delta_R(y, B)$
   $\acute{\delta_R}(f(x), f(M)) = \acute{\delta_R}(-x, \{-m\}) = inf_{m\in M}\mathcal{F}_{-x,\ -m}(t) = inf_{-m\in M}\mathcal{F}_{x,\ m}(t) = \acute{\delta_R}(x, M)$

   Hence, integer numbers with the usual addition is A-Random approach group

□

Definition 5.4 *Let $(\Omega, \delta_R, *)$ be A-Random approach group and $B \subset \Omega$. Then, $(B,\ \delta_R,\ *)$ is said to be A-Random approach sub- group, if satisfy:*

1. *(B, δ_B) is A-Random approach space.*
2. *$(B,\ *)$ is sub- group.*
3. *$\vartheta :\ B \times B \to B$ with $\vartheta(x, y) = x * y^{-1}$ is δ_R-contraction.*

Proposition 5.3 *Let $\mathcal{Z}$ be the set of all integer numbers and sub set of R with usual distance δ_R ,*

$$\delta_R(x, M) = \begin{cases} \inf_{x\ \in M} \mathcal{F}_{x,y}\ (t)M\ \neq\ \emptyset \\ K_0(\infty),\ M = \emptyset \end{cases}$$

Then $(\mathcal{Z},\ \delta_R, +)$ is A-Random approach sub- group.

Proof: $(\mathcal{Z},\ \delta_R, +)$ is A-Random approach space (it is clear).
And $(\mathcal{Z}, +)$ is sub group of $(R, +)$.

1. $(\mathcal{Z}, \delta_R)$ is A-Random approach space

We will prove $+:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\left(n,m^{-1}\right)\mapsto n-m$ is δ_R-contraction
If $N=\emptyset, B=\emptyset$
$\acute{\delta_R}\left(f\left(x,y\right),f\left(N,B\right)\right)=\acute{\delta_R}\left(x-y,N-B\right)\geq\min\left\{\delta_R\left(x-y,N\right),\delta_R\left(x-y,N\cap B^c\right)\right\}=K_0(t)$
If $N\neq\emptyset$
$\acute{\delta_R}\left(x-y,N-B\right)=inf_{a\in N,b\in B}\mathcal{F}_{x-y-(a\ -b)}\left(t\right)\geq inf_{a\in N,b\in B}\mathcal{F}_{x-a+(-y\ -b)}\left(t\right)=inf_{a\in N}\mathcal{F}_{x-a}\left(t\right)+inf_{b\in B}\mathcal{F}_{-(y+b)}=\acute{\delta_R}\left(x,N\right)+\acute{\delta_R}\left(y,B\right)$
Thus is$\acute{\delta_R}$-contraction.
Then $(\mathcal{Z},\ \delta_R,+)$ is A-Random appr. subgroup .

□

Definition 5.5 *Let Ω be a non-empty set with binary operations: addition and scalar multiplication, δ_R is distance on Ω. We said $(\Omega,\delta_R,*,\odot)$ to be A-Random approach vector space if satisfy: A-Random approach.*

1. $(\Omega,\ \delta_R\ *)$ *is A-Random approach group.*
2. $\alpha.\ a\ \in\ \Omega.$
3. $\alpha\left(a+\ b\right)=\ \alpha\ a\ +\ \alpha\ b$ *for all* $\alpha\ \in\ F$ *for all* $a,\ b\ \in\Omega.$
4. $(a+b)\alpha\ =a\ \alpha+b\ \alpha$ *for all* $\alpha\ \in\ F$ *for all* $a,\ b\ \in\Omega.$
5. $(\lambda.\ \alpha).a=\ \lambda\ (\alpha.\ a),$ *for all* $a\ \in\Omega$ *and* $\lambda,\ \alpha\ \in F$.
6. $\odot:\ F\times\Omega\to\Omega,\odot(\alpha,\ a)=\alpha.a$ *is* δ_R *– contraction* .
7. $a=a,a\in\Omega$

Proposition 5.4 *The Euclidean space. $(R^n,\delta_R,+)$ with usual distance δ_R, addition and scalar multiplication $\odot$ is A-Random approach vector space.*

Proof: $(R^n,\delta_R,+))$ is A-Random approach group with usual distance and addition for $i=1,\ldots,n$ For all $X\in R^n$, $M\in 2^{R^n}$ $(R^n,\ \delta_R\ )$ A-Random approach space

1. It is clear $\propto x_i\in R^n$.
2. $\acute{\delta_R}\left(x_i+y_i,N+B\right)=inf_{a\in N,b\in B}\mathcal{F}_{x_i+y_i-(a+b)}\left(t\right)=inf_{a\in N,b\in B}\mathcal{F}_{x_i+y_i-a-b}\left(t\right)$
$\geq inf_{a\in N}\mathcal{F}_{x_i-a}\left(t\right)+inf_{b\in B}\mathcal{F}_{(y_i-b)}=\acute{\delta_R}\left(x_i,N\right)+\acute{\delta_R}\left(y_i,B\right).$
3. $\left(x_i+y_i\right)\propto=x_i\propto+y_i\propto.$
4. $\left(\propto\beta\right)x_i=\alpha\left(\beta x_i\right).$
5. $1x_i=x_i.$

Then $(R^n,\ \delta_R,\ +)$ is A-Random approach vector space □

Proposition 5.5 *If$\mathcal{N}$is A-Random approach vector space, then$\mathcal{N}$is vector space.*

Proof: The proof is straight forward. According to definition of A-Random approach vector space, $\mathcal{N}$satisfy the condition of vector space. □

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