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A study of $*-$Ricci solitons in the framework of trans-Sasakian 3-manifolds

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ABSTRACT: The purpose of this work is to study $*-$Ricci solitons in trans-Sasakian 3-manifolds. First, $*-$Ricci solitons in trans-Sasakian 3-manifolds admitting cyclic $\eta-$recurrent Ricci tensor, and the curvature condition $R \cdot Q = 0$ are considered. Furthermore, the manifolds admitting $*-$Ricci solitons obeying certain conditions on the projective curvature tensor are being inspected. Finally, we construct an example of trans-Sasakian 3-manifolds and proved that $*-$Ricci soliton on a trans-Sasakian 3-manifold is steady.

Key Words: $*-$Ricci solitons, Projective curvature tensor, Einstein manifold, Trans-Sasakian manifolds.

Contents

1. Introduction

The study of Ricci soliton (in short, RS) is of great importance due to its wide spread usage in quantum field theory, cosmology, general relativity, string theory, etc. A RS on a Riemannian manifold (M, g) is specified by [13]

$$\pounds_\zeta g + 2\Lambda g + 2S = 0, \tag{1.1}$$

where S and $\pounds_\zeta$ mean the Ricci tensor and the Lie derivative operator along the vector field ζ on M, respectively; and $\Lambda \in \mathbb{R}$ (the set of real numbers). Since for $\zeta = 0$, the equation (1.1) reduces to the Einstein equation, that's why, an Einstein manifold is a particular case of RS. The nature of RS depends on the values of Λ, according to $\Lambda > 0$: expanding soliton; $\Lambda < 0$: shrinking soliton; and $\Lambda = 0$: steady soliton.

The concept of $*-$Ricci tensor on almost Hermitian manifolds was proposed by Tachibana [25], and in the following years this concept began to be widely used in the fields of physics and mathematics. Further, Hamada [12] defined and studied the $*-$Ricci tensor of real hypersurfaces in the non-flat complex space forms, while in contact metric manifolds $*-$Ricci tensor defined by Blair [3] is given by

$$S^*(\zeta_1, \zeta_2) = g(Q^*\zeta_1, \zeta_2) = \text{Trace}\{\varphi \circ R(\zeta_1, \varphi\zeta_2)\} \tag{1.2}$$

for any vector fields ζ_1, ζ_2 on M, where S^* is a tensor field of type $(0, 2)$ and Q^* is the $*-$Ricci operator. Recently, $*-$Ricci tensor on $\alpha-$cosymplectic manifolds was studied in [1].

In [22], Oubina defined a new class of almost contact manifolds called "trans-Sasakian manifold" with the product manifold $M \times \mathbb{R}$ belonging to the class W_4. The local structures of trans-Sasakian manifolds

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was carried by Marrero [21]. It is to be noticed that the trans-Sasakian structures of kind $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are $\alpha-$Sasakian [18], $\beta-$Kenmotsu [18] and cosymplectic [2], respectively.

Definition 1.1 *[19] A Riemannian metric g on an (M, g) is called a $*$-Ricci soliton (in short, $*$-RS) if*

$$(\pounds_\xi g)(\zeta_1, \zeta_2) + 2S^*(\zeta_1, \zeta_2) + 2\varLambda g(\zeta_1, \zeta_2) = 0 \tag{1.3}$$

for all ζ_1, ζ_2 on M.

An abundant of work on RS has been carried out by various geometers in several ways to a different extent, for instance, we refer to the papers [4,5,7,8,9,11,14,15,17,20,23,24,26] and the references therein.

In this study, we handle the scrutiny of a trans-Sasakian 3-manifold admitting a $*$-RS. The article is unfolded as follows: Preliminaries on trans-Sasakian 3-manifolds are the focus of Section 2. In Section 3, we confer the $*$-RS in trans-Sasakian 3-manifolds having cyclic $\eta-$recurrent Ricci tensor and cyclic parallel Ricci tensor. Section 4 deals with the study of $*$-RS in trans-Sasakian 3-manifolds satisfying the curvature condition $R \cdot Q = 0$. Sections 5 and 6 are dedicated to confering the $*$-RS in trans-Sasakian 3-manifolds confessing the constraint of curvature conditions $Q \cdot \mathcal{P} = 0$ and $\mathcal{P}(\xi, \zeta_1) \cdot S = 0$, respectively. In section 7, we have shown that a $\varphi-$projectively flat trans-Sasakian 3-manifold admitting $*$-RS is an Einstein manifold and the Ricci soliton is steady. At last, we model a trans-Sasakian 3-manifold example which helps to examine the existence of $*$-RS on the manifold.

2. Preliminaries

A manifold M^{2n+1} (dimension $M = 2n+1$) is said to be an almost contact metric manifold if there is a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and g is a Riemannian metric (compatible) such that [2]

$$\varphi^2\zeta_1 = -\zeta_1 + \eta(\zeta_1)\xi, \quad \eta(\xi) - 1 = 0, \tag{2.1}$$

$$g(\varphi\zeta_1, \varphi\zeta_2) = g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2), \tag{2.2}$$

for all $\zeta_1, \zeta_2 \in \chi(M^{2n+1})$; where $\chi(M^{2n+1})$ is the Lie algebra of vector fields on M^{2n+1}. In addition, we have

$$\varphi\xi = 0, \quad \eta(\varphi\zeta_1) = 0, \quad g(\zeta_1, \xi) = \eta(\zeta_1). \tag{2.3}$$

The fundamental 2-form Φ of M^{2n+1} is defined by

$$\Phi(\zeta_1, \zeta_2) = g(\zeta_1, \varphi\zeta_2) \tag{2.4}$$

for any $\zeta_1, \zeta_2 \in \chi(M^{2n+1})$.

A structure (φ, ξ, η, g) on M^{2n+1} is known as a trans-Sasakian structure [22], if $(M^{2n+1} \times \mathbb{R}, J, G)$ belongs to the class W_4 [10], where J is the almost complex structure on $M^{2n+1} \times \mathbb{R}$ demarcated by smooth functions f on $M^{2n+1} \times \mathbb{R}$ and $J(\zeta_1, f\frac{d}{dt}) = (\varphi\zeta_1 - f\xi, \eta(\zeta_1)\frac{d}{dt})$ for all ζ_1 on M^{2n+1}. The condition that might be used to express this is as follows:

$$(\nabla_{\zeta_1}\varphi)\zeta_2 = \alpha(g(\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\zeta_1) + \beta(g(\varphi\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\varphi\zeta_1). \tag{2.5}$$

Here, we say that α and β are the smooth functions on M^{2n+1} with trans-Sasakian structure is of type (α, β). From (2.5), it follows that

$$\nabla_{\zeta_1}\xi = -\alpha\varphi\zeta_1 + \beta(\zeta_1 - \eta(\zeta_1)\xi), \tag{2.6}$$

$$(\nabla_{\zeta_1}\eta)\zeta_2 = -\alpha g(\varphi\zeta_1, \zeta_2) + \beta g(\varphi\zeta_1, \varphi\zeta_2), \tag{2.7}$$

where ∇ stands for the Levi-Civita connection of g.

In a trans-Sasakian 3-manifold (say M^3), we have [6]

$$\begin{aligned} R(\zeta_1, \zeta_2)\xi \quad = \quad & (\alpha^2 - \beta^2)(\eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2) \\ & + 2\alpha\beta(\eta(\zeta_2)\varphi\zeta_1 - \eta(\zeta_1)\varphi\zeta_2) \\ & + (\zeta_2\alpha)\varphi\zeta_1 - (\zeta_1\alpha)\varphi\zeta_2 \\ & + (\zeta_2\beta)\varphi^2\zeta_1 - (\zeta_1\beta)\varphi^2\zeta_2, \end{aligned} \tag{2.8}$$

$$\begin{aligned} R(\xi,\zeta_1)\zeta_2 &= (\alpha^2-\beta^2)(g(\zeta_1,\zeta_2)\xi-\eta(\zeta_2)\zeta_1) \\ &\quad +2\alpha\beta(g(\varphi\zeta_1,\zeta_2)\xi-\eta(\zeta_2)\varphi\zeta_1) \\ &\quad +(\zeta_2\alpha)\varphi\zeta_1+g(\varphi\zeta_2,\zeta_1)(grad\ \alpha) \\ &\quad +(\zeta_2\beta)(\zeta_1-\eta(\zeta_1)\xi)-g(\varphi\zeta_1,\varphi\zeta_2)(grad\ \beta), \end{aligned} \tag{2.9}$$

$$2\alpha\beta+\xi\alpha=0, \tag{2.10}$$

$$S(\zeta_1,\xi)=(2(\alpha^2-\beta^2)-\xi\beta)\eta(\zeta_1)-\zeta_1\beta-(\varphi\zeta_1)\alpha, \tag{2.11}$$

where R and S represent the curvature tensor and the Ricci tensor of M^3, respectivelty. Moreover, in an M^3 of type (α,β), we have [6]

$$grad\ \beta=\varphi(grad\ \alpha). \tag{2.12}$$

For constants α and β, we obtain from (2.10) and (2.12) that

$$R(\xi,\zeta_1)\zeta_2 = (\alpha^2-\beta^2)(g(\zeta_1,\zeta_2)\xi-\eta(\zeta_2)\zeta_1), \tag{2.13}$$

$$R(\xi,\zeta_1)\xi = (\alpha^2-\beta^2)(\eta(\zeta_1)\xi-\zeta_1), \tag{2.14}$$

$$R(\zeta_1,\zeta_2)\xi = (\alpha^2-\beta^2)(\eta(\zeta_2)\zeta_1-\eta(\zeta_1)\zeta_2), \tag{2.15}$$

$$\eta(R(\zeta_1,\zeta_2)\zeta_3)=(\alpha^2-\beta^2)(g(\zeta_2,\zeta_3)\eta(\zeta_1)-g(\zeta_1,\zeta_3)\eta(\zeta_2)), \tag{2.16}$$

$$S(\zeta_1,\xi)=2(\alpha^2-\beta^2)\eta(\zeta_1), \tag{2.17}$$

$$(\pounds_\xi g)(\zeta_1,\zeta_2)=2\beta(g(\zeta_1,\zeta_2)-\eta(\zeta_1)\eta(\zeta_2)) \tag{2.18}$$

for all $\zeta_1,\zeta_2,\zeta_3\in\chi(M^3)$. In the paper, throughout we consider $\alpha=\beta=$constant.

Definition 2.1 *[27] An M^3 is said to be an η-Einstein if its S is of the form*

$$S(\zeta_1,\zeta_2)=\sigma_1 g(\zeta_1,\zeta_2)+\sigma_2\eta(\zeta_1)\eta(\zeta_2),$$

where σ_1 and σ_2 are smooth functions on M^3. Furthermore, M^3 is said to be an Einstein if $\sigma_2=0$.

Definition 2.2 *[27] The projective curvature tensor $\mathcal{P}$ in an M^3 is defined by*

$$\mathcal{P}(\zeta_1,\zeta_2)\zeta_3=R(\zeta_1,\zeta_2)\zeta_3-\frac{1}{2}[S(\zeta_2,\zeta_3)\zeta_1-S(\zeta_1,\zeta_3)\zeta_2], \tag{2.19}$$

where $\zeta_1,\zeta_2,\zeta_3\ \in\chi(M^3)$.

Now, we recall the results of Haseeb et al. [16] obtained on a trans-Sasakian 3-manifold admitting $*$-RS (g,ξ,Λ):

Lemma 2.1 *[16] In an M^3, the $*$-Ricci tensor S^* is given by*

$$S^*(\zeta_1,\zeta_2)=S(\zeta_1,\zeta_2)-(\alpha^2-\beta^2)g(\zeta_1,\zeta_2)-(\alpha^2-\beta^2)\eta(\zeta_1)\eta(\zeta_2) \tag{2.20}$$

for any $\zeta_1,\zeta_2\in\chi(M^3)$.

Theorem 2.1 *[16] An M^3 admitting $*$-RS (g,ξ,Λ) is an η-Einstein manifold of the form*

$$S(\zeta_1,\zeta_2)=(\alpha^2-\beta^2-\beta-\Lambda)g(\zeta_1,\zeta_2)+(\alpha^2-\beta^2+\beta)\eta(\zeta_1)\eta(\zeta_2) \tag{2.21}$$

and the $*$-Ricci soliton is steady.

By contracting (2.21), we find the scalar curvature

$$r=4\alpha^2-4\beta^2-2\beta-3\Lambda. \tag{2.22}$$

Again from (2.21), we obtain

$$Q\zeta_1=(\alpha^2-\beta^2-\beta-\Lambda)\zeta_1+(\alpha^2-\beta^2+\beta)\eta(\zeta_1)\xi, \tag{2.23}$$

where Q is the Ricci operator related with S by $g(Q\zeta_1,\zeta_2)=S(\zeta_1,\zeta_2)$.

3. $*$-RS on an M^3 admitting cyclic η-recurrent Ricci tensor

Definition 3.1 *An M^3 is said to admit cyclic η-recurrent Ricci tensor, if the following relation holds on M^3:*

$$(\nabla_{\zeta_1}S)(\zeta_2,\zeta_3)+(\nabla_{\zeta_2}S)(\zeta_3,\zeta_1)+(\nabla_{\zeta_3}S)(\zeta_1,\zeta_2) \tag{3.1}$$

$$=\eta(\zeta_1)S(\zeta_2,\zeta_3)+\eta(\zeta_2)S(\zeta_3,\zeta_1)+\eta(\zeta_3)S(\zeta_1,\zeta_2)$$

for all $\zeta_1,\zeta_2,\zeta_3\in\chi(M^3)$.

The covariant differentiation of (2.21) respecting to ζ_1 leads to

$$(\nabla_{\zeta_1}S)(\zeta_2,\zeta_3)=(\alpha^2-\beta^2+\beta)\{(\nabla_{\zeta_1}\eta)(\zeta_2)\eta(\zeta_3)+\eta(\zeta_2)(\nabla_{\zeta_1}\eta)(\zeta_3)\}. \tag{3.2}$$

By using (2.7) in (3.2), we have

$$\begin{aligned}(\nabla_{\zeta_1}S)(\zeta_2,\zeta_3) \quad = \quad & (\alpha^2-\beta^2+\beta)\{-\alpha g(\varphi\zeta_1,\zeta_2)\eta(\zeta_3)-\alpha g(\varphi\zeta_1,\zeta_3)\eta(\zeta_2)\\ &+\beta g(\varphi\zeta_1,\varphi\zeta_2)\eta(\zeta_3)+\beta g(\varphi\zeta_1,\varphi\zeta_3)\eta(\zeta_2)\}.\end{aligned} \tag{3.3}$$

If the manifold M^3 has cyclic η-recurrent Ricci tensor, then in view of (2.21) and (3.3), the relation (3.1) takes the form

$$\begin{aligned}&\{2\beta(\alpha^2-\beta^2+\beta)-(\alpha^2-\beta^2-\beta-\Lambda)\}\{g(\zeta_2,\zeta_3)\eta(\zeta_1)+g(\zeta_1,\zeta_3)\eta(\zeta_2)\\ &\quad+g(\zeta_1,\zeta_2)\eta(\zeta_3)\}-\{(6\beta+3)(\alpha^2-\beta^2+\beta)\}\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)=0,\end{aligned}$$

which by putting $\zeta_2=\zeta_3=\xi$ and using the equations (2.1) and (2.3) reduces to

$$(2(\alpha^2-\beta^2)-\Lambda)\eta(\zeta_1)=0.$$

This gives

$$\Lambda=2(\alpha^2-\beta^2), \tag{3.4}$$

where $\eta(\zeta_1)\neq 0$. By means of the fact that, in an M^3 the $*$-RS is steady, that is, $\Lambda=0$, we obtain from (3.4) that $\alpha=\pm\beta$. Therefore, we have the following theorem:

Theorem 3.1 *Let an M^3 admit $*$-RS. If M^3 has a cyclic η-recurrent Ricci tensor, then $\alpha=\pm\beta$ on M^3.*

Next, we consider that if M^3 admits cyclic parallel Ricci tensor, then (3.1) reduces to

$$(\nabla_{\zeta_1}S)(\zeta_2,\zeta_3)+(\nabla_{\zeta_2}S)(\zeta_3,\zeta_1)+(\nabla_{\zeta_3}S)(\zeta_1,\zeta_2)=0. \tag{3.5}$$

In view of (2.2) and (3.3), (3.5) leads to

$$2\beta(\alpha^2-\beta^2+\beta)\{g(\zeta_1,\zeta_2)\eta(\zeta_3)+g(\zeta_1,\zeta_3)\eta(\zeta_2)+g(\zeta_2,\zeta_3)\eta(\zeta_1)-2\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)\}=0.$$

Putting $\zeta_3=\xi$ in the foregoing equation, we have

$$2\beta(\alpha^2-\beta^2+\beta)g(\varphi\zeta_1,\varphi\zeta_2)=0.$$

Thus, we have either $\beta=0$ or $\alpha^2-\beta^2+\beta=0 \implies \alpha=\beta=0$, or $\alpha=0,\beta=1$ (where $g(\varphi\zeta_1,\varphi\zeta_2)\neq 0$). Therefore, we have the following corollary:

Corollary 3.1 *Let an M^3 admit $*$-RS. If M^3 admits a cyclic parallel Ricci tensor, then either M^3 is an α-Sasakian or cosymplectic or Kenmotsu manifold.*

4. ∗-RS on M^3 satisfying $R \cdot Q = 0$

Let an M^3 admit ∗-RS and satisfies the condition $(R(\zeta_1, \zeta_2) \cdot Q)\zeta_3 = 0$. Then, we have

$$R(\zeta_1, \zeta_2)Q\zeta_3 - Q(R(\zeta_1, \zeta_2)\zeta_3) = 0 \tag{4.1}$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \chi(M^3)$. In view of (2.23), from (4.1), we have either $\alpha^2 - \beta^2 + \beta = 0$, or

$$\eta(R(\zeta_1, \zeta_2)\zeta_3)\xi - \eta(\zeta_3)R(\zeta_1, \zeta_2)\xi = 0. \tag{4.2}$$

Since, in a 3-dimensional Riemannian manifold, the conformal curvature tensor vanishes. Thus, the curvature tensor R can be expressed as

$$\begin{aligned} R(\zeta_1, \zeta_2)\zeta_3 \quad = \quad & g(\zeta_2, \zeta_3)Q\zeta_1 - g(\zeta_1, \zeta_3)Q\zeta_2 + S(\zeta_2, \zeta_3)\zeta_1 - S(\zeta_1, \zeta_3)\zeta_2 \\ & -\frac{r}{2}(g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2). \end{aligned}$$

Utilizing (2.21)-(2.23) in the last equation, it follows that

$$\begin{aligned} R(\zeta_1, \zeta_2)\zeta_3 \quad = \quad & (-\beta - \frac{\Lambda}{2})(g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2) \\ & +(\alpha^2 - \beta^2 + \beta)(\eta(\zeta_2)\eta(\zeta_3)\zeta_1 - \eta(\zeta_1)\eta(\zeta_3)\zeta_2) \\ & +(\alpha^2 - \beta^2 + \beta)(g(\zeta_2, \zeta_3)\eta(\zeta_1)\xi - g(\zeta_1, \zeta_3)\eta(\zeta_2)\xi), \end{aligned} \tag{4.3}$$

from which, we find

$$\eta(R(\zeta_1, \zeta_2)\zeta_3) = (\alpha^2 - \beta^2 - \frac{\Lambda}{2})(g(\zeta_2, \zeta_3)\eta(\zeta_1) - g(\zeta_1, \zeta_3)\eta(\zeta_2)), \tag{4.4}$$

$$R(\zeta_1, \zeta_2)\xi = (\alpha^2 - \beta^2 - \frac{\Lambda}{2})(\eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2). \tag{4.5}$$

Now using (4.4) and (4.5) in (4.2), we have

$$\begin{aligned} (\alpha^2 - \beta^2 - \frac{\Lambda}{2})&(g(\zeta_2, \zeta_3)\eta(\zeta_1)\xi - g(\zeta_1, \zeta_3)\eta(\zeta_2)\xi \\ & -\eta(\zeta_2)\eta(\zeta_3)\zeta_1 + \eta(\zeta_1)\eta(\zeta_3)\zeta_2) = 0. \end{aligned} \tag{4.6}$$

Replacing ζ_3 by $\varphi^2\zeta_3$, (4.6) turns to

$$(\alpha^2 - \beta^2 - \frac{\Lambda}{2})(g(\zeta_1, \zeta_3)\eta(\zeta_2)\xi - g(\zeta_2, \zeta_3)\eta(\zeta_1)\xi) = 0,$$

which on replacing ζ_2 by ξ takes the form

$$(\alpha^2 - \beta^2 - \frac{\Lambda}{2})(g(\zeta_1, \zeta_3)\xi - \eta(\zeta_1)\eta(\zeta_3)\xi) = 0. \tag{4.7}$$

By contracting (4.7), we get

$$(\alpha^2 - \beta^2 - \frac{\Lambda}{2}) = 0. \tag{4.8}$$

Since, ∗-RS in an M^3 is steady, we find from (4.8) that $\alpha = \pm\beta$. This helps us to state the following theorem:

Theorem 4.1 *Let an M^3 admit ∗-RS. If M^3 satisfies $R \cdot Q = 0$, then $\alpha = \pm\beta$ on M^3.*

5. $*$-RS on M^3 satisfying $Q \cdot \mathcal{P} = 0$

Let an M^3 admit $*$-RS and satisfies the condition $Q \cdot \mathcal{P} = 0$. Then, we have

$$Q(\mathcal{P}(\zeta_1, \zeta_2)\zeta_3) - \mathcal{P}(Q\zeta_1, \zeta_2)\zeta_3 - \mathcal{P}(\zeta_1, Q\zeta_2)\zeta_3 - \mathcal{P}(\zeta_1, \zeta_2)Q\zeta_3 = 0 \tag{5.1}$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \chi(M^3)$. In view of (2.19), (5.1) leads to

$$\begin{aligned} &Q(R(\zeta_1, \zeta_2)\zeta_3) - R(Q\zeta_1, \zeta_2)\zeta_3 - R(\zeta_1, Q\zeta_2)\zeta_3 \\ &-R(\zeta_1, \zeta_2)Q\zeta_3 + S(\zeta_2, Q\zeta_3)\zeta_1 - S(\zeta_1, Q\zeta_3)\zeta_2 = 0. \end{aligned}$$

Now by putting $\zeta_2 = \xi$ and then taking the inner product with ξ, the foregoing equation takes the form

$$\begin{aligned} &\eta(Q(R(\zeta_1, \xi)\zeta_3)) - \eta(R(Q\zeta_1, \xi)\zeta_3) - \eta(R(\zeta_1, Q\xi)\zeta_3) \\ &-\eta(R(\zeta_1, \xi)Q\zeta_3) + S(\xi, Q\zeta_3)\eta(\zeta_1) - S(\zeta_1, Q\zeta_3)\xi = 0. \end{aligned} \tag{5.2}$$

From (2.21), (2.22) and (4.3), we can easily find the followings:

$$\eta(Q(R(\zeta_1, \xi)\zeta_3)) = \{\Lambda - 2(\alpha^2 - \beta^2)\}(\alpha^2 - \beta^2 - \frac{\Lambda}{2})g(\varphi\zeta_1, \varphi\zeta_3), \tag{5.3}$$

$$\begin{aligned} \eta(R(Q\zeta_1, \xi)\zeta_3) = &-\{\Lambda - 2(\alpha^2 - \beta^2)\}(\alpha^2 - \beta^2 - \frac{\Lambda}{2})\eta(\zeta_1)\eta(\zeta_3) \\ &-(\alpha^2 - \beta^2 - \frac{\Lambda}{2})S(\zeta_1, \zeta_3), \end{aligned} \tag{5.4}$$

$$\eta(R(\zeta_1, Q\xi)\zeta_3) = \{\Lambda - 2(\alpha^2 - \beta^2)\}(\alpha^2 - \beta^2 - \frac{\Lambda}{2})g(\varphi\zeta_1, \varphi\zeta_3), \tag{5.5}$$

$$\begin{aligned} \eta(R(\zeta_1, \xi)Q\zeta_3) = &-\{\Lambda - 2(\alpha^2 - \beta^2)\}(\alpha^2 - \beta^2 - \frac{\Lambda}{2})\eta(\zeta_1)\eta(\zeta_3) \\ &-(\alpha^2 - \beta^2 - \frac{\Lambda}{2})S(\zeta_1, \zeta_3), \end{aligned} \tag{5.6}$$

$$S(\xi, Q\zeta_3) = \{2(\alpha^2 - \beta^2) - \Lambda\}^2\eta(\zeta_3). \tag{5.7}$$

Plugging (5.3)-(5.7) in (5.2), it follows that

$$S(\zeta_1, Q\zeta_3) = 2(\alpha^2 - \beta^2 - \frac{\Lambda}{2})S(\zeta_1, \zeta_3). \tag{5.8}$$

In view of (2.21) from (5.8) we find either $\alpha^2 - \beta^2 + \beta = 0$, or $S(\zeta_1, \zeta_3) = \{2(\alpha^2 - \beta^2) - \Lambda\}\eta(\zeta_1)\eta(\zeta_3)$, which by using the fact that $*$-RS in an M^3 is steady reduces to

$$S(\zeta_1, \zeta_3) = 2(\alpha^2 - \beta^2)\eta(\zeta_1)\eta(\zeta_3). \tag{5.9}$$

Thus, we have the following theorem:

Theorem 5.1 *Let an M^3 admit $*$-RS. If M^3 satisfies $Q \cdot \mathcal{P} = 0$, then either $\alpha^2 - \beta^2 + \beta = 0$ or the manifold is a special type of η-Einstein manifold.*

6. $*$-RS on M^3 satisfying $\mathcal{P}(\xi,\zeta_1)\cdot S=0$

We consider an M^3 admitting $*$-RS and satisfies the condition $\mathcal{P}(\xi,\zeta_1)\cdot S=0$. Then, we have

$$S(\mathcal{P}(\xi,\zeta_1)\zeta_2,\zeta_3)+S(\zeta_2,\mathcal{P}(\xi,\zeta_1)\zeta_3)=0 \tag{6.1}$$

for all $\zeta_1,\zeta_2,\zeta_3\in\chi(M^3)$. From (2.19), we have

$$\mathcal{P}(\xi,\zeta_1)\zeta_2=R(\xi,\zeta_1)\zeta_2-\frac{1}{2}[S(\zeta_1,\zeta_2)\xi-S(\xi,\zeta_2)\zeta_1],$$

which by using $R(\xi,\zeta_1)\zeta_2=(\alpha^2-\beta^2-\frac{\Lambda}{2})(g(\zeta_1,\zeta_2)\xi-\eta(\zeta_2)\zeta_1)$ and $S(\xi,\zeta_2)=2(\alpha^2-\beta^2-\frac{\Lambda}{2})\eta(\zeta_2)$ takes the form

$$\mathcal{P}(\xi,\zeta_1)\zeta_2=-\frac{1}{2}S(\zeta_1,\zeta_2)\xi+(\alpha^2-\beta^2-\frac{\Lambda}{2})g(\zeta_1,\zeta_2)\xi. \tag{6.2}$$

By making use of (6.2) in (6.1), we have either $\alpha^2-\beta^2=0$, or

$$S(\zeta_1,\zeta_2)\eta(\zeta_3)+S(\zeta_1,\zeta_3)\eta(\zeta_2)-2(\alpha^2-\beta^2-\frac{\Lambda}{2})(g(\zeta_1,\zeta_2)\eta(\zeta_3)+g(\zeta_1,\zeta_3)\eta(\zeta_2))=0,$$

which by putting $\zeta_3=\xi$ and using $\Lambda=0$ leads to

$$S(\zeta_1,\zeta_2)=(\alpha^2-\beta^2)g(\zeta_1,\zeta_2).$$

This helps us to state the following theorem:

Theorem 6.1 *Let an M^3 admit $*$-RS. If M^3 satisfies $\mathcal{P}(\xi,\zeta_1)\cdot S=0$, then either $\alpha=\pm\beta$ or M^3 is an Einstein manifold.*

7. $\varphi-$projectively flat trans-Sasakian 3-manifolds admitting $*$-RS

In this section, we consider a $\varphi-$projectively flat M^3 admitting $*$-RS, that is, $g(\mathcal{P}(\varphi\zeta_1,\varphi\zeta_2)\varphi\zeta_3,\varphi\zeta_4)=0$ holds on M^3. Thus, in view of (2.21), we have

$$g(R(\varphi\zeta_1,\varphi\zeta_2)\varphi\zeta_3,\varphi\zeta_4)=\frac{1}{2}(S(\varphi\zeta_2,\varphi\zeta_3)g(\varphi\zeta_1,\varphi\zeta_4)-S(\varphi\zeta_1,\varphi\zeta_3)g(\varphi\zeta_2,\varphi\zeta_4)),$$

which by taking $\zeta_1=\zeta_4=e_i$ and summing over $i=1,2$, we have

$$\sum_{i=1}^{2}g(R(\varphi e_i,\varphi\zeta_2)\varphi\zeta_3,\varphi e_i) = \frac{1}{2}\sum_{i=1}^{2}[S(\varphi\zeta_2,\varphi\zeta_3)g(\varphi e_i,\varphi e_i) -S(\varphi e_i,\varphi\zeta_3)g(\varphi\zeta_2,\varphi e_i)]. \tag{7.1}$$

In an M^3, we have

$$\sum_{i=1}^{2}g(R(\varphi e_i,\varphi\zeta_2)\varphi\zeta_3,\varphi e_i)=S(\varphi\zeta_2,\varphi\zeta_3)-(\alpha^2-\beta^2)g(\varphi\zeta_2,\varphi\zeta_3), \tag{7.2}$$

$$\sum_{i=1}^{2}g(\varphi e_i,\varphi e_i)=2, \tag{7.3}$$

$$\sum_{i=1}^{2}S(\varphi e_i,\varphi\zeta_3)g(\varphi\zeta_2,\varphi e_i)=S(\varphi\zeta_2,\varphi\zeta_3). \tag{7.4}$$

Utilizing (7.2)-(7.4) in (7.1), we obtain

$$S(\varphi\zeta_2,\varphi\zeta_3)=2(\alpha^2-\beta^2)g(\varphi\zeta_2,\varphi\zeta_3),$$

which in view of (2.21) turns to $(\alpha^2-\beta^2+\beta+\Lambda)g(\varphi\zeta_2,\varphi\zeta_3)=0$, this implies $\Lambda=-(\alpha^2-\beta^2+\beta)$, where $g(\varphi\zeta_2,\varphi\zeta_3)\neq 0$. By using the fact that, in M^3 the $*$-RS is steady, thus we have $\alpha^2-\beta^2+\beta=0$. This implies either $\alpha=\beta=0$ or $\alpha=0,\beta=1$. Thus, we state the following theorem:

Theorem 7.1 *Let M^3 be a $\varphi-$projectively flat M^3 and admit the $*$-RS. Then either M^3 is a cosymplectic manifold or a Kenmotsu manifold.*

Example: We consider a manifold $M^3 = \{(u, v, w) \in \mathbb{R}^3\}$, where (u, v, w) are the standard coordinates in $\mathbb{R}^3$. Let ϱ_1, ϱ_2 and ϱ_3 be the vector fields on M^3 given by

$$\varrho_1 = \frac{\partial}{\partial w} + 2v\frac{\partial}{\partial u}, \quad \varrho_2 = \frac{\partial}{\partial v}, \qquad \varrho_3 = 2\frac{\partial}{\partial u} = \xi.$$

Let g be the Riemannian metric defined by

$$g(\varrho_k, \varrho_l) = \begin{cases} 1, & 1 \leq k = l \leq 3, \\ 0, & 1 \leq k \neq l \leq 3. \end{cases}$$

Let φ be the $(1, 1)$ tensor field on M^3 defined by

$$\varphi\varrho_1 = -\varrho_2, \qquad \varphi\varrho_2 = \varrho_1, \qquad \varphi\varrho_3 = 0.$$

Let η be the 1-form on M^3 defined by $\eta(\zeta_1) = g(\zeta_1, \varrho_3)$ for all $\zeta_1 \in \mathfrak{X}(M^3)$. By applying the linearity of φ and g, we have

$$\eta(\xi) = 1, \quad \varphi^2 = -\zeta_1 + \eta(\zeta_1)\xi, \quad \eta(\varphi\zeta_1) = 0,$$

$$g(\zeta_1, \xi) = \eta(\zeta_1), \quad g(\varphi\zeta_1, \varphi\zeta_2) = g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)$$

for $\zeta_1, \zeta_2 \in \chi(M^3)$. Then, we have

$$[\varrho_1, \varrho_2] = -\varrho_3, \qquad [\varrho_2, \varrho_3] = [\varrho_3, \varrho_1] = 0.$$

By using well-known Koszul's formula, one can derive

$$\nabla_{\varrho_1}\varrho_1 = 0, \qquad \nabla_{\varrho_1}\varrho_2 = -\frac{1}{2}\varrho_3, \qquad \nabla_{\varrho_1}\varrho_3 = \frac{1}{2}\varrho_2,$$

$$\nabla_{\varrho_2}\varrho_1 = \frac{1}{2}\varrho_3, \qquad \nabla_{\varrho_2}\varrho_2 = 0, \qquad \nabla_{\varrho_2}\varrho_3 = -\frac{1}{2}\varrho_1,$$

$$\nabla_{\varrho_3}\varrho_1 = \frac{1}{2}\varrho_2, \qquad \nabla_{\varrho_3}\varrho_2 = -\frac{1}{2}\varrho_1, \qquad \nabla_{\varrho_3}\varrho_3 = 0.$$

It can be easily exposed that M^3 is a trans-Sasakian manifold of type $(\frac{1}{2}, 0)$. Further, the following components of the curvature tensor can be easily obtained:

$$R(\varrho_1, \varrho_2)\varrho_2 = -\frac{3}{4}\varrho_1, \quad R(\varrho_1, \varrho_3)\varrho_3 = \frac{1}{4}\varrho_1, \quad R(\varrho_2, \varrho_1)\varrho_1 = -\frac{3}{4}\varrho_2,$$

$$R(\varrho_2, \varrho_3)\zeta_3 = \frac{1}{4}\varrho_2, \quad R(\varrho_3, \varrho_1)\varrho_1 = \frac{1}{4}\varrho_3, \quad R(\varrho_3, \varrho_2)\varrho_2 = \frac{1}{4}\varrho_3.$$

From these components we calculate $S(\varrho_3, \varrho_3) = \frac{1}{2}$. Also from (2.21), we have $S(\varrho_3, \varrho_3) = 2(\alpha^2 - \beta^2) - \Lambda$. By equating both the values of $S(\varrho_3, \varrho_3)$, we obtain $2(\alpha^2 - \beta^2) - \Lambda = \frac{1}{2}$, which for $\alpha = \frac{1}{2}$ and $\beta = 0$ gives $\Lambda = 0$. Thus, a $*$-RS (g, ξ, Λ) on a trans-Sasakian 3-manifold is steady.

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