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Decomposition of $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity

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ABSTRACT: In this new research paper we introduce and investigate the new kind of open sets α-$\mathcal{H}_g$-open, σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open, β-$\mathcal{H}_g$- open and S-β-$\mathcal{H}_g$- open sets in hereditary generalized topological spaces. Moreover we introduce and study some new types of sets in $\mathcal{HGTS}$. Also, we obtained a decomposition of $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity and decompositions of (g_μ, λ)-continuity.

Key Words: hereditary generalized topology, α-$\mathcal{H}_g$-open, σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open sets and β-$\mathcal{H}_g$-open sets.

Contents

1. Introduction and Preliminaries

In the year 2002, Csaszar [6] introduced very usefull notions of generalized topology and generalized continuity. Consider $\mathcal{Z}$ be a nonempty set and μ be a collection from the subsets of $\mathcal{Z}$. Then μ is called a *generalized topology* (briefly GT) if $\emptyset \in \mu$ and an arbitrary union of elements from μ belongs to μ. The generalized-closure of a subset A of X, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the interior of A, denoted by $i_\mu(A)$, is the union of all μ-open sets contained in A. A subset $\mathcal{L}$ of a space $(\mathcal{Z}, \mu)$ is called as *μ-α-open* [7] (resp. μ-σ-open [7], μ-π-open [7],μ-b-open [16], μ-β-open [7]) if $\mathcal{L} \subset i_\mu c_\mu i_\mu(\mathcal{L})$ (resp. $\mathcal{L} \subset c_\mu i_\mu(\mathcal{L})$, $\mathcal{L} \subset i_\mu c_\mu(\mathcal{L})$, $\mathcal{L} \subset i_\mu c_\mu \cup c_\mu i_\mu(\mathcal{L})$, $\mathcal{L} \subset c_\mu i_\mu c_\mu(\mathcal{L})$). A space $\mathcal{Z}$ is called a C_0-space [17], if $C_0 = \mathcal{Z}$, where C_0 is the set of all representative elements of sets of μ and x is called a represent element of $u \in \mu$ if $u \subset v$ for each $v \in \mu(x)$. A subset A of a space (X, μ) is said to be *μ-semi-open* [10], if $A \subset c_\mu i_\mu(A)$. A subset A of generalized topological space (X, μ) is said to be *g_μ-closed* [10] (resp. *ω_μ-closed* [12]), if $c_\mu(A) \subseteq M$ whenever $A \subseteq M$ and M is μ-open (resp. μ-σ-open) in X. The complement of ω_μ-closed (resp. g_μ-closed) is ω_μ-open [12] (resp. g_μ-open [10]). The g_μ-interior (resp. ω_μ-interior) is the largest g_μ-open (resp. ω_μ-open) set contained in A and it is denoted by $i_g(A)$ (resp. $i_\omega(A)$). A nonempty family $\mathcal{H}$ of subsets of $\mathcal{Z}$ is called as a *hereditary class* [8], if $\mathcal{L} \in \mathcal{H}$ and $B \subset \mathcal{L}$, then $B \in \mathcal{H}$. For each $\mathcal{L} \subseteq \mathcal{Z}$, $\mathcal{L}^*(\mathcal{H}, \mu)=\{\mathcal{Z} \in \mathcal{Z} : \mathcal{L} \cap V \notin \mathcal{H}$ for all $V \in \mu$ such that $\mathcal{Z} \in V\}$ [8]. For $\mathcal{L} \subset \mathcal{Z}$, define $c^*_\mu(\mathcal{L}) = \mathcal{L} \cup \mathcal{L}^*(\mathcal{H}, \mu)$ and $\mu^* = \{\mathcal{L} \subset \mathcal{Z} : \mathcal{Z} - \mathcal{L} = c^*_\mu(\mathcal{Z} - \mathcal{L})\}$. If $\mathcal{H}$ is a hereditary class on $\mathcal{Z}$ then $(\mathcal{Z}, \mu, \mathcal{H})$ is called a hereditary generalized topological space and it is denoted by $\mathcal{HGTS}$. Also papers [1-5] have introduced some property related to minimal spaces with hereditary classes.

Definition 1.1 [8] A subset $\mathcal{L}$ of a $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is said to be α-$\mathcal{H}$-open (resp.σ-$\mathcal{H}$-open, π-$\mathcal{H}$-open, β-$\mathcal{H}$-open, β^*-$\mathcal{H}$-open, μ^*-closed), if $\mathcal{L} \subseteq i_\mu c^*_\mu i_\mu(\mathcal{L})$ (resp.$\mathcal{L} \subseteq c^*_\mu i_\mu(\mathcal{L})$, $\mathcal{L} \subseteq i_\mu c^*_\mu(\mathcal{L})$, $\mathcal{L} \subseteq c_\mu i_\mu c^*_\mu(\mathcal{L})$, $\mathcal{L} \subseteq c^*_\mu i_\mu c^*_\mu(\mathcal{L})$, $c^*_\mu(\mathcal{L}) \subset \mathcal{L}$).

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2. α-$\mathcal{H}_g$-open sets

Definition 2.1 A subset $\mathcal{L}$ of a $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is called as

1. α-$\mathcal{H}_g$-open, if $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L})$.
2. σ-$\mathcal{H}_g$-open, if $\mathcal{L} \subseteq c^*_\mu i_g(\mathcal{L})$.
3. π-$\mathcal{H}_g$-open, if $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$.
4. β-$\mathcal{H}_g$-open, if $\mathcal{L} \subseteq c_\mu i_g c^*_\mu(\mathcal{L})$.
5. $\mathcal{S}$-β-$\mathcal{H}_g$-open, if $\mathcal{L} \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L})$.

Proposition 2.1 *In $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$, the following holds:*

1. *Every μ-open set is α-$\mathcal{H}_g$-open.*
2. *Every μ-open set is σ-$\mathcal{H}_g$-open.*
3. *Every μ-open set is π-$\mathcal{H}_g$-open.*
4. *Every μ-open set is β-$\mathcal{H}_g$-open.*
5. *Every μ-open set is $\mathcal{S}$-β-$\mathcal{H}_g$-open.*

Proof: (1). Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ be μ-open. Then, $\mathcal{L} = i_\mu(\mathcal{L})$. Now $\mathcal{L} \subseteq i_\mu(\mathcal{L}) = i_g(\mathcal{L}) \subseteq i_g c^*_\mu(\mathcal{L}) = i_g c^*_\mu i_\mu(\mathcal{L}) \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is α-$\mathcal{H}_g$-open.

Proof of 2, 3 4 and 5 are similar to proof 1.

The converse of Proposition 2.1 need not be true from the following example. □

Example 2.1 *Consider $\mathcal{Z} = \{1, 2, 3, 4\}$, $\mu = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}, Z\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{3\}\}$. Then $\mathcal{L} = \{1, 2, 3\}$ is α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open, β-$\mathcal{H}_g$-open, $\mathcal{S}$-β-$\mathcal{H}_g$-open) but not μ-open.*

Proposition 2.2 *In $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, the following are holds:*

1. *Every g_μ-open set is α-$\mathcal{H}_g$-open.*
2. *Every g_μ-open set is σ-$\mathcal{H}_g$-open.*
3. *Every g_μ-open set is π-$\mathcal{H}_g$-open.*
4. *Every g_μ-open set is β-$\mathcal{H}_g$-open.*
5. *Every g_μ-open set is S-β-$\mathcal{H}_g$-open.*

Proof: (1). Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is g_μ-open. Then, $\mathcal{L} = i_g(\mathcal{L})$. Now $\mathcal{L} \subseteq i_g(\mathcal{L}) \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is α-$\mathcal{H}_g$-open.

Proof of 2, 3, 4 and 5 are similar to proof 1.

□

Proposition 2.3 *In $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, the following are holds:*

1. *Every ω_μ-open set is α-$\mathcal{H}_g$-open.*

2. *Every ω_μ-open set is σ-$\mathcal{H}_g$-open.*

3. *Every ω_μ-open set is π-$\mathcal{H}_g$-open.*

4. *Every ω_μ-open set is β-$\mathcal{H}_g$-open.*

5. *Every ω_μ-open set is S-β-$\mathcal{H}_g$-open.*

Proof: (1). Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is ω_μ-open. Then, $\mathcal{L} = i_\omega(\mathcal{L})$. Now $\mathcal{L} = i_\omega(\mathcal{L}) \subseteq i_g(\mathcal{L}) \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is α-$\mathcal{H}_g$-open.

Proof of 2, 3 4 and 5 are similar to proof 1.

The converse of Propositions 2.2 and 2.3 need not be true from the following example. □

Example 2.2 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{3\}\}$. Then $\mathcal{L} = \{1,2,3\}$ is α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open, β-$\mathcal{H}_g$-open, $\mathcal{S}$-β-$\mathcal{H}_g$-open) but neither g_μ-open nor ω_μ-open.*

The notions of α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open, β-$\mathcal{H}_g$-open) and μ-α-open (resp. μ-σ-open, μ-π-open, μ-β-open) are independent.

Example 2.3 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,2,3\}, \{3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{2\}\}$. Then $\mathcal{L} = \{3\}$ is α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open) but not μ-α-open (resp. μ-σ-open) and $\mathcal{M} = \{2,3,4\}$ is μ-α-open (resp. μ-σ-open) but not α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open).*

Example 2.4 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{3\}\}$. Then $\mathcal{L} = \{4\}$ is π-$\mathcal{H}_g$-open (resp. β-$\mathcal{H}_g$-open) but not μ-π-open (resp. μ-β-open).*

Example 2.5 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1\}, \{1,2,3\}, \{3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{3\}\}$. Then $\mathcal{L} = \{2,4\}$ is μ-π-open but not π-$\mathcal{H}_g$-open and $\mathcal{M} = \{1,4\}$ is μ-β-open but not β-$\mathcal{H}_g$-open.*

Proposition 2.4 *For a subset of a $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ the following hold:*

1. *Every α-$\mathcal{H}$-open is α-$\mathcal{H}_g$-open set.*

2. *Every σ-$\mathcal{H}$-open is σ-$\mathcal{H}_g$-open set.*

3. *Every π-$\mathcal{H}$-open is π-$\mathcal{H}_g$-open set.*

4. *Every β-$\mathcal{H}$-open is β-$\mathcal{H}_g$-open set.*

5. *Every β^*-$\mathcal{H}$-open is $\mathcal{S}$-β-$\mathcal{H}_g$-open set.*

Proof: (1). Consider $\mathcal{L}$ be a α-$\mathcal{H}$-open set. Then, we have $\mathcal{L} \subseteq i_\mu c^*_\mu i_\mu(\mathcal{L}) \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is α-$\mathcal{H}_g$-open.

(2). Consider $\mathcal{L}$ be a σ-$\mathcal{H}$-open. Then, we have $\mathcal{L} \subseteq c^*_\mu i_\mu(\mathcal{L}) \subseteq c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is σ-$\mathcal{H}_g$-open.

(3). Consider $\mathcal{L}$ be a π-$\mathcal{H}$-open. Then, we have $\mathcal{L} \subseteq i_\mu c^*_\mu(\mathcal{L}) \subseteq i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is π-$\mathcal{H}_g$-open.

(4). Consider $\mathcal{L}$ be a β-$\mathcal{H}$-open. Then, we have $\mathcal{L} \subseteq c_\mu i_\mu c^*_\mu(\mathcal{L}) \subseteq c_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is β-$\mathcal{H}_g$-open.

(5). Consider $\mathcal{L}$ be a Strongly β-$\mathcal{H}$-open. Then, we have $\mathcal{L} \subseteq c^*_\mu i_\mu c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is $\mathcal{S}$-β-$\mathcal{H}_g$-open.

□

The converse of Proposition 2.4 need not be correct from the following examples.

Example 2.6 *Consider* $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$,$\mathcal{H}$=$\{\emptyset, \{1,2\}\}$. *Then* $\mathcal{L} = \{1\}$ *is* α-$\mathcal{H}_g$*-open (resp.* σ-$\mathcal{H}_g$*-open,* π-$\mathcal{H}_g$*-open,* β-$\mathcal{H}_g$*-open,* $\mathcal{S}$-β-$\mathcal{H}_g$*-open) but not* α-$\mathcal{H}$*-open (*σ-$\mathcal{H}$*-open,* π-$\mathcal{H}$*-open,* β-$\mathcal{H}$*-open,* β^*-$\mathcal{H}$*-open).*

Proposition 2.5 *In* $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, *every* α-$\mathcal{H}_g$*-open is* σ-$\mathcal{H}_g$*-open.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is α-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Now $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L}) \subseteq c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is σ-$\mathcal{H}_g$-open. □

The converse part of the Proposition 2.5 need not be true from the following counter example.

Example 2.7 *Consider* $\mathcal{Z} = \{1,2,3,4,5\}$, $\mu = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$ $\mathcal{H} = \{\emptyset, \{2\}\}$. *Then* $\mathcal{Z}$ *is* σ-$\mathcal{H}_g$*-open but not* α-$\mathcal{H}_g$*-open.*

Proposition 2.6 *In* $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, *every* α-$\mathcal{H}_g$*-open set is* π-$\mathcal{H}_g$*-open.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is α-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Now $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L}) \subseteq i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is π-$\mathcal{H}_g$-open. □

The converse part of the Proposition 2.6 need not be true from the following counter example.

Example 2.8 *Consider* $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. *Then,* $\mathcal{L} = \{1,2\}$ *is* π-$\mathcal{H}_g$*-open but not* α-$\mathcal{H}_g$*-open.*

Proposition 2.7 *In* $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, *every* σ-$\mathcal{H}_g$*-open set is* β-$\mathcal{H}_g$*-open but not conversely.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is σ-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq c^*_\mu i_g(\mathcal{L})$. Now, $\mathcal{L} \subseteq c_\mu i_g(\mathcal{L}) \subseteq c_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is β-$\mathcal{H}_g$-open set. □

Example 2.9 *Consider* $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. *Then* $\mathcal{L} = \{1,2\}$ *is* β-$\mathcal{H}_g$*-open but not* σ-$\mathcal{H}_g$*-open.*

Proposition 2.8 *In* $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, *every* π-$\mathcal{H}_g$*-open set is* β-$\mathcal{H}_g$*-open but not conversely.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is π-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$. Now, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq c_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is β-$\mathcal{H}_g$-open set. □

Example 2.10 *Consider* $\mathcal{Z} = \{1,2,3,4,5\}$, $\mu = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$ $\mathcal{H} = \{\emptyset, \{2\}\}$. *Then* $\mathcal{Z}$ *is* β-$\mathcal{H}_g$*-open but not* π-$\mathcal{H}_g$*-open.*

Proposition 2.9 *In* $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, *every* π-$\mathcal{H}_g$*-open set is* S-β-$\mathcal{H}_g$*-open but not conversely.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is π-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$. Now, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is S-β-$\mathcal{H}_g$-open set. □

Example 2.11 *Consider* $\mathcal{Z} = \{1,2,3,4,5\}$, $\mu = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$ $\mathcal{H} = \{\emptyset, \{2\}\}$. *Then* $\mathcal{Z}$ *is* S-β-$\mathcal{H}_g$*-open but not* π-$\mathcal{H}_g$*-open.*

Proposition 2.10 *In $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, every σ-$\mathcal{H}_g$-open set is S-β-$\mathcal{H}_g$-open but not conversely.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}$ $(\mathcal{Z}, \mu, \mathcal{H})$ is σ-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq c^*_\mu i_g(\mathcal{L})$. Now, $\mathcal{L} \subseteq c_\mu i_g(\mathcal{L}) \subseteq c_\mu i_g c^*_\mu(\mathcal{L})$. Hence, $\mathcal{L}$ is S-β-$\mathcal{H}_g$-open set. □

Example 2.12 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. Then $\mathcal{L} = \{1,2\}$ is S-β-$\mathcal{H}_g$-open but not σ-$\mathcal{H}_g$-open.*

Theorem 2.1 *A subset $\mathcal{L}$ of $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is σ-$\mathcal{H}_g$-open if and only if $c^*_\mu(\mathcal{L}) = c^*_\mu i_g(\mathcal{L})$.*

Proof: Consider a subset $\mathcal{L}$ of $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is σ-$\mathcal{H}_g$-open. Then, $\mathcal{L} \subseteq c^*_\mu i_g(\mathcal{L})$. Now, $c^*_\mu(\mathcal{L}) \subseteq c^*_\mu c^*_\mu i_g(\mathcal{L}) \subseteq c^*_\mu i_g(\mathcal{L})$. Always $c^*_\mu i_g(\mathcal{L}) \subseteq c^*_\mu(\mathcal{L})$. Hence, $c^*_\mu(\mathcal{L}) = c^*_\mu i_g(\mathcal{L})$. Conversely, Consider $c^*_\mu(\mathcal{L}) = c^*_\mu i_g(\mathcal{L})$. Then, $\mathcal{L} \subseteq c^*_\mu(\mathcal{L}) = c^*_\mu i_g(\mathcal{L})$. Hence, $\mathcal{L}$ is σ-$\mathcal{H}_g$-open. □

Theorem 2.2 *For subset $\mathcal{L}$ of a $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, the following results are equivalent.*

1. *$\mathcal{L}$ is α-$\mathcal{H}_g$-open.*
2. *$\mathcal{L}$ is σ-$\mathcal{H}_g$-open and π-$\mathcal{H}_g$-open.*

Proof: (1) $\Rightarrow$ (2). Consider $\mathcal{L}$ is α-$\mathcal{H}_g$-open. Then by Proposition 2.5 and 2.6, $\mathcal{L}$ is σ-$\mathcal{H}_g$-open and π-$\mathcal{H}_g$-open.

(2) $\Rightarrow$ (1). Consider $\mathcal{L}$ is both σ-$\mathcal{H}_g$-open and π-$\mathcal{H}_g$-open. Then $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq i_g c^*_\mu c^*_\mu i_g(\mathcal{L}) \subseteq i_g c^*_\mu i_g(\mathcal{L})$. Hence $\mathcal{L}$ is α-$\mathcal{H}_g$-open.

□

The notions of σ-$\mathcal{H}_g$-open and π-$\mathcal{H}_g$-open are independent.

Example 2.13 *Consider $\mathcal{Z} = \{1,2,3,4,5\}$, $\mu = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$ $\mathcal{H} = \{\emptyset, \{2\}\}$. Then $\mathcal{Z}$ is σ-$\mathcal{H}_g$-open but not π-$\mathcal{H}_g$-open.*

Example 2.14 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. Then $\mathcal{L} = \{1,2\}$ is π-$\mathcal{H}_g$-open but not σ-$\mathcal{H}_g$-open.*

Proposition 2.11 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$. If $\mathcal{L}$ is an π-$\mathcal{H}_g$-open subset of $\mathcal{Z}$ such that $U \subseteq \mathcal{L} \subseteq c^*_\mu(U)$, and $U \subseteq \mathcal{Z}$, then U is π-$\mathcal{H}_g$-open set.*

Proof: Since $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$ and $c^*_\mu(\mathcal{L}) \subseteq c^*_\mu(U)$, then we have $U \subseteq \mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L}) \subseteq i_g c^*_\mu(U)$. Thus U is π-$\mathcal{H}_g$-open set. □

Proposition 2.12 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$. A subset $\mathcal{L}$ is a σ-$\mathcal{H}$-open if and only if $\mathcal{L}$ is S-β-$\mathcal{H}_g$-open and $i_g c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_\mu(\mathcal{L})$.*

Proof: Let $\mathcal{L}$ be a σ-$\mathcal{H}$-open. Then, $\mathcal{L} \subseteq c^*_\mu i_\mu(\mathcal{L}) \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L})$ and hence, $\mathcal{L}$ is S-β-$\mathcal{H}_g$-open. In addition $c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_\mu(\mathcal{L})$ and hence $i_g c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_\mu(\mathcal{L})$. Conversely, let $\mathcal{L}$ be S-β-$\mathcal{H}_g$-open and $i_g c^*_\mu(\mathcal{L}) \subseteq c^*_\mu i_\mu \subseteq (\mathcal{L})$. Then, $\mathcal{L} \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L}) \subseteq c^*_\mu c^*_\mu i_\mu(\mathcal{L}) = c^*_\mu i_\mu(\mathcal{L})$ and hence, $\mathcal{L}$ is σ-$\mathcal{H}$-open. □

Lemma 2.1 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$ and $\mathcal{L} \subseteq \mathcal{Z}$. If U is μ-open subset of $\mathcal{Z}$ then, $U \cap c^*_\mu \mathcal{L} \subseteq c^*_\mu(U \cap \mathcal{L})$.*

Proposition 2.13 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$. Then the intersection of π-$\mathcal{H}_g$-open set and an μ-open set is π-$\mathcal{H}_g$-open.*

Proof: Let $\mathcal{L}$ is π-$\mathcal{H}_g$-open set and U be a μ-open set. Then, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$. Since every μ-open set is g_μ-open, $U \cap \mathcal{L} \subseteq i_g(U) \cap i_g c^*_\mu(\mathcal{L}) = i_g(U \cap c^*_\mu(\mathcal{L})) \subseteq i_g c^*_\mu(U \cap \mathcal{L})$ by Lemma 2.1. This shows that $U \cap \mathcal{L}$ is π-$\mathcal{H}_g$-open. □

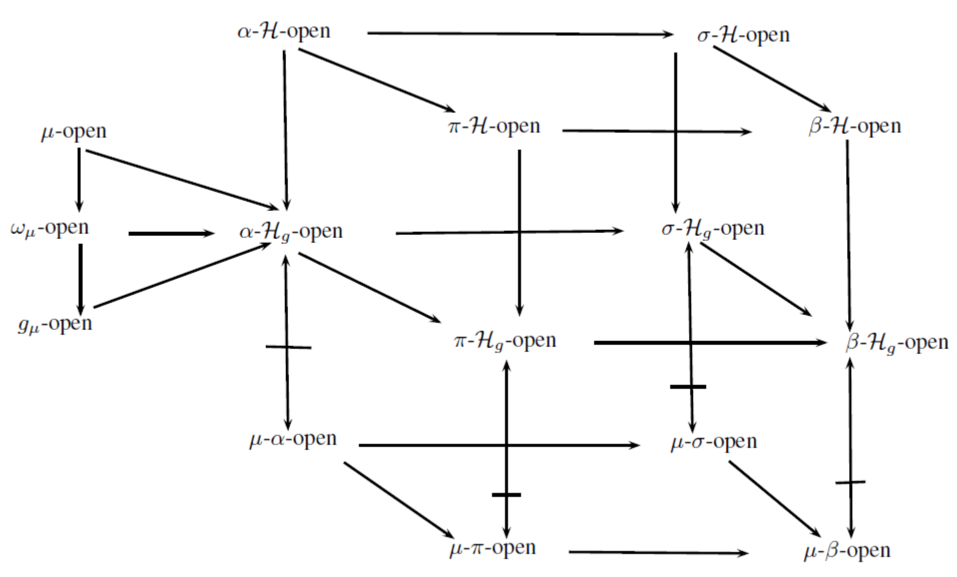
Proposition 2.14 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$. Then the intersection of S-β-$\mathcal{H}_g$-open set and an μ-open set is S-β-$\mathcal{H}_g$-open.*

Proof: Let $\mathcal{L}$ is S-β-$\mathcal{H}_g$-open set and U be a μ-open set. Then, $\mathcal{L} \subseteq c^*_\mu i_g c^*_\mu(\mathcal{L})$. Since every μ-open set is g_μ-open, $U \cap \mathcal{L} \subseteq U \cap c^*_\mu i_g(U) c^*_\mu \subseteq c^*_\mu(U \cap i_g c^*_\mu(\mathcal{L})) \subseteq c^*_\mu(i_g(U) \cap i_g c^*_\mu(\mathcal{L})) = c^*_\mu i_g(U \cap c^*_\mu(\mathcal{L})) \subseteq c^*_\mu i_g c^*_\mu(U \cap \mathcal{L})$ by Lemma 2.1 This shows that $U \cap \mathcal{L}$ is S-β-$\mathcal{H}_g$-open. □

Proposition 2.15 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a $\mathcal{HGTS}$. Then the intersection of α-$\mathcal{H}_g$-open set and an μ-open set is α-$\mathcal{H}_g$-open.*

Proof: Let $\mathcal{L}$ is α-$\mathcal{H}_g$-open set and U be a μ-open set. Then $\mathcal{L} \subseteq i_g c^*_\mu i_g(\mathcal{L})$ and $U = i_g(U)$ Since every μ-open set is g_μ-open, $U \cap \mathcal{L} \subseteq i_g(U) \cap i_g c^*_\mu i_g(\mathcal{L}) = i_g(U \cap c^*_\mu i_g(\mathcal{L})) \subseteq i_g(c^*_\mu(U \cap i_g \mathcal{L})) = i_g c^*_\mu i_g(U \cap \mathcal{L})$. This shows that $U \cap \mathcal{L}$ is α-$\mathcal{H}_g$-open. □

For several sets defined above, we have the following implications.



Definition 2.2 A subset $\mathcal{L}$ of a$\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is said to be σ-$\mathcal{H}_g$-closed, if $i_g c^*_\mu(\mathcal{L}) \subseteq \mathcal{L}$.

Definition 2.3 A subset $\mathcal{L}$ of a $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$ is said to be $\mathcal{R}^{*g}$-set, if $A = U \cap V$, where U is g_μ-open and V is σ-$\mathcal{H}_g$-closed.

Proposition 2.16 *In $\mathcal{HGTS}(\mathcal{Z}, \mu, \mathcal{H})$, every g_μ-open is $\mathcal{R}^{*g}$-set but not conversely.*

Proof: Obvious. □

Example 2.15 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1\}, \{2\}\}$. Then $L = \{2\}$ is $\mathcal{R}^{*g}$-set but not g_μ-open.*

Theorem 2.3 *Let $(\mathcal{Z}, \mu, \mathcal{H})$ be a strong $\mathcal{HGTS}$, where $\mathcal{Z}$ is C_0-space and $\mathcal{L} \subset \mathcal{Z}$. Then the following conditions are equivalent.*

1. *$\mathcal{L}$ is g_μ-open.*
2. *$\mathcal{L}$ is α-$\mathcal{H}_g$-open and $\mathcal{R}^{*g}$-set.*
3. *$\mathcal{L}$ is π-$\mathcal{H}_g$-open and $\mathcal{R}^{*g}$-set.*

Proof: (1) $\Rightarrow$ (2). Let a subset $\mathcal{L}$ of $\mathcal{Z}$ is g_μ-open. Then it is α-$\mathcal{H}_g$-open and R^{*g}-set by Propositions 2.2 and 2.16.

(2) $\Rightarrow$ (3). Let a subset $\mathcal{L}$ of $\mathcal{Z}$ is both α-$\mathcal{H}_g$-open and R^{*g}-set. Then it is both π-$\mathcal{H}_g$-open and R^{*g}-set.

(3) $\Rightarrow$ (1). Let a subset $\mathcal{L}$ of $\mathcal{Z}$ is both π-$\mathcal{H}_g$-open and R^{*g}-set. Then $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L})$ and $\mathcal{L} = U \cap V$, where U is g_μ-open and V is σ-$\mathcal{H}_g$-closed.
Now, $\mathcal{L} \subseteq i_g c^*_\mu(\mathcal{L}) = i_g c^*_\mu(U \cap V) \subseteq i_g(c^*_\mu(U) \cap c^*_\mu(V)) \subseteq i_g c^*_\mu(U) \cap i_g c^*_\mu(V)$.
Since V is σ-$\mathcal{H}_g$-closed, $i_g c^*_\mu(V) \subseteq V \Rightarrow i_g c^*_\mu(V) \subseteq i_g(V)$.
Hence $\mathcal{L} \subseteq i_g c^*_\mu(U) \cap i_g(V)$.
Now as $\mathcal{L} \subseteq U$, we have

$$\begin{aligned}\mathcal{L} &= U \cap \mathcal{L}\\ &\subseteq U \cap i_g c^*_\mu(U) \cap i_g(V)\\ &= [U \cap i_g c^*_\mu(U)] \cap i_g(V)\\ &= U \cap i_g(V)\\ &= i_g(U) \cap i_g(V)\\ &= i_g(U \cap V)\\ &= i_g(\mathcal{L}).\end{aligned}$$

Therefor $\mathcal{L} \subseteq i_g(\mathcal{L})$. Hence $\mathcal{L}$ is g_μ-open. □

The notions of α-$\mathcal{H}_g$-open (resp. π-$\mathcal{H}_g$-open) and R^{*g}-set are independent.

Example 2.16 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. Then $\mathcal{L} = \{1,2,4\}$ is R^{*g}-set but neither α-$\mathcal{H}_g$-open nor π-$\mathcal{H}_g$-open.*

Example 2.17 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{1,3,4\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{4\}\}$. Then $\mathcal{L} = \{3,4\}$ is π-$\mathcal{H}_g$-open but not R^{*g}-set.*

Example 2.18 *Consider $\mathcal{Z} = \{1,2,3,4\}$, $\mu = \{\emptyset, \{3,4\}, \{1,2,3\}, \mathcal{Z}\}$, $\mathcal{H} = \{\emptyset, \{1,2\}\}$. Then $\mathcal{L} = \{1,3,4\}$ is α-$\mathcal{H}_g$-open but not R^{*g}-set.*

3. Decomposition of $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity

Definition 3.1 A map $f : (\mathcal{Z}, \mu, \mathcal{H}) \to (W, \lambda)$ is $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity (resp.$(\sigma\text{-}\mathcal{H}_g, \lambda)$-continuity, $(\pi\text{-}\mathcal{H}_g, \lambda)$-continuity), if $j^{-1}(V)$ is α-$\mathcal{H}_g$-open (resp. σ-$\mathcal{H}_g$-open, π-$\mathcal{H}_g$-open) for each λ- open set V in (W, λ).

Definition 3.2 A map $f : (\mathcal{Z}, \mu, \mathcal{H}) \to (W, \lambda)$ is (R^{*g}, λ)-continuity, if $f^{-1}(V)$ is R^{*g}set for each λ-open set V in (W, λ).

Theorem 3.1 *For a map $f : (\mathcal{Z}, \mu, \mathcal{H}) \to (W, \lambda)$, the following results are equivalent.*

1. *f is $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity.*
2. *f is $(\sigma\text{-}\mathcal{H}_g, \lambda)$-continuity and $(\pi\text{-}\mathcal{H}_g, \lambda)$-continuity.*

Proof: This is an immediate consequence of Theorem 2.2. □

Theorem 3.2 *For a map $f : (\mathcal{Z}, \mu, \mathcal{H}) \to (W, \lambda)$ where $\mathcal{Z}$ is C_0-space, the following results are equivalent.*

1. *f is (g_μ, λ)-continuity.*
2. *f is $(\alpha\text{-}\mathcal{H}_g, \lambda)$-continuity and (R^{*g}, λ)-continuity.*
3. *f is $(\pi\text{-}\mathcal{H}_g, \lambda)$-continuity and (R^{*g}, λ)-continuity.*

Proof: This is an immediate consequence of Theorem 2.3.

□

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