Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.67179

Topological Degree Methods for Non-local Fractional $\mathfrak{p}(\mathrm{x},.)$- Laplacian Elliptic Equation

Abdesslam Ouaziz and Ahmed Aberqi

ABSTRACT: The purpose of this paper is mainly to investigate the existence of weak solutions to a class of fractional $\mathfrak{p}(\mathrm{x},.)$- Laplacian problem as follows:

$$\begin{cases} (\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v}(\mathrm{x}) &= \lambda\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x}) + f(\mathrm{x},\mathrm{v}(\mathrm{x})) \quad \text{in} \quad U, \\ \mathrm{v} &= 0 \quad \text{in} \quad \mathbb{R}^N\backslash U, \end{cases} \tag{0.1}$$

where $f : U \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, U is a smooth bounded domain of $\mathbb{R}^{\mathbb{N}}$, and $(\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}$ is the fractional $\mathfrak{p}(\mathrm{x},.)$- Laplacian operator. The main tool used here is the topological method introduced by Berkovits for demi-continuous operators of generalized type (S_+), which is based on transforming our problem to an abstract Hammerstein equation.

Key Words: operator of (S_+) type, quasimonotone operator, degree theory, Berkovits topological degree, elliptic equation

Contents

1. Introduction

Recently, great attention has been paid to the study of problems involving Laplacian operator $\mathfrak{p}(\mathrm{x},.)-$Laplacian operator. These types of problems appear in many applications and many fields in mathematics, such as elastic mechanics, electro-rheological, ultra-materials, phase transitions, thin obstacle problems, optimization, minimal surfaces, stratified materials, game theory, constrained heating, image processing, conservation laws, mathematical finance, stabilization of Lévy processes, flame propagation, fluid potentials, population dynamics, crystal dislocation, etc. We refer to [3,12,13,17,18,19,20, 26,27,29,36] and the references therein for further details.
The purpose of this paper is to investigate the existence of non-trivial solutions for equations driven by non-local integrodifferential operator $(\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}$ with homogeneous Dirichlet boundary conditions. Specifically, we consider the following problem:

$$\begin{cases} (\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v}(\mathrm{x}) &= \lambda\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x}) + f(\mathrm{x},\mathrm{v}(\mathrm{x})) \quad \text{in} \quad U, \\ \mathrm{v} &= 0 \quad \text{in} \quad \mathbb{R}^N\backslash U, \end{cases} \tag{1.1}$$

where λ is a real parameter positive, $s \in (0,1)$, U is a smooth bounded domain of $\mathbb{R}^{\mathbb{N}}$, β is a suitable potential function in $(0,+\infty)$ such that $\beta \in L^\infty(U)$, $r \in C(U,(1,+\infty))$, $\mathfrak{p} \in C(U \times U,(1,+\infty))$ with $s\mathfrak{p}(\mathrm{x},\mathrm{y}) < N$, we asssume $\mathfrak{p}$ satisfies the following conditions:

$$1 < \mathfrak{p}^- = \min_{(\mathrm{x},\mathrm{y})\in U^2} \mathfrak{p}(\mathrm{x},\mathrm{y}) \leq \mathfrak{p}(\mathrm{x},\mathrm{y}) < \mathfrak{p}^+ = \max_{(\mathrm{x},\mathrm{y})\in U^2} \mathfrak{p}(\mathrm{x},\mathrm{y}), \tag{1.2}$$

$$\mathfrak{p}(\mathrm{x},\mathrm{y}) = \mathfrak{p}(\mathrm{y},\mathrm{x}) \qquad \text{for any } (\mathrm{x},\mathrm{y}) \in U^2, \tag{1.3}$$

Submitted February 18, 2023. Published August 10, 2025
2010 *Mathematics Subject Classification:* 8J10, 58J20, 35J66.

$$\mathfrak{p}(\mathrm{x}-\mathrm{a},\mathrm{y}-\mathrm{a})=\mathfrak{p}(\mathrm{x},\mathrm{y}) \qquad \text{for any } (\mathrm{x},\mathrm{y},\mathrm{a})\in U^3, \tag{1.4}$$

$f: U\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the following assumption:

$(\mathcal{H}_1)$ There exist $\alpha>0$ and $1<\mathfrak{q}(\mathrm{x})<\mathfrak{p}_s^*=\frac{N\mathfrak{p}(\mathrm{x},\mathrm{x})}{(N-s\mathfrak{p}(\mathrm{x},\mathrm{x}))}$ such that

$$|f(\mathrm{x},\theta)|\leq\alpha\left(1+|\theta|^{\mathfrak{q}(\mathrm{x})-1}\right),$$

for a.e. $\mathrm{x}\in U,\theta\in\mathbb{R}$,

where $\mathfrak{q}: U\to(1,+\infty)$ is a continuous function such that $r^+=\max_{\mathrm{x}\in U} r(\mathrm{x})\leq\mathfrak{q}^-\leq\mathfrak{p}_s^*$, and $(\mathcal{L}_{\mathrm{K}}^s)_{\mathfrak{p}(\mathrm{x},.)}$ is the fractional $\mathfrak{p}(\mathrm{x},.)$-Laplacian operator defined as follows:

$$(\mathcal{L}_{\mathrm{K}}^s)_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v}(\mathrm{x})=2\lim_{\varepsilon\to0^+}\int_{U\setminus B_\varepsilon(\mathrm{x})}|\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})dz,$$

for all $\mathrm{x}\in U$. Here $B_\varepsilon(\mathrm{x})$ is the ball of U of radius ε and center x and $\mathrm{K}(\mathrm{x},\mathrm{z})=\frac{1}{|\mathrm{x}-\mathrm{z}|^{s+N\mathfrak{p}(\mathrm{x},\mathrm{z})}}$.
The topological degree theory is one of the most efficient methods for study the existence, uniqueness, multiplicity, and boudedness of solutions to the non-lineair equations. The degree function has fundamental properties such as normalization, additivity, existence, and homotopie invariance. These properties play an essential role in the study of non-linear differential and integral equations.
In 1912, Brouwer [15] first introduced a topological degree for continuous maps in the Euclidian space. In 1934, the authors [25] generalized the degree theory for compact operators in infinite dimensional Banach space. Moreover, the authors in [16] introduced a topological degree for nonlinear operators of monotone type in reflexive Banach space. Recently, Berkovits studied in [6] an extension of the Leray-Schauder degree for operators of generalized monotone type. In 2015, the authors in [24] we introduce a topological degree theory for a class of demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces. Moreover, they investigated the following Dirichlet boundary problem:

$$\begin{cases} -\Delta_{\mathfrak{p}}\mathrm{v}=\mathrm{v}+f(\mathrm{x},\mathrm{v},\nabla\mathrm{v}) & \text{in } \Omega,\\ \mathrm{v}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a smooth bounded of $\mathbb{R}^n$, $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function with f has the following growth condition:

$$f(\mathrm{x},s,w)\leq c(k(\mathrm{x})+|s|^{\mathfrak{q}-1}+|w|^{\mathfrak{q}-1}),$$

c is a positive constant, $\mathfrak{q}\in(1,\mathfrak{p})$, $k\in L^{p'}(\Omega)$, and $\Delta_{\mathfrak{p}}$ is the classical $\mathfrak{p}-$ Laplacian operator. For more generalization of properties of the degree topological, we recommend the papers [7,8,9,10,11,?,16,28].
The topological degree methods is based on solving the equation:

$$\mathrm{v}+S\circ T\mathrm{v}=0.$$

This equation is called an abstract Hammerstein equation, where Y is a reflexive Banach space with dual space Y^* and $T: Y\to Y^*$, and $S: Y^*\to Y$ are two monotones operators. The abstract Hammerstein equation was studied by the authors in [39] when S is an operator lineair. Moreover, when S is quasi-monotonne and T satisfies conditions (S_+), it was treated in [6].
On the other side, attention has been paied to the study of fractional Sobolev space and nonlocal problems driven by the operator $(\mathcal{L}_{\mathrm{K}}^s)_{\mathfrak{p}(\mathrm{x},.)}$. For example, in 2017 the authors [23] first introduced type of fractional Sobolev spaces with variable exponents $W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U)$ defined as

$$W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U)=\{\mathrm{v}\in L^{\mathfrak{q}(\mathrm{x})}(U),\int_{U\times U}\frac{|\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}}{|\mathrm{x}-\mathrm{z}|^{n+s\mathfrak{p}(\mathrm{x},\mathrm{z})}}dxdz<+\infty\},$$

where $s\in(0,1)$, U is an open boubded of $\mathbb{R}^n$, $\mathfrak{q}\in C^+(\bar{U})$, and $\mathfrak{p}\in C^+(\bar{U}\times\bar{U})$. Moreover, they proved the compact embedding $W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U)\hookrightarrow\hookrightarrow L^{\mathfrak{a}(\mathrm{x})}(U)$, where $\mathfrak{a}\in C^+(\bar{U})$ and $\mathfrak{a}(\mathrm{x})\in(1,\frac{n\mathfrak{p}(\mathrm{x},\mathrm{z})}{n-s\mathfrak{p}(\mathrm{x},\mathrm{z})})$ for

$\mathrm{x} \in U$. In addition, they used to the direct methods of calculs of variation for solve the following elliptic problem:

$$\begin{cases} \mathcal{L}\mathrm{v}(\mathrm{x}) + |\mathrm{v}(\mathrm{x})|^{\mathfrak{q}(\mathrm{x})-2}\mathrm{v}(\mathrm{x}) & = f(\mathrm{x}) \quad \text{in} \quad U, \\ \mathrm{v} & = 0 \quad \text{in} \quad \partial U, \end{cases} \tag{1.6}$$

with $f \in L^{a(\mathrm{x})}(U), a(\mathrm{x}) > 1$. Next, Bahrouni and Repovs [5] used Ekeland's variational principale in order to prove the existence of a solution to the following problem:

$$\begin{cases} \mathcal{L}\mathrm{v}(\mathrm{x}) + |\mathrm{v}(\mathrm{x})|^{q(\mathrm{x})-2}\mathrm{v}(x) & = \lambda|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(x) \quad \text{in} \quad U, \\ \mathrm{v} & = 0 \quad \text{in} \quad \mathbb{R}^n\backslash U, \end{cases} \tag{1.7}$$

where U is a smooth bounded domain in $\mathbb{R}^n$, λ is a positive real parameter, $r(\mathrm{x}) < \mathfrak{p}^- = \min_{(\mathrm{x},\mathrm{z})\in U\times U} \mathfrak{p}(\mathrm{x},\mathrm{z})$. Bahrouni [4] contenied the study of problem involving $(\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}$ operator, but he used the sub-supersolution methods for investigate the existence of a solution to the following equation:

$$\begin{cases} (\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v} & = h(\mathrm{x},\mathrm{v}) \quad \text{in} \quad U, \\ \mathrm{v} & = 0 \quad \text{in} \quad \mathbb{R}^n\backslash U, \end{cases} \tag{1.8}$$

where U is a smooth open bounded domain, $s \in (0,1)$, $\mathfrak{p}$ is a continuous function, and h satisfies the following assumption

$$|h(\mathrm{x},t)| \leq A_1|t|^{r(\mathrm{x})-1} + A_2, \quad \text{for all } (\mathrm{x},t) \in \mathbb{R}^N \times \mathbb{R},$$

where $r \in C(\mathbb{R}^N,\mathbb{R})$ and $1 < r(\mathrm{x}) < \mathfrak{p}^\star_s(\mathrm{x})$, for any $\mathrm{x} \in \mathbb{R}^N$. The authors [37] proved the existence and uniqueness result of weak solutions for the fractional $\mathfrak{p}(\mathrm{x},.)$-Laplacian problem:

$$\begin{cases} \mathrm{u} + (\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}(\mathrm{v}-\theta(\mathrm{v})) + \alpha(\mathrm{v}) & = h \quad \text{in} \quad U, \\ \mathrm{v} & = 0 \quad \text{in} \quad \mathbb{R}^n\backslash U, \end{cases} \tag{1.9}$$

using the classical method ([37] Theorem 2.8), where $h \in L^\infty(U)$, θ is a continuous functions defined $\mathbb{R}$ to $\mathbb{R}$, and α is a non decreasing continuous and real function defined on $\mathbb{R}$. The novelty of this work is to study the existence of the weak solution to following problem:

$$\begin{cases} (\mathcal{L}^s_{\mathrm{K}})_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v}(\mathrm{x}) & = \lambda\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x}) + f(\mathrm{x},\mathrm{v}(\mathrm{x})) \quad \text{in} \quad U, \\ \mathrm{v} & = 0 \quad \text{in} \quad \mathbb{R}^N\backslash U, \end{cases} \tag{1.10}$$

using the Berkovits topological degree for a generalized class of type (S_+). The importance of this technique is that it allows us to prove the existence of a solution to the our problem without using many conditions on the function f such as Ambrosetti-Rabinowitz condition, convergence uniforme, and limit uniforme. See [1,2,5,22,31,32,33,37]. It should be mentioned that the results in this paper are generalised to papers [4,5,23,24,37].

The outline is as follows: In Section 2, we recall some mathematical preliminaries and essential background concerning the recent Berkovits degree theory. In Section 3, we present some auxiliary lemmas necessary for the proof of the existence theorem. In Section 4, we state our main theorem. By transforming an PPE to an abstract Hammerstein equation (3.8) we prove the solvability of the nonlocal elliptic problem (1.10).

2. Mathematical preliminaries

In this section, we review some properties and definitions of Lebesgue spaces with variable exponent and the space of type $W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U)$. Secondly, we state some classes of mappings and topological degrees which will be used later. For more background, we refer to [2,5,6,21,24,34] and the references therein.

2.1. The fractional Sobolev space with variable exponent

Let
$$C^+(\bar{U}) := \left\{\mathfrak{q} : \bar{U} \to \mathbb{R}^+ : \mathfrak{q} \text{ is a continuous function, } 1 < \mathfrak{q}^- < \mathfrak{q}^+ < +\infty\right\}.$$
For any $\mathfrak{q} \in C^+(\bar{U})$, we denote $\mathfrak{q}^- = \inf_{\mathrm{x}\in\bar{U}} \mathfrak{q}(\mathrm{x})$ and $\mathfrak{q}^+ = \sup_{\mathrm{x}\in\bar{U}} \mathfrak{q}(\mathrm{x})$. Let $\mathcal{M}(U)$ be the vector space of measurable function from U into $\mathbb{R}$. The Lebesgue space with variable exponent $L^{\mathfrak{q}(\mathrm{x})}(U)$ is defined as follows:
$$L^{\mathfrak{q}(\mathrm{x})}(U) = \{\mathrm{v} \in \mathcal{M}(U) \text{ such that } \int_U |\mathrm{v}(\mathrm{x})|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x} < +\infty\}.$$
Endowed with the Luxemburg norm
$$|\mathrm{v}|_{\mathfrak{q}(\mathrm{x})} = \inf\{\alpha > 0, \int_U |\frac{\mathrm{v}(\mathrm{x})}{\alpha}|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x} \leq 1\}.$$

Lemma 2.1 *[21] The space $(L^{\mathfrak{q}(\mathrm{x})}(U), |.|_{\mathfrak{q}(\mathrm{x})})$ is a separable, uniformly convex Banach space for any $\mathfrak{q} \in C^+(\bar{U})$.*

Lemma 2.2 *(Hölder's inequality, [21]) For every $\mathfrak{q} \in C^+(\bar{U})$, the following inequality holds:*
$$|\int_U g(\mathrm{x})\mathrm{v}(\mathrm{x}) d\mathrm{x}| \leq (\frac{1}{\mathfrak{q}^-} + \frac{1}{\mathfrak{q}'^-})|g|_{\mathfrak{q}(\mathrm{x})}|\mathrm{v}|_{\mathfrak{q}'(\mathrm{x})} \leq 2|g|_{\mathfrak{q}(\mathrm{x})}|\mathrm{v}|_{\mathfrak{q}'(\mathrm{x})},$$
for all $(g, \mathrm{v}) \in L^{\mathfrak{q}(\mathrm{x})}(U) \times L^{\mathfrak{q}'(\mathrm{x})}(U)$, with $1 = \frac{1}{\mathfrak{q}(\mathrm{x})} + \frac{1}{\mathfrak{q}'(\mathrm{x})}$.

The important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{\mathfrak{q}(\mathrm{x})}(U)$ space, which is the mapping
$\rho_{\mathrm{v}(\mathrm{x})} : L^{\mathfrak{q}(\mathrm{x})}(U) \to \mathbb{R}$ defined by
$$\rho_{\mathfrak{q}(\mathrm{x})}(\mathrm{v}) = \int_U |\mathrm{v}|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x}.$$

Lemma 2.3 *[21] Let $\mathrm{v}, (\mathrm{v}_n)_{n\in\mathbb{N}} \in L^{\mathfrak{q}(\mathrm{x})}(U)$, then*

1) $|\mathrm{v}|_{\mathfrak{q}(\mathrm{x})} < 1$ ($=1$, > 1) $\Leftrightarrow \rho_{\mathfrak{q}(\mathrm{x})}(\mathrm{v}) < 1$ ($= 1$, > 1).

2) $|\mathrm{v}|_{\mathfrak{q}(\mathrm{x})} \leq 1$ *then* $|\mathrm{v}|^{\mathfrak{q}^+}_{\mathfrak{q}(\mathrm{x})} \leq \rho_{\mathfrak{q}(\mathrm{x})}(\mathrm{v}) \leq |\mathrm{v}|^{\mathfrak{q}^-}_{\mathfrak{q}(\mathrm{x})}$.

3) $|\mathrm{v}|_{\mathfrak{q}(\mathrm{x})} \geq 1$ *then* $|\mathrm{v}|^{\mathfrak{q}^-}_{\mathfrak{q}(\mathrm{x})} \leq \rho_{\mathfrak{q}(\mathrm{x})}(\mathrm{v}) \leq |\mathrm{v}|^{\mathfrak{q}^+}_{\mathfrak{q}(\mathrm{x})}$.

4) $\lim_{n\to+\infty} |\mathrm{v}_n - \mathrm{v}|_{L^{\mathfrak{q}(\mathrm{x})}(U)} = 0 \quad \Leftrightarrow \lim_{n\to+\infty} \rho_{\mathfrak{q}(\mathrm{x})}(\mathrm{v}_n - \mathrm{v}) = 0.$

Let $s \in (0, 1)$, $\mathfrak{q} \in C^+(\bar{U})$, and $\mathfrak{p} \in C^+(\bar{U} \times \bar{U})$ satisfies conditions (1.2)- (1.3). The fractional Sobolev space with variable exponent is introduced as follows:
$$W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U) = \{\mathrm{v} \in L^{\mathfrak{q}(\mathrm{x})}(U), \int_{U\times U} \frac{1}{\mu^{\mathfrak{p}(\mathrm{x},\mathrm{z})}}|\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}\mathrm{K}(\mathrm{x}, \mathrm{z}) d\mathrm{x} d\mathrm{z} < +\infty\}$$
endowed with the norm
$$\|\mathrm{v}\|_X = [\mathrm{v}]_{s,\mathfrak{p}(\mathrm{x},\mathrm{z})} + |\mathrm{v}|_{\mathfrak{q}(\mathrm{x})},$$
where the term $[\mathrm{v}]_{s,\mathfrak{p}(\mathrm{x},\mathrm{z})}$ defined by
$$[\mathrm{v}]_{s,\mathfrak{p}(\mathrm{x},\mathrm{z})} = \inf\{\mu > 0, \int_{U\times U} \frac{1}{\mu^{\mathfrak{p}(\mathrm{x},\mathrm{z})}}|\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}\mathrm{K}(\mathrm{x}, \mathrm{z}) d\mathrm{x} d\mathrm{z} < 1\},$$
is the so-called Gagliardo semi-norm of v. For the sake of simplicity, we note $X = W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U)$ and $\mathrm{K}(\mathrm{x}, \mathrm{z}) = \frac{1}{|\mathrm{x}-\mathrm{z}|^{N+s\mathfrak{p}(\mathrm{x},\mathrm{z})}}$.

Lemma 2.4 *[5] Let $U \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $s \in (0,1)$. Let $(\mathfrak{q},\mathfrak{p}) \in C^+(\bar{U}) \times C^+(\bar{U} \times \bar{U})$. We assume $\mathfrak{p}$ is symmetric, then the space X is a separable Banach space, reflexive space.*

Theorem 1 *[5] Let $U \subset \mathbb{R}^N$ be a Lipschitz bounded domain, $s \in (0,1)$, $\mathfrak{p} \in C(U \times U, (1,+\infty))$ and $(\mathfrak{q}, r) \in C(U, (1,+\infty))$ with $s\mathfrak{p}(\mathrm{x},\mathrm{z}) < N, \mathfrak{q}(\mathrm{x}) > \mathfrak{p}(\mathrm{x},\mathrm{x})$, and*

$$\mathfrak{p}^*(\mathrm{x}) = \frac{N\mathfrak{p}(\mathrm{x},\mathrm{x})}{N - s\mathfrak{p}(\mathrm{x},\mathrm{x})} > r(\mathrm{x}) \geq r_- > 1, \text{ for all } x \in U,$$

then $W^{s,\mathfrak{q}(\mathrm{x}),\mathfrak{p}(\mathrm{x},\mathrm{z})}(U) \hookrightarrow L^{r(\mathrm{x})}(U)$ is a compact embedding for any $r(\mathrm{x}) \in (1, \mathfrak{p}^*(\mathrm{x}))$.

Lemma 2.5 *[21] We define the modular $\phi : X \to \mathbb{R}$ by*

$$\phi(\mathrm{v}) = \int_U \int_U |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x} d\mathrm{z} + \int_U |v(\mathrm{x})|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x},$$

then we have

1) $\phi(\mathrm{v}) > 1 (= 1; < 1) \Leftrightarrow \|\mathrm{v}\|_X > 1 (= 1; < 1)$.
2) *If* $\|\mathrm{v}\|_X > 1$, *then* $\|\mathrm{v}\|_X^{\mathfrak{p}^-} \leq \phi(\mathrm{v}) \leq \|\mathrm{v}\|_X^{\mathfrak{p}^+}$.
3) *If* $\|\mathrm{v}\|_X < 1$, *then* $\|\mathrm{v}\|_X^{\mathfrak{p}^+} \leq \phi(\mathrm{v}) \leq \|\mathrm{v}\|_X^{\mathfrak{p}^-}$.

Lemma 2.6 *(Simon inequality, [38]) For every $t, a \in \mathbb{R}^N$, the following holds:*

$$\begin{cases} |t-a|^n \leq c_n \left(|t|^{n-2}t - |a|^{n-2}a\right) \cdot (t-a), & n \geq 2 \\ |t-a|^n \leq C_n \left[\left(|t|^{n-2}t - |a|^{n-2}a\right) \cdot (t-a)\right]^{\frac{n}{2}} \left(|t|^n + |a|^n\right)^{\frac{2-n}{2}}, & 1 < n < 2 \end{cases}$$

where $c_n = (\frac{1}{2})^{-n}$ and $C_n = \frac{1}{n-1}$.

2.2. Classes of mappings and topological degree

Let E be a real, separable Banach space, and E^* its dual. Given $U \subset E$ a nonempty set, denoted by $\bar{U}$ and ∂U the closure and the boundary of U, respectively.

Definition 1 *Let $f : U \subset E \to E^*$ be an operator and $\{z_n\}_{n\in\mathbb{N}} \subset U$.*

1) *We say that f is an (S_+)-map if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have*

$$z_n \overset{weakly}{\rightharpoonup} z \text{ in } E \quad \text{and} \quad \limsup_{n\to\infty} \langle f z_n, z_n - z\rangle \leq 0 \quad \Rightarrow \quad z_n \to z.$$

2) *We say that f is a quasi-monotone operator if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have*

$$z_n \overset{weakly}{\rightharpoonup} z \text{ in } E \Rightarrow \limsup_{n\to\infty} \langle f z_n, z_n - z\rangle \geq 0.$$

Definition 2 *Let $U_1 \subset E$ such that $U \subset U_1$, $\mathfrak{B} : U_1 \to E^*$ a bounded operator, and $f : U \subset E \to E$ an operator.*

1) *We say that f satisfies condition $(S_+)_{\mathfrak{B}}$ if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have*

$$\begin{cases} z_n \overset{weakly}{\rightharpoonup} z \text{ in } E \\ a_n := \mathfrak{B}(z_n) \overset{weakly}{\rightharpoonup} a \\ \limsup\limits_{n\to+\infty} \langle f z_n, a_n - a\rangle \geq 0 \end{cases} \tag{2.1}$$

then $z_n \to z$.

2) (Property $(QM)_{\mathfrak{B}}$). We say that f satisfies condition $(QM)_{\mathfrak{B}}$ if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{weakly}{\rightharpoonup} z \text{ and } a_n := \mathfrak{B}z_n \overset{weakly}{\rightharpoonup} a \Rightarrow \limsup_{n\to\infty} \langle fz_n, a - a_n\rangle \geq 0\cdot$$

We consider the following set:

$$\mathcal{F}_1^{\star}(U){:=}\{g : U \subset E \to E^* \text{ such that g is a bounded, demicontinuous, and satisfies condition } (S_+)\}\,.$$

$$\mathcal{F}_{\mathfrak{B},1}(U) := \{g : U \subset E \to E \mid g \text{ is bounded, demicontinuous, and satisfies condition } (S_+)_{\mathfrak{B}}\}\,.$$

$$\mathcal{F}_{\mathfrak{B}}(U) := \{f : U \subset E \to E \mid f \text{ is demicontinuous and satisfies condition } (S_+)_{\mathfrak{B}}\}\,.$$

Let $U \subset D_f$ and $\mathfrak{B} \in \mathcal{F}_1^{\star}(U)$, D_f denotes f's domain.
The following operators are considered:

$$\begin{aligned}
&\mathcal{F}_{S_+}(E) := \{f \in \mathcal{F}_1^{\star}(\omega) \mid \omega \in \mathcal{N}\}\,,\\
&\mathcal{F}_B(E) := \{f \in \mathcal{F}_{\mathfrak{B},1}(\overline{\omega}) \mid \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_1^{\star}(\overline{\omega})\}\,,\\
&\mathcal{F}(E) := \{f \in \mathcal{F}_{\mathfrak{B}}(\overline{\omega}) \mid \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_1^{\star}(\overline{\omega})\}
\end{aligned}$$

where $\mathcal{N}$ denotes the collection of all bounded open sets in E. Here, $\mathfrak{B} \in \mathcal{F}_1^{\star}(\bar{\omega})$ is called an essential inner map to f.

Lemma 2.7 *[24] Let ω be a bounded open set in uniformaly convex Banach space E, $\mathfrak{B} : \overline{\omega} \to E^*$ be a bounded operator, and $f{:}\overline{\omega} \to E$. Then, we have that:*

1) If f is locally bounded and verify condition $(S_+)_{\mathfrak{B}}$ and $\mathfrak{B}$ is continuous, then f has the property $(QM)_{\mathfrak{B}}$.

2) The operator f has the property $(QM)_{\mathfrak{B}}$, if $z_n \overset{weakly}{\rightharpoonup} z$ and $a_n := \mathfrak{B}z_n \overset{weakly}{\rightharpoonup} a$, then

$$\liminf \langle fz_n, a_n - a\rangle \geq 0 \text{ for any } \{z_n\}_{n\in\mathbb{N}} \subset U.$$

3) If $f_1, f_2 : \overline{\omega} \to E$ be two operators satisfy $(QM)_{\mathfrak{B}}$ condition, then $f_1 + f_2$ and αf_1 satisfy also $(QM)_{\mathfrak{B}}$ condition, for all α is a positive number.

4) Let $f : \overline{\omega} \to E$ be an operator of the type $(S_+)_{\mathfrak{B}}$ and $s : \overline{\omega} \to E$ be an ather operator has the property $(QM)_{\mathfrak{B}}$, then $f + s$ satisfies condition $(S_+)_{\mathfrak{B}}$.

Lemma 2.8 *[6] Let B be a bounede open set in E. Let $\mathfrak{B} \in \mathcal{F}_1^{\star}(\overline{B})$ be continuous and $g : D_g \subset E^* \to E$ an demicontinuous operator such that $\mathfrak{B}(\bar{B}) \subset D_g$. Then, we have that*

a) If g is quasimonotone operator , then $i + go\mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\bar{B})$, with i denotes the identity operator.

b) If g is an operator of type (S_+), then $go\mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\overline{B})$.

Definition 3 *Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_1^{\star}(\overline{B})$ be continuous, and $\mathrm{f}, \mathrm{g} \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. The map $\mathrm{H} : [0,1] \times \overline{E} \to E$ given by*

$$\mathrm{H}(s, \mathrm{v}) := (1 - s)\mathrm{fv} + s\mathrm{gv} \quad \text{for every} \quad (s, \mathrm{v}) \in [0,1] \times \overline{B}$$

is called an admissible affine homotopy.

Lemma 2.9 *[6] Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_1^{\star}(\overline{B})$ be continuous and $\mathrm{f}, \mathrm{g} \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. The homotopy $\mathrm{H}(a, .)$ satisfies the condition $(S_+)_{\overline{B}}$.*

Theorem 2 *[6] There exists a unique degree function*

$$d:\left\{(f,F,a) \text{ such that } F\in\mathcal{N},\mathfrak{B}\in\mathcal{F}_1^{\star}(\overline{B}), f\in\mathcal{F}_{\mathfrak{B},1}(\overline{F}), a\notin f(\partial F)\right\}\to\mathbb{Z}$$

that satisfies the following properties:

1) *Let $a\in F$, then $d(i,F,a)=1$.*
2) *If $G:[0,1]\times\overline{B}\to F$ is a bounded admissible affine homotopy with a common continuous essential inner map and $b:[0,1]\to F$ is a continuous mapping in E. Then, $d(G(x,\cdot),F,b(x))$ is constant for all $x\in[0,1]$ and $b(x)\notin G(t,\partial F)$.*
3) *Let F_1 and F_2 be two disjointed open subsets of F with $a\notin f\left(\overline{F}\backslash(F_1\cup F_2)\right)$. So,*

$$d(f,F,a)=d\left(f,F_1,a\right)+d\left(f,F_2,a\right).$$

4) *The equation $f\mathrm{v}=a$ has a solution in F if $d(f,F,a)\neq 0$.*

3. Main result

In this part, we give an existence result to the following problem:

$$\begin{cases} (\mathcal{L}_{\mathrm{K}}^{s})_{\mathfrak{p}(\mathrm{x},.)}\mathrm{v}(\mathrm{x}) &= \lambda\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x})+f(\mathrm{x},\mathrm{v}(\mathrm{x})) \quad \text{in} \quad U,\\ \mathrm{v} &= 0 \quad \text{in} \quad \mathbb{R}^N\backslash U, \end{cases} \tag{3.1}$$

by using the topological degree methods.

Definition 4 *Let $\mathrm{v}\in\mathcal{M}(U)$. We say that $\mathrm{v}\in X$ is a weak solution of problem (3.1) if for all $h\in X^*$ we have,*

$$\begin{aligned}&\int_{U\times U}|\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z}))(h(\mathrm{x})-h(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})d\mathrm{x}d\mathrm{z}\\ &=\lambda\int_U\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x})h(\mathrm{x})d\mathrm{x}+\int_U f(\mathrm{x},\mathrm{v}(\mathrm{x}))h(\mathrm{x})d\mathrm{x}.\end{aligned}$$

We consider the functional $\zeta:X\to\mathbb{R}$ defined by

$$\begin{aligned}\zeta(\mathrm{v})=&\int_{U\times U}\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}|\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}\mathrm{K}(\mathrm{x},\mathrm{z})d\mathrm{x}d\mathrm{z}\\ &-\lambda\int_U\frac{1}{r(\mathrm{x})}\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x}-\int_U F(\mathrm{x},\mathrm{v}(\mathrm{x}))d\mathrm{x}.\end{aligned}$$

Then from [5,23] it follows that $\zeta\in C^1(X,\mathbb{R})$, and

$$\begin{aligned}\langle\zeta^{'}(\mathrm{v}),h\rangle=&\int_{U\times U}|\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{z}))(h(\mathrm{x})-h(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})d\mathrm{x}d\mathrm{z}\\ &-\lambda\int_U\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x})h(\mathrm{x})d\mathrm{x}-\int_U f(\mathrm{x},\mathrm{v}(\mathrm{x}))hd\mathrm{x}\\ &:=<L(\mathrm{v}),h>-<S_1(\mathrm{v}),h>-<S_2(\mathrm{v}),h>.\end{aligned}$$

Lemma 3.1 *Under assumption $(\mathcal{H}_1)$ the operator $L:X\to X^*$ is continuous, bounded, and strictly monotone operator,*

i) *L is an operator of type (S_+),*
ii) *L is a mapping homeomorphism.*

Proof: It's obvious that L is bounded. We show that L is continuous. Assume that $\mathrm{v}_n \to \mathrm{v}$ in X, and we show that $L(\mathrm{v}_n) \to L(\mathrm{v})$ in X^*. Indeed,

$$
\begin{aligned}
&< L\mathrm{v}_n - L\mathrm{v}, \varphi >= \\
&\int_{U\times U} [(|\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})) - |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z}))] \\
&\times \mathrm{K}(\mathrm{x},\mathrm{z})(\varphi(\mathrm{x}) - \varphi(\mathrm{z}))d\mathrm{x}d\mathrm{z} \\
&= \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{\mathfrak{p}(\mathrm{x},\mathrm{z})-1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \\
&- |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{\mathfrak{p}(\mathrm{x},\mathrm{z})-1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}}] \\
&\times (\varphi(\mathrm{x}) - \varphi(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} d\mathrm{x}d\mathrm{z}.
\end{aligned}
$$

Let

$$G_n(\mathrm{x},\mathrm{z}) = |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{\mathfrak{p}(\mathrm{x},\mathrm{z})-1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \in L^{\mathfrak{p}'(\mathrm{x},\mathrm{z})}(U\times U),$$

$$G(\mathrm{x},\mathrm{z}) = |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2}(\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{\mathfrak{p}(\mathrm{x},\mathrm{z})-1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \in L^{\mathfrak{p}'(\mathrm{x},\mathrm{z})}(U\times U),$$

and

$$F(\mathrm{x},\mathrm{z}) = (\varphi(\mathrm{x}) - \varphi(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \in L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U),$$

where $\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})} + \frac{1}{\mathfrak{p}'(\mathrm{x},\mathrm{z})} = 1$. Thanks to Hölder's inequality, we have

$$< L(\mathrm{v}_n) - L(\mathrm{v}), \varphi >\leq 2|G_n(\mathrm{x},\mathrm{z}) - G(\mathrm{x},\mathrm{z})|_{L^{\mathfrak{p}'(\mathrm{x},\mathrm{z})}(U\times U)}|F|_{L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U)}.$$

Thus

$$\|L(\mathrm{v}_n) - L(\mathrm{v})\|_{X^*} \leq 2|G_n(\mathrm{x},\mathrm{z}) - G(\mathrm{x},\mathrm{z})|_{L^{\mathfrak{p}'(\mathrm{x},\mathrm{z})}(U\times U)}.$$

Let $V_n(\mathrm{x},\mathrm{z}) = (\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \in L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U)$, and $V(\mathrm{x},\mathrm{z}) = (\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z})^{\frac{1}{\mathfrak{p}(\mathrm{x},\mathrm{z})}} \in L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U)$. Since $\mathrm{v}_n \to \mathrm{v}$ in X, we have $V_n \to V$ in $L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U)$. So, there exists a subsequence of $\{V_n\}_{n\in\mathbb{N}}$ and $h(\mathrm{x},\mathrm{z}) \in L^{\mathfrak{p}(\mathrm{x},\mathrm{z})}(U\times U)$ such that $V_n \to V$ a.e in $U\times U$ and $|V_n| \leq h(\mathrm{x},\mathrm{z})$. So, we have $G_n \to G$ a.e in $U\times U$ and $|G_n(\mathrm{x},\mathrm{z})| = |V_n(\mathrm{x},\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-1} \leq h(\mathrm{x},\mathrm{z})^{\mathfrak{p}(\mathrm{x},\mathrm{z})-1}$. We use the dominated converge theorem, we have

$$G_n \to G \text{ in } L^{\mathfrak{p}'(\mathrm{x},\mathrm{z})}(U\times U).$$

We use to lemma2.6, we get L is an operator strictly monotonne. Now, we show that L is mapping of type (S_+). Let $\{\mathrm{v}_n\}_{n\in\mathbb{N}} \subset X$ be a sequence with $\mathrm{v}_n \rightharpoonup \mathrm{v}$ in X and $\limsup_{n\to+\infty} \langle L(\mathrm{v}_n) - L(\mathrm{v}), \mathrm{v}_n - \mathrm{v}\rangle \leq 0$.

We use (i), we have

$$0 = \lim_{n\to+\infty} \langle L\mathrm{v}_n - L\mathrm{v}, \mathrm{v}_n - \mathrm{v}\rangle \tag{3.2}$$

By Theorem 1, we have that

$$\mathrm{v}_n(\mathrm{x}) \to \mathrm{v}(\mathrm{x}) \text{ a.e in } U.$$

This, in combination with Fatou's lemma, gives us

$$\liminf_{n\to+\infty} \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}\mathrm{K}(\mathrm{x},\mathrm{z})d\mathrm{x}d\mathrm{z} \geq \int_{U\times U} |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})}\mathrm{K}(\mathrm{x},\mathrm{z})d\mathrm{x}d\mathrm{z}. \tag{3.3}$$

From the other hand, we've got

$$\lim_{n\to+\infty} \langle L(\mathrm{v}_n), \mathrm{v}_n - \mathrm{v}\rangle = \lim_{n\to+\infty} \langle L(\mathrm{v}_n) - L(\mathrm{v}), \mathrm{v}_n - \mathrm{v}\rangle = 0. \tag{3.4}$$

Using Young's inequality, we can see there is a positive constant c such that.

$$\begin{aligned}\langle L(\mathrm{v}_n), \mathrm{v}_n - \mathrm{v}\rangle &= \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz \\ &- \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-2} (\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z}))(\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z}))\mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz \\ &\geq \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz \\ &- \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})-1} |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})| \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz \\ &\geq c\int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz \\ &- c\int_{U\times U} |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz.\end{aligned}$$

We combine with (3.2) - (3.4) and inequality above, we have

$$\lim_{n\to+\infty} \int_{U\times U} |\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz = \int_{U\times U} |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x}dz. \tag{3.5}$$

Now from (3.2), (3.5), and the Brezis-Lieb lemma [14], our result is proved.
We show that L is a heomeomorphism. It is easy to see that L is coercive and injective. Thanks to Minty-Browder's Theorem [39], L is a surjection. So, L is a bijection. There exists $G : X^* \to X.$ such that $G \circ L = id_X$ and $L \circ G = id_{X*}$. We show that G is continuous. Let $g_n, g \in X$ such that $g_n \to g$ in X. Let $t_n = G(g_n), v = G(g)$, then $L(v_n) = g_n$ and $L(v) = g$. Since $\{t_n\}_{n\in\mathbb{N}}$ is bounded in X, we have $t_n \rightharpoonup v$ in X. It follows that

$$\lim_{n\to+\infty} < L(t_n) - L(v), t_n - v >= \lim_{n\to+\infty} < g_n, t_n - v >= 0.$$

We use (i) in Lemma 3.1, we have that $t_n \to v$ in X. This achevied our proof. □

Lemma 3.2 *Assume that hypothesis $(\mathcal{H}_1)$ is fulfilled. Then the operator $S : X \to X^*$ setting by*

$$< S\mathrm{v}(\mathrm{x}), w(\mathrm{x}) >= -\lambda \int_U \beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x})w(\mathrm{x})d\mathrm{x} - \int_U f(\mathrm{x}, \mathrm{v}(\mathrm{x}))w(\mathrm{x})d\mathrm{x}$$

is compact, for all $w \in X^*$.

Proof: Let

$$\begin{aligned} S_1 : X &\to L^{q'(\mathrm{x})}(U) \\ \mathrm{v} &\longmapsto S_1\mathrm{v} = -\lambda\beta(\mathrm{x})|\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{v}(\mathrm{x}) \end{aligned} \qquad \begin{aligned} S_2 : X &\to L^{q'(\mathrm{x})}(U) \\ \mathrm{v} &\longmapsto S_2\mathrm{v} = -f(\mathrm{x}, \mathrm{v}(\mathrm{x})). \end{aligned}$$

We will show that S_1 and S_2 are boundes and continuous.
• For every $\mathrm{v} \in X$,

$$\begin{aligned} |S_1\mathrm{v}|_{\mathfrak{q}'(\mathrm{x})} &= \lambda \int_U |\beta(\mathrm{x})|\mathrm{v}|^{r(\mathrm{x})-2}\mathrm{v}|^{\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\ &\leq \lambda\|\beta\|_\infty \int_U |\mathrm{v}|^{(r(\mathrm{x})-1)\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\ &\leq \lambda C\|\beta\|_\infty \int_U |\mathrm{v}|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x}. \end{aligned}$$

This implies S_1 is bounded in X.
• By condition $(\mathcal{H}_1)$, there exists $\alpha > 0$ such that

$$|f(\mathrm{x}, \mathrm{v}(\mathrm{x}))| \leq \alpha(1 + |\mathrm{v}(\mathrm{x})|^{\mathfrak{q}(\mathrm{x})-1}).$$

So,

$$\begin{aligned}
|S_2\mathrm{v}|_{\mathfrak{q}'(\mathrm{x})}^{\mathfrak{q}'(\mathrm{x})} &= \int_U |f(\mathrm{x}, \mathrm{v}(\mathrm{x})|^{\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\
&\leq \int_U \alpha(1 + |\mathrm{v}(\mathrm{x})|^{\mathfrak{q}(\mathrm{x})-1})^{\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\
&\leq 2^{\mathfrak{q}'^{+}}(|U| + \int_U |\mathrm{v}(\mathrm{x})|^{(\mathfrak{q}(\mathrm{x})-1)\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\
&\leq \alpha c'(U, \mathfrak{q}(\mathrm{x})) \int_U |\mathrm{v}(\mathrm{x})|^{(\mathfrak{q}(\mathrm{x})-1)\mathfrak{q}'(\mathrm{x})} d\mathrm{x} \\
&\leq \alpha c'(U, \mathfrak{q}(\mathrm{x})) |\mathrm{v}(\mathrm{x})|_{L^{\mathfrak{q}'(\mathrm{x})}},
\end{aligned}$$

So, S_2 is bounded in X. Now, we show S_2 is continuous. Let $\mathrm{v}_n \in X$ such that $\mathrm{v}_n \to \mathrm{v}$ in X. Then, we have $\mathrm{v}_n \to \mathrm{v}$ in $L^{\mathfrak{q}(\mathrm{x})}(U)$. Hence, there exist a subsequence still denote by v_n and measurable function g in $L^{\mathfrak{q}(\mathrm{x})}(U)$ such that $\mathrm{v}_n(\mathrm{x}) \to \mathrm{v}$ and $|\mathrm{v}_n(\mathrm{x})| \leq g(\mathrm{x})$ a.e in U. Since f is the Carathéodory condition, we have

$$f(\mathrm{x}, \mathrm{v}_n) \to f(\mathrm{x}, \mathrm{v}) \text{ a.e in } U. \tag{3.6}$$

According to condition $(\mathcal{H}_1)$, we have that

$$|f(\mathrm{x}, \mathrm{v}_n(\mathrm{x})| \leq \alpha(1 + g(\mathrm{x}))^{\mathfrak{q}(\mathrm{x})-1} \in L^{\mathfrak{q}'(\mathrm{x})}(U).$$

We using (3.6), we have that

$$\int_U |f(\mathrm{x}, \mathrm{v}_n(\mathrm{x})) - f(\mathrm{x}, \mathrm{v}(\mathrm{x}))|^{q'(\mathrm{x})} d\mathrm{x} \to 0 \qquad \text{as } n \to \infty.$$

The dominated converge theorem implies that

$$S_2\mathrm{v}_n \to S_2\mathrm{v} \text{ in } L^{q'(\mathrm{x})}(U).$$

Finally, S_2 is continue in X.
• Because $i : X \to L^{\mathfrak{q}(\mathrm{x})}(U)$ is a compact embedding, the adjoint operator $i^* : L^{\mathfrak{q}'(\mathrm{x})}(U) \to X^*$ is also compact, as is well known. As a result, the $i^* \circ S_2$ and $S_2 \circ i^*$ compositions are compact. We come to the conclusion that the operator S is compact. This brings our proof to a close. □

Now, we state our existence theorem.

Theorem 3 *Under assumption $(\mathcal{H}_1)$ problem (1.10) has a weak solution* v *in X.*

Proof: Let $\mathrm{v} \in X$, it's obvious to say v is a weak solution to the (1.10) if and only if

$$L\mathrm{v} + S\mathrm{v} = 0, \tag{3.7}$$

where L, S are the operators defined in Lemmas 3.1 - 3.2. Since S is bounded continuous and quasi-monotone (see Lemma 3.1) and L is strictly monotone, thanks to Minty-Browder's Theorem (see [39], Theorem 26 A,) we have that $L^{-1} = G$ is bounded continuous of type (S_+).
So, equation (3.7) is equivalent to

$$\mathrm{v} = Gh \text{ and } h + S \circ Gh = 0. \tag{3.8}$$

For solve equation (3.8), we will use the Berkovits topological degree introduced in section 2. To do this, we first show that the see

$$D := \{h \in E^* \text{ such that } h + tS \circ Gh = 0 \text{ for some } t \in [0,1]\},$$

is bounded. Let $h \in D$ and take $\mathrm{v} := Gh$. We use the growth condition, Hölder's inequality, Young's inequality and continuous embedding $L^{\mathfrak{q}(\mathrm{x})}(U) \hookrightarrow X$, we get

$$\begin{aligned}
\|Gh\|_X &\leq \int_{U\times U} |\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{z})|^{\mathfrak{p}(\mathrm{x},\mathrm{z})} \mathrm{K}(\mathrm{x},\mathrm{z}) d\mathrm{x} dz \\
&= < L\mathrm{v}, h > \\
&= < h, Gh > \\
&\leq |t| < S \circ Gh, Gh > \\
&\leq \lambda \int_U \beta(\mathrm{x}) |\mathrm{v}(\mathrm{x})|^{r(\mathrm{x})} d\mathrm{x} + \int_U f(\mathrm{x},\mathrm{v})\mathrm{v} d\mathrm{x} \\
&\leq \lambda \|\beta\|_\infty C_1 \|\mathrm{v}\|_X^{r^+} + C_2 (\int_U |f(\mathrm{x},\mathrm{v})|^{\mathfrak{q}'(\mathrm{x})} d\mathrm{x})^{\frac{1}{\mathfrak{q}'(\mathrm{x})}} + C_3 (\int_U |\mathrm{v}|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x})^{\frac{1}{\mathfrak{q}(\mathrm{x})}} \\
&\leq \lambda \|\beta\|_\infty C_1 \|\mathrm{v}\|_X^{r^+} + C_2 \beta \int_U ((1 + |\mathrm{v}(\mathrm{x})|^{(\mathfrak{q}(\mathrm{x})-1)\mathfrak{q}'(\mathrm{x})}) d\mathrm{x})^{\frac{1}{\mathfrak{q}'(\mathrm{x})}} \\
&+ C_3 (\int_U |\mathrm{v}|^{\mathfrak{q}(\mathrm{x})} d\mathrm{x})^{\frac{1}{\mathfrak{q}(\mathrm{x})}} \\
&\leq \lambda \|\beta\|_\infty C_1 \|\mathrm{v}\|_X^{r^+} + 2^{\mathfrak{q}^+} C_4 \|\beta\| \|\mathrm{v}\|_X + C_4 \|\mathrm{v}\|_X,
\end{aligned}$$

where $r^+ = \sup_{\mathrm{x}\in U} r(\mathrm{x})$. Because the operator S is bounded, it follows the set D is bounded in X^*. As a result, there exists a positive constant η such that

$$\|h\|_{X^*} < \eta \text{ for all } h \in D.$$

As a result, $h + xS \circ Gh \neq 0$ for all $(h, x) \in \partial B_\eta(0) \times [0,1]$. We use Lemma 2.7, and $i + S \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ and $i = L \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ are present. $i + S \circ G$ is also bounded because the operator i, S and G are all bounded. We come to the conclusion that

$$i + S \circ G \in \mathcal{F}_{\mathfrak{B},B}(\overline{B_\eta(0)}) \text{ and } i \in \mathcal{F}_{\mathfrak{B},B}(\overline{B_\eta(0)}).$$

We consider the map $\mathrm{H} : [0,1] \times \overline{B_\eta(0)} \to X^*$ given by

$$H(x, v) := v + xS \circ Lv.$$

By the statement (1)-(2) in Theorem 2, we can deduce

$$d(i + S \circ G, B_\eta(0), 0) = d(i, B_\eta(0), 0) = 1,$$

by applying the homotopy invariance and normalization properties of the degree d stated in Theorem 2. Then, there exists $v \in B_\eta(0)$ such that

$$h + S \circ Gh = 0.$$

We deduce $\mathrm{v} = Gh$ is a weak solution to the problem (1.10) in X. This completes the proof. □

To create the Acknowledgments

Acknowledgments

We thank the referee by your suggestions.

References

1. Aberqi, A., Ouaziz, A., Morse's theory and local linking for a fractional $(p_1(x,.), p_2(x,.))$: Laplacian problems on compact manifolds. J. Pseudo-Differ. Oper. Appl. 14, 41 (2023).
2. Aberqi, A., Ouaziz, A.: Blow-up and global existence of solutions for a new class of parabolic $p(x,.)-$ Kirchhoff equation involving non linearity logarithmic. Journal of Pseudo-Differential Operators and Applications 16 (13), (2025)
3. Applebaum, D., Lévy Processes and Stochastic Calculus, 2nd ed., Cambridge Stud. Adv. Math. 116, Cambridge University Press, Cambridge, (2009).
4. Bahrouni, A., Comparison and sub-supersolution principles for the fractional $p(x)$-Laplacian. J. Math. Anal. Appl. 458 (2018),1363–1372.
5. Bahrouni, A., Rădulescu, V., On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent. Discrete Contin. Dyn. Syst., Ser. S 11 (2018), 379–389.
6. Berkovits, J., Extension of the Leray Schauder degree for abstract Hammerstein type mappings. Journal of Differential Equations, vol. 234(2007), 289-310.
7. Berkovits, J., On degree theory for mappings of monotone, Ann. Acad. Sci. Fenn. Ser. A1 Diss. (1986), 58.
8. Berkovits, J. Topological degree theory and multiplication theorem for a class of nonlinear mappings, Bull. London Math. Soc. , 23 (1991), 596-606.
9. Berkovits, J., On the Leray-Schauder formula and bifurcation, J. Differential Equation, 173 (2001), 451-469.
10. Berkovits, J., Fabry, C., An extension of the topological degree theory in Hilbert space, Abstr. Appl. Anal. 6 (2005)
11. Berkovits, J., Fabry, C., Semilineair problems with a non-symmetric lineair part having an infinite -dimensional kernel, Port. Math. 61(2004), 439-460.
12. Bertoin, J., Lévy Processes, Cambridge Tracts in Math. 121, Cambridge University Press, Cambridge, 1996
13. Bogdan, K., Sztonyk, P., Harnack's inequality for stable L evy processes, Potential Anal. 22(2005), 133–150.
14. Brezis, H., Lieb, E.H., A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc., 88 (1983), 486-490.
15. Brouwer, L.E.J., Über Abbildungen von Mannigfaltigkeiten. Math. Ann. 71(1912), 97-115.
16. Browder, F.E., Fixed point theory and nonlinear problems. Bull. Amer. Math. Soc. (N.S.), 9(1983), 1-39.
17. Caffarelli, L., Nonlocal equations, drifts and games. Nonlinear Partial Differential Equations. Abel Symposia, vol. 7(2012), 37–52.
18. Cont, R., Tankov, P., Financial Modelling with Jump Processes, Chapman Hall/CRC, Boca Raton, (2004).
19. Choi, E.B., Kim, J. M., Kim, Y.H., Infinitely many solutions for nonlinear elliptic equations of $p(x)$-Laplace type without the Ambrosetti and Rabinowitz condition. Proc. R. Soc. Edinb. 148(2018), 1–31.
20. Drabek, P., Kufner, A., Mustonen, V., Pseudo-monotonicity and degenerate or singular elliptic operators. Bull. Austral. Math. Soc. 58(1998), 213-221.
21. Fan, X.L., Zhao, D., On the spaces $L^{p(x)}(U)$ and $W^{k,p(x)}(U)$. J. Math. Anal. Appl. 263(2001), 424–446.
22. Guo, L., *Fractional p-Laplacian equations on Riemannian manifolds*, Electronic Journal of Differential Equations, 156(2018), 1-–17.
23. Kaufmann, U., Rossi, J.D., Vidal, R., Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians. Electron. J. Qual. Theory Differ. Equ. (2017), 76.
24. Kim, I.S., Hong, S.J., A topological degree for operators of generalized (S_+) type. Fixed Point Theory and Appl. 2015:194 (2015)
25. Leray, J., Schauder, J., Topologie et quations fonctionnelles. Ann. Sci. Ec. Norm., 51(1934), 45-78.
26. Ito, K., Lectures on Stochastic Processes, 2nd ed., Tata Inst. Fund. Res. Lecture Math. Phys. 24, Tata Institute of Fundamental Research, Bombay, (1984)
27. Majda, A., Tabak, E., A two-dimensional model for quasigeostrophic flow: Comparison with the two-dimensional Euler flow, Phys. 98(1996), 515–522.
28. Mawhin, J., Equivalence theorem for nonlinear operator equation and coincidence degree theory for some mapping in locally convex topological vector spaces 12(1972), 610-636.
29. M. Mihailescu, M., Rădulescu, V., A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. Ser. 462(2006), 2625–2641.

30. Melian, J.G., Sabina de Lis, J., On the perturbation of eigenvalues for the p-Laplacian. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, 10 (2001), 893-898.

31. Ouaziz, A. and Aberqi, A., Infinitely many solutions to a Kirchhoff-type equation involving logarithmic nonlinearity via Morse's theory. Boletín de la Sociedad Matemática Mexicana, 30(2024), p.10.

32. Ouaziz, A., Aberqi, A.: Singular fractional double-phase problems with variable exponent viaMorse's theory. Filomat 38(21), 7579–7595 (2024).

33. Ouaziz, A., Aberqi, A. and Benslimane, O., On some weighted fractional $p(\cdot,\cdot)-$ Laplacian problems. Palestine Journal of Mathematics, 12(2023), 56-66.

34. Rădulescu, V.D., Repovš, D., Partial Differential Equations with Variable Exponents, Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Taylor and Francis Group, Boca Raton (2015).

35. Servadei, R., Valdinoci, E., Mountain pass solutions for non-local elliptic operators. J. Math. Anal. Appl. 389(2012), 887–898.

36. M. Ružicka, M., Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, (2000), vol. 1748. Springer, Berlin.

37. Sabri, A., Jamea, A., Talibi, A.H., Weak Solution for Fractional $\mathfrak{p}(x)$-Laplacian Problem With Dirichlet-Type Boundary Condition, Methods of Functional Analysis and Topology 26(2020), 272-282.

38. Simon, J., Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$. Journées d'Analyse Non Linéaire (Proc. Conf., Besan¸con, 1977, pp. 205-227, Lecture Notes in Math., 665, Springer, Berlin.

39. Zeidler, E. Nonlinear Functional Analysis and its Applications. II/B : Nonlinear Monotone Operators, Springer-Verlag, New York, (1990).

Abdesslam Ouaziz,

Applied Sciences and Emergent Technologies Laboratory, National School of Applied Sciences,

Sidi Mohamed Ben Abdellah University, Morocco.

E-mail address: abdesslam.ouaziz1994@gmail.com

and

Ahmed Aberqi,

Applied Sciences and Emergent Technologies Laboratory, National School of Applied Sciences,

Sidi Mohamed Ben Abdellah University, Morocco.

E-mail address: ahmed.aberqi@usmba.ac.ma