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A primal-dual interior-point algorithm for absolute value equation based on a novel parametric kernel function

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ABSTRACT: In this work, we provide a primal-dual interior point algorithm based on a novel parametric kernel function for the absolute value equation $Ay - |y| = b$. We may describe the NP-hard absolute value equation as a monotone linear complementarity problem when the singular values of the matrix A are greater than 1, demonstrating that the previous assumption guarantees the existence and uniqueness of the solution to the monotone linear complementarity problem, which is equivalent to the existence and uniqueness of the solution to the absolute value equation. The new algorithm has $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ and $O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$ iteration for large- and small-update methods, respectively, with a given value of its parameter p. Finally, we provide some numerical results to clarify the usefulness of the suggested algorithm.

Key Words: Absolute value equation, kernel function, primal-dual interior-point algorithm, large- and small-update methods.

Contents

1. Introduction

The absolute value equation (AVE) is the process of determining a vector $y \in \mathbb{R}^n$ that satisfies a given system of equality. AVE is defined as follows

$$Ay - |y| = b \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $|y|$ denote the vector whose components are the absolute values of the components of the vector y. This type is a special form, so it can be obtained from the general form, $Ay + B|y| = b$. which was first introduced by *Rohn* in [25], then investigated in a more general context by *Mangasarian* in [18].

The fact that the general NP-hard linear complementarity problem, which includes a number of mathematical programming issues, may be restated as (1.1), further proves that (1.1) is NP-hard in its broader form (see [9]- [11]).

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Subsequently, (1.1) is considered the focus of the attention of many researchers, who are particularly interested in studying ways to solve it and, before that, whether it has a solution or not. For the existence of a solution to (1.1), we have different results about it, which are given by *Mangasarian et al.* [21] and *Rohn et al.* [26]. As for how to solve it, their research showed that there are a number of methods, such as, the reformulation of (1.1) as a standard linear complementarity problem, where *Mangasarian* [18] and *Yong* [28] showed that this is possible under a suitable assumption but *Hu et al.* [15] proved that this satisfies without any assumption, a concave minimization optimization method [19], a generalized Newton method [20], an iterative method [22], [26], a smoothing Newton method with global and quadratic convergence [8] and a smoothing-type algorithm [16].

Recently, one of the most important and widely used methods for solving the different problems in articles is the primal-dual interior-point method (IPM), since it has polynomial complexity, can solve very large problems and is of the Newton type (leading to digital efficiency). But because the purpose of this type of research is always to achieve the best-known complexity for large-update methods, the authors based their work on a new class of functions called kernel functions to obtain it. In 1997, *Roos et al.* [27] proposed the classical logarithmic functions. Then, in 2002, *Peng et al.* [23] introduced the so-called "self-regular proximity function". In 2004, *Bai et al.* [4] introduced a new class of kernel functions that has some simple conditions on the kernel function and its derivatives and is based on a proximity function. Whereas *Zhang* [29] (2012), *Achache et al.* [1] (2015), *Djeffal et al.* [12] (2018), *Bounibane et al.* [7] (2019) and *Benterki et al.* [6] (2021) have all used this last class.

Our objective is to provide a novel parametric kernel function (which combines the terms of the classical kernel functions) to solve the absolute value equation through primal-dual IPMs based on the Newton direction. The technique we employ in this research is the same as that in [4]. The existence and uniqueness of the solution to LCP, which is similar to the existence and uniqueness of the solution to (1.1), may be demonstrated by converting the absolute value equation (1.1) into a monotone linear complementarity problem. Then, in order to solve (1.1), we developed a workable interior-point algorithm using the new kernel function. Our algorithm's small- and large-update methods complexity results are shown at the end.

This article is organized as follows : The presentation of the problem after the reformulation, some basic results show the existence of a solution to (1.1) and LCP in Sect. 2. We describe the feasible interior-point algorithm for solving LCP, present the new parametric kernel function and study some of its properties in Sect. 3. A theoretical study to analyze the complexity of our algorithm in Sect. 4. We report numerical results with different values of the parameter p and evaluate the efficiency of our kernel function in Sect. 5. In Sect. 6, we give some remarks and the topics that attract our attention for future work.

Right now, we are explaining our notations. We indicate the real space with n-dimensions $\mathbb{R}^n$ and $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : u \geq 0\}$. The all-one vector of length n denotes e, and all vectors are considered to be column vectors until they are transposed to be row vectors. The matrix I stands for the identity matrix. For $u, v \in \mathbb{R}^n$, $U = diag(u_i)$ represents the diagonal matrix whose diagonal members are u_i. The Euclidean norm of a vector u is indicated by $\|u\| = \sqrt{\langle u, u\rangle}$, and we use $\lambda_{max}(\lambda_{min})$ to signify the maximal (the minimum) eigenvalue of A. The *Hadamard* product of the vectors u and v is uv and $u^T v = \langle u, v\rangle = u_1 v_1 + \cdots + u_n v_n$ is their usual inner product.

2. Preliminaires and problem setting

We will review several key concepts and lemmas in this part that we will require in the future. We take for granted that AVE meets the following requirement throughout this thesis.

Assumption 1 *The singular values of A exceed* 1. $(\lambda_{min}(A^T A) > 1)$.

Definition 2.1 A matrix $M \in \mathbb{R}^{n\times n}$ is called a positive-definite matrix if $x^T M x > 0; \quad \forall 0 \neq x \in \mathbb{R}^n$.

Proposition 2.1 *Under Assumption 1, the matrix $(A-I)^{-1}(A+I)$ is positive-definite.*

Proof: According to Assumption 1 and the eigenvalue characteristics of a matrix, the matrix $A^T A - I$ is positive definite because $\lambda_{min}(A^T A - I) \geq \lambda_{min}(A^T A) - \lambda_{max}(I) = \lambda_{min}(A^T A) - 1 > 0$.

Due to the fact that the eigenvalues of the two matrices A^TA and AA^T are equal, we get that the matrix $AA^T - I$ is also positive-definite. As a result, we find that for all nonzero $x \in \mathbb{R}^n$

$$x^T(A+I)(A^T-I)x = x^T(AA^T-I)x > 0$$

letting $z = (A^T - I)x \neq 0$, we obtain

$$x^T(A+I)(A^T-I)x = ((A^T-I)^{-1}z)^T(A+I)z = z^T(A-I)^{-1}(A+I)z$$

thus, the matrix $(A-I)^{-1}(A+I)$ is positive-definite. □

Definition 2.2 A matrix $M \in \mathbb{R}^{n\times n}$ is called $\mathscr{P}$-matrix if all its minors are positive.

Corollary 2.1 *If $M \in \mathbb{R}^{n\times n}$ is positive-definite, then M is $\mathscr{P}$-matrix.*

Definition 2.3 For any vector $y \in \mathbb{R}^n$, we can define the following two vectors y^+ and y^-

$$y_+ = \max(y, 0); \qquad y_- = \max(0, -y).$$

such that

$$\begin{cases} y = y_+ - y_-, & (y_-, y_+) \in \mathbb{R}^{2n}_+ \\ |y| = y_+ + y_- \\ {y_+}^T y_- = 0 \end{cases} \tag{2.1}$$

Definition 2.4 The linear complementarity problem (LCP) consists of finding the pair $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{cases} s = Mx + q, & (x, s) \in \mathbb{R}^{2n}_+ \\ x^T s = 0 \end{cases} \tag{2.2}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The so-called complimentary condition is the second constraint, whereas the first constraint stands for feasibility.

Proposition 2.2 *If the matrix $M \in \mathbb{R}^{n\times n}$ is positive-semidefinite (positive-definite), then the LCP is monotone (strictly monotone).*

Remark 2.1 Assumption 1 and Proposition 2.1 imply the strict monotonicity of the system (2.2).

For the unique solvability of LCPs, we cite the following theorem, which was proved by *Cottle*, *Pang* and *Stone*.

Theorem 2.1 *(Theorem 3.3.7, [11]) A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathscr{P}$-matrix if and only if the system (2.4) has a unique solution for every $q \in \mathbb{R}^n$.*

The process of converting the absolute value equation into a monotone linear complementarity problem under a reasonable assumption is described in the next part.

Proposition 2.3 *Under Assumption 1, the system (1.1) is written as (2.2).*

Proof: Using (2.1), the system (1.1) can be converted to the following system

$$\begin{cases} (A-I)y_+ - (A+I)y_- = b, & (y_-, y_+) \in \mathbb{R}^{2n}_+ \\ {y_+}^T y_- = 0 \end{cases} \tag{2.3}$$

We begin by demonstrating that the matrix $(A-I)$ is nonsingular since, if not for a nonzero vector $x \in \mathbb{R}^n$, we would have $(A-I)x = 0$. Then, $\sigma^2_{min}(A) = \lambda_{min}(A^TA) = \min_{\|x\|=1} x^TA^TAx \leq \|Ax\|^2 = \|x\|^2 = 1$, this is a contradiction with Assumption 1. Consequently, the matrix $(A-I)$ is invertible and the system (2.3) can be reduced to the following

$$\begin{cases} y_+ - (A-I)^{-1}(A+I)y_- = (A-I)^{-1}b, & (y_-, y_+) \in \mathbb{R}^{2n}_+ \\ {y_-}^T y_+ = 0 \end{cases} \tag{2.4}$$

Corresponding to (2.2), we find

$$M = (A-I)^{-1}(A+I), \quad q = (A-I)^{-1}b, \quad s = y_+ \quad and \quad x = y_-. \tag{2.5}$$

this completes the proof. □

We also give the following corollary, which illustrates how the solutions to the two systems AVE and LCP relate to one another, in addition to their equivalence.

Corollary 2.2 *The system (1.1) is uniquely solvable for any $b \in \mathbb{R}^n$ if and only if the system (2.2) is an $\mathscr{P}$-LCP.*

Proof: We suppose that the system (2.2) is an $\mathscr{P}$-LCP. Due to Theorem 2.1 for any $q = (A-I)^{-1}b \in \mathbb{R}^n$, the system (2.4) has a unique solution noted by (y_-^*, y_+^*) and satisfies the following

$$\begin{cases} y_+^* = (A-I)^{-1}(A+I)y_-^* + (A-I)^{-1}b, \quad (y_-^*, y_+^*) \in \mathbb{R}_+^{2n} \\ {y_-^*}^T y_+^* = 0 \end{cases}$$

we now simplify the feasible constraint that we have obtained

$$\begin{aligned} y_+^* = (A-I)^{-1}(A+I)y_-^* + (A-I)^{-1}b \quad &\Leftrightarrow \quad (A-I)y_+^* = (A+I)y_-^* + b \\ &\Leftrightarrow \quad (A-I)y_+^* - (A+I)y_-^* = b \\ &\Leftrightarrow \quad A(y_+^* - y_-^*) - (y_+^* + y_-^*) = b \end{aligned}$$

substituting in the above system and using (2.1), we get $Ay^* - |y^*| = b$. Hence, y^* is the unique solution of the system (1.1) for any $b \in \mathbb{R}^n$. □

Remark 2.2 From the above results, the condition "The system (2.2) is an $\mathscr{P}$-*LCP*" used in Corollary 2.2 can be reduce to "The matrix $M = (A-I)^{-1}(A+I)$ is positive-definite".

Now, we present a theorem that shows the unique solvability of (1.1).

Theorem 2.2 *If Assumption 1 is satisfied, the AVE is uniquely solvable for every $b \in \mathbb{R}^n$.*

Proof: Since Assumption 1 is satisfied, the matrix $M = (A-I)^{-1}(A+I)$ is positive-definite (see Proposition 2.1). So, using Corollary 2.1 we obtain that M is a $\mathscr{P}$-matrix. Consequently, the system (2.2) has a unique solution for each $q = (A-I)^{-1}b$ from Theorem 2.1. As a result of Corollary 2.2, we can conclude that (1.1) is uniquely solvable for every $b \in \mathbb{R}^n$. □

Remark 2.3 Under Assumption 1, the system (1.1) is uniquely solvable for any $q = (A-I)^{-1}b$.

We conclude this section with the following corollary.

Corollary 2.3 *(Corollary 2.12, [2]) Under Assumption 1, the pair of vectors (y_-^*, y_+^*) is the unique solution of (2.2) if and only if the vector $y^* = y_+^* - y_-^*$ is the unique solution of (1.1).*

Proof: Assumption 1 ensures that (2.2) is an $\mathscr{P}$-LCP based on Theorem 2.1 and Corollary 2.2. Then, due to Theorem 2.2 we get that (y_-^*, y_+^*) is the unique solution of (2.2) if and only if $y^* = y_+^* - y_-^*$ is the unique solution of (1.1). □

3. Solving AVE via the feasible interior-point algorithm

To solve the absolute value equations (1.1) using primal-dual interior-point methods, we provide a parametric kernel function in this section (IPMs). In addition to calculating the distance between the provided iteration and the center, this function is crucial in establishing the new search directions. The fundamental goal of our theoretical investigation is to create a primal-dual interior-point algorithm based on the kernel function to solve the linear complementarity problem (2.4). We started by introducing the subsequent sets

$$\mathscr{F} = \{(y_-, y_+) \in \mathbb{R}^{2n} : y_+ = (A-I)^{-1}(A+I)y_- + (A-I)^{-1}b \ ; \ y_- \geq 0 \ , \ y_+ \geq 0\},$$
$$\mathscr{F}^* = \{(y_-, y_+)) \in \mathscr{F} \ ; \ y_- > 0 \ , \ y_+ > 0\} \quad \text{and} \quad \mathscr{F}_{sol} = \{(y_-, y_+) \in \mathscr{F} : y_-^T y_+ = 0\}$$

where $\mathscr{F}$, $\mathscr{F}^*$ and $\mathscr{F}_{sol}$ are the feasible set, the strictly feasible set of (2.4) and its solution set , respectively. In addition to the following condition

Assumption 2 *The interior point condition (IPC), i.e* $\mathscr{F}^* \neq \varnothing$.

We are aware that the central path of the system (2.2) is well-defined (exists) under Assumption 2 and the hypothesis "LCP is strictly monotone". The system (2.4) is uniquely solvable for any $(A-I)^{-1}b \in \mathbb{R}^n$ according to Remark 2.1 The following system of equations must be solved in order to solve system (2.4).

$$\begin{cases} y_+ - (A-I)^{-1}(A+I)y_- = (A-I)^{-1}b, \quad (y_-, y_+) \in \mathbb{R}^{2n}_+ \\ y_- y_+ = 0 \end{cases} \tag{3.1}$$

The basic idea of the feasible interior-point methods (IPMs) is to replace the complementary condition of the system (3.1) with the centring condition $y_- y_+ = \mu e$ and we find the following system

$$\begin{cases} y_+ - (A-I)^{-1}(A+I)y_- = (A-I)^{-1}b, \quad (y_-, y_+) \in \mathbb{R}^{2n}_+ \\ y_- y_+ = \mu e \end{cases} \tag{3.2}$$

where $\mu > 0$ denotes a positive parameter and $e = (1, \ldots, 1)^T$ denotes an all-one vector. Under the two assumptions, the new system (3.2) has a unique solution denoted by $(y_-(\mu), y_+(\mu))$ for all $\mu > 0$ and is called the μ-center of the system (2.4). *Kojima et al.* [17] proved that the limit of μ-centres when μ goes to zero exists and converges to the solution of (2.4).

3.1. New search directions

The IPMs are iterative methods, therefore in order to obtain new iterations, we must locate the displacements $(\Delta y_-, \Delta y_+)$. In order to do this, we use Newton's method on the system (3.2) with fixed μ to produce the following system

$$\begin{cases} \Delta y_+ - (A-I)^{-1}(A+I)\Delta y_- = 0 \\ y_+ \Delta y_- + y_- \Delta y_+ = \mu e - y_- y_+ \end{cases} \tag{3.3}$$

the system (3.3) has a unique solution under Assumption 1. The scaling directions (d_{y_-}, d_{y_+}), as well as the two scaled vectors v and d, are defined below to make the analysis simpler

$$d_{y_-} = \frac{v\Delta y_-}{y_-}, \quad d_{y_+} = \frac{v\Delta y_+}{y_+}, \quad v = \sqrt{\frac{y_- y_+}{\mu}} \quad \textit{and} \quad d = \sqrt{\frac{y_-}{y_+}}. \tag{3.4}$$

Hence, we have

$$d_{y_-} = \frac{\Delta y_-}{d\sqrt{\mu}}, \quad d_{y_+} = \frac{d\Delta y_+}{\sqrt{\mu}}, \quad v = \frac{y_-}{d\sqrt{\mu}} = \frac{dy_+}{\sqrt{\mu}}$$

and

$$d_{y_-} d_{y_+} = \frac{\Delta y_- \Delta y_+}{\mu}, \quad d_{y_-} + d_{y_+} = \frac{y_+ \Delta y_- + y_- \Delta y_+}{\mu v}. \tag{3.5}$$

The system (3.3) may be represented as the following equivalent system by utilizing (3.4) and the second part of (3.5)

$$\begin{cases} d_{y_+} - \Lambda d_{y_-} = 0 \\ d_{y_-} + d_{y_+} = v^{-1} - v \end{cases} \tag{3.6}$$

where $\Lambda = D(A-I)^{-1}(A+I)D$ and $D = diag(d)$.

3.2. Proximity measure

Any approximation of $(y_-(\mu), y_+(\mu))$ must be monitored for the algorithm's analysis by its quality. To that aim, we provide the next logarithmic barrier function in its classical form $\Psi(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$

$$\Psi(y_-, y_+; \mu) = \Psi(v) = \sum_{i=1}^{n} \psi_l(v_i) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right) \tag{3.7}$$

The right-hand side of the second equation in (3.6) easily equals minus the derivative of $\Psi(v)$. The system (3.6) can be rewritten as follows

$$\begin{cases} d_{y_+} - \Lambda d_{y_-} = 0 \\ d_{y_-} + d_{y_+} = -\nabla\Psi(v) \end{cases} \tag{3.8}$$

Moreover, the proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is given by

$$\delta(y_-, y_+; \mu) = \delta(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\|d_{y_-} + d_{y_+}\| \tag{3.9}$$

in order to make it obvious that if $(y_-, y_+) = (y_-(\mu), y_+(\mu))$, we have $y_- y_+ = \mu e \Leftrightarrow v = e \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow \Psi(v) = 0$ if not, using (3.7) and (3.9) we get $\Psi(v) > 0$.

Remark 3.1 Our algorithm uses the barrier function $\Psi(v)$ to measure the distance between the iterates (y_-, y_+) and the μ-center $(y_-(\mu), y_+(\mu))$.

3.3. Generic interior-point algorithm for lcp

Algorithm 1 *Feasible interior-point algorithm for solving* (2.4)

Input :

An accuracy parameter $\varepsilon > 0$,a threshold parameter $\tau \geq 1$,
an update parameter $0 < \theta < 1$, a strictly feasible point (y_-^0, y_+^0)
and $\mu^0 > 0$ with $\mu^0 = \frac{(y_-^0)^T y_+^0}{n}$ such that $\Psi(v^0) \leq \tau$.

Begin :

$y_- = y_-^0$, $y_+ = y_+^0$, $\mu = \mu^0$.
While $(n\mu \geq \varepsilon)$ ***do***
Begin :
Update of μ : $\mu = (1-\theta)\mu$.
While $(\Psi(v) > \tau)$ ***do***
Begin :
Solve system (3.3) *using* (3.6) *to obtain* $(\Delta y_-, \Delta y_+)$,
determine a step size $\alpha > 0$,
$(y_-, y_+) = (y_-, y_+) + \alpha(\Delta y_-, \Delta y_+)$.
End.
End.

End.

The feasible interior-point algorithm used to solve the LCP works as follows: begins with a strictly feasible initial interior point (y_-^0, y_+^0) $\left((y_-^0, y_+^0) \in \mathscr{F}^*\right)$ in a $\tau-$neighborhood $(\Psi(v) \leq \tau)$ and set of parameters, the ones that changes their value are the parameter $\mu > 0$ and the scaled vector v, which are fixed throughout the algorithm are the accuracy parameter ε is greater than zero, the proximity parameter $\tau \geq 1$ and barrier update $\theta \in]0, 1[$ (plays an important role in the theory and practice of IPMs). We reduced μ to $(1-\theta)\mu$ and v to $\frac{v}{\sqrt{1-\theta}}$ in each outer iteration. If $\Psi(v) > 0$, we enter the inner iteration first using the system (3.3) and (3.6) to obtain the unique search directions $(\Delta y_-, \Delta y_+)$. Then, we have to determine the value for step size α, which we choose in such a way that by applying this algorithm, we get the solution with the best complexity. So the only thing left is to find the next iteration. According to our assumptions, the LCP has a unique solution, which means that we repeat the outer iteration until we find a point that satisfy $n\mu < \varepsilon$ and $\Psi(v) \leq \tau$. This point is called the optimal solution to (2.4) and from it and (2.5), we deduce the optimal solution to (1.1).

In this paper, we replace the logarithmic barrier function ψ_l defined in (3.7) by a new barrier function, whose kernel function ψ will be introduced in the next section.

3.4. The new generic kernel function

In this section, we look in depth at our kernel function by demonstrating some fundamental properties that are important in the complexity analysis. Before presenting the results, we mention the following concept.

Definition 3.1 $\psi(t) : \mathbb{R}_{++} \to \mathbb{R}_+$ is a kernel function if ψ is twice differentiable and satisfies the following conditions

Cond 1 $\psi(1) = \psi^{'}(1) = 0.$

Cond 2 $\lim_{t\to 0^+} \psi(t) = \lim_{t\to+\infty} \psi(t) = +\infty.$

Cond 3 $\psi(t)$ *is strictly convex function for all* $t > 0$ $\left(\psi^{''}(t) > 0, \quad \forall t > 0\right).$

The function that we are going to suggest is based on the kernel function provided by *Bai et al.* in [5], namely, $\psi_b(t) = \dfrac{t^2-1}{2} + \dfrac{e^{p(1-t)}-1}{p}$ for any $p \geq 1$, where $\psi_b(t)$ fails to satisfy the first property of Cond 2. We tried to alter it in order to follow the analysis employed in [4]. The new parameterized kernel function, which is neither a self-regular nor a logarithmic barrier function, shall be defined for every $t > 0$.

$$\psi(t) = \frac{t^2-1}{2} + \frac{e^{(p-1)(1-t)} + t^{-p+1} - 2}{2(p-1)}, \quad \forall p > 1. \tag{3.10}$$

the three first derivatives of the function (3.10) for all $t > 0$ are given as follows

$$\psi^{'}(t) = t - \frac{1}{2}\left(e^{(p-1)(1-t)} + \frac{1}{t^p}\right). \tag{3.11}$$

$$\psi^{''}(t) = 1 + \frac{1}{2}\left((p-1)e^{(p-1)(1-t)} + \frac{p}{t^{p+1}}\right). \tag{3.12}$$

$$\psi^{'''}(t) = -\frac{1}{2}\left((p-1)^2 e^{(p-1)(1-t)} + \frac{p(p+1)}{t^{p+2}}\right). \tag{3.13}$$

The three conditions Cond 1, Cond 2 and Cond 3 are now checked by this function, as shown by (3.10), (3.11) and (3.12). Consequently, it follows that the function described in (3.10) is a kernel function.

Remark 3.2 From Cond 3 it is clear that $\psi(t)$ is coercive and has the barrier property. Moreover, (3.12) implies that

$$\psi^{''}(t) \geq 1 > 0, \qquad \forall t > 0. \tag{3.14}$$

The next lemma serves to prove that our new kernel function (3.10) is an eligible kernel function.

Lemma 3.1 *Let* $\psi(t)$ *be as defined in* (3.10). *Then, for all* $p > 1$ *the function* $\psi(t)$ *satisfies*

(a) $t\psi^{''}(t) + \psi^{'}(t) > 0, \qquad 0 < t < 1.$

(b) $t\psi^{''}(t) - \psi^{'}(t) > 0, \qquad t > 1.$

(c) $\psi^{'''}(t) < 0, \qquad t > 0.$

(d) $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) > 0, \quad 0 < t < 1.$

Proof: These inequalities follow from some elementary calculations (3.10), (3.11), (3.12) and (3.13).

(a) $t\psi^{''}(t) + \psi^{'}(t) = 2t + \dfrac{1}{2}\left(\left((p-1)t - 1\right)e^{(p-1)(1-t)} + \dfrac{p-1}{t^p}\right) > 0, \quad \forall t \in]0,1[.$

(b) $t\psi^{''}(t) - \psi^{'}(t) = \dfrac{1}{2}\left(\left((p-1)t + 1\right)e^{(p-1)(1-t)} + \dfrac{p+1}{t^p}\right) > 0, \quad \forall t > 1.$

(c) We can conclude from the positivity of t, p and $e^{(p-1)(1-t)}$ that $\psi^{'''}(t) < 0$ for all $t > 0$.

$$(d)\quad 2\psi^{''}(t)^2-\psi^{'}(t)\psi^{'''}(t)= 2+\frac{(p-1)^2}{4}e^{2(p-1)(1-t)}+\frac{p^2-p}{4t^{2(p+1)}}+\frac{p^2+5p}{2t^{p+1}}$$
$$+e^{(p-1)(1-t)}\left(\frac{t(p-1)^2}{4}+2(p-1)-\frac{(p-1)^2}{4t^p}\right.$$
$$\left.+\frac{(p-1)p}{t^{p+1}}-\frac{p(p+1)}{4t^{p+2}}\right)>0,\quad \forall t\in]0,1[.$$

Therefore, the kernel function (3.10) is efficient. □

Remark 3.3 The following is a statement that our function (3.10) fulfills as a result of Lemma 2.4, Lemma 2.5, Lemma 2.6 in [4] and Lemma 1 in [23].

$$\psi^{''}(t)\psi^{'}(\eta t)-\eta\psi^{'}(t)\psi^{''}(\eta t)>0,\quad t>1,\quad \eta>1.$$
$$\frac{1}{2}\psi^{''}(t)(t-1)^2<\psi(t)<\frac{1}{2}\psi^{''}(1)(t-1)^2,\quad t>1. \tag{3.15}$$
$$\frac{1}{2}\psi^{''}(1)(t-1)^2<\psi(t)<\frac{1}{2}\psi^{''}(t)(t-1)^2,\quad t<1.$$
$$\psi(\sqrt{t_1t_2})\leq\frac{1}{2}\left(\psi(t_1)+\psi(t_2)\right),\quad t_1,t_2>0. \tag{3.16}$$

the last inequality means that $\psi(t)$ is exponentially convex.

Corollary 3.1 *For $p>1$, we have $\psi(t)\leq\dfrac{t^2-1}{2}$.*

Proof: For all $t\geq 1$ and $p>1$, we find $\psi(t)-\dfrac{t^2-1}{2}=\phi(t)=\dfrac{e^{(p-1)(1-t)}+t^{-p+1}-2}{2(p-1)}\leq 0$ this last inequality follows from the fact that the function $\phi(t)$ is monotonically decreasing with respect to t, so we get $\phi(t)\leq\phi(1)=0$. Then for all $t\geq 1$, $\phi(t)\leq 0$. □

Lemma 3.2 *For our function $\psi(t)$, we have*

$$\frac{1}{2}(t-1)^2\leq\psi(t)\leq\frac{1}{2}\left(\psi^{'}(t)\right)^2,\quad \textit{for all } t>0 \textit{ and } p>1. \tag{3.17}$$
$$\psi(t)\leq\frac{2p+1}{4}(t-1)^2,\quad \textit{for all } t\geq 1 \textit{ and } p>1. \tag{3.18}$$

Proof: For (3.17). Using Cond 1 and (3.14), we get $\psi(t)=\displaystyle\int_1^t\int_1^\xi\psi^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi$, for all $t>0$. Then, it is easy to verify the first equality

$$\psi(t)=\int_1^t\int_1^\xi\psi^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi\geq\int_1^t\int_1^\xi\mathrm{d}z\,\mathrm{d}\xi=\frac{1}{2}(t-1)^2,\quad\forall t>0.$$

for the second equality, we have

$$\psi(t)=\int_1^t\int_1^\xi\psi^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi\quad\leq\quad\int_1^t\int_1^\xi\psi^{''}(\xi)\psi^{''}(z)\,\mathrm{d}z\,\mathrm{d}\xi$$
$$=\quad\int_1^t\psi^{''}(\xi)\psi^{'}(\xi)\,\mathrm{d}\xi=\frac{1}{2}\left(\psi^{'}(t)\right)^2.$$

For (3.18), from the right part of the inequality (3.15) and (3.12), we obtain

$$\psi(t)\leq\frac{1}{2}\psi^{''}(1)(t-1)^2=\frac{1}{2}\left(p+\frac{1}{2}\right)(t-1)^2,\quad t\geq 1$$

this completes the proof. □

Lemma 3.3 *For all $t \geq 1$, let $\gamma(s) : [0, +\infty[\to [1, +\infty[$ be the inverse function of (3.10). Then, we have*

$$1 + \sqrt{\frac{4s}{2p+1}} \leq \gamma(s) \leq 1 + \sqrt{2s}, \quad p > 1. \tag{3.19}$$

Proof: Let $s = \psi(t), \quad t \geq 1 \Leftrightarrow \gamma(s) = t, \quad s \geq 0$. From the left part of the inequality (3.17), we have

$$s = \psi(t) \geq \frac{1}{2}(t-1)^2 \Rightarrow t = \gamma(s) \leq 1 + \sqrt{2s}$$

therefore from (3.18), we get

$$s = \psi(t) \leq \frac{2p+1}{4}(t-1)^2 \Rightarrow t = \gamma(s) \geq 1 + \sqrt{\frac{4s}{2p+1}}.$$

This completes the proof. □

Lemma 3.4 *For all $t \in]0, 1]$, let $\rho(s) : [0, +\infty[\to]0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$. Then, we have*

$$\rho(s) \geq \frac{1}{(4s+2)^{\frac{1}{p}}}, \quad p > 1. \tag{3.20}$$

Proof: Let $s = -\frac{1}{2}\psi'(t), \quad 0 < t \leq 1 \Leftrightarrow \rho(s) = t, \quad s \geq 0$. From (3.11), we get

$$\begin{aligned} s = -\frac{1}{2}\psi'(t) \quad &\Rightarrow \quad s = -\frac{1}{2}\left(t - \frac{1}{2}\left(e^{(p-1)(1-t)} + \frac{1}{t^p}\right)\right) \\ &\Rightarrow \quad 2s + t = \frac{1}{2}\left(e^{(p-1)(1-t)} + \frac{1}{t^p}\right) \\ &\Rightarrow \quad \frac{1}{t^p} \leq 4s + 2 \\ &\Rightarrow \quad t \geq \frac{1}{(4s+2)^{\frac{1}{p}}} \end{aligned} \tag{3.21}$$

where the third inequality follows from $2s + t - \dfrac{e^{(p-1)(1-t)}}{2} \leq 2s + 1$. □

The theorem that follows estimates the effect of a μ-update on the value of $\Psi(v)$.

Theorem 3.1 *(Theorem 3.2, [4]) Let $\gamma(s) : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi(t)$ for all $t \geq 1$. Then we have for any positive vector v and any $\beta \geq 1$ that $\Psi(\beta v) \leq n\psi\left(\beta\gamma\left(\dfrac{\Psi(v)}{n}\right)\right)$.*

For the growth of $\Psi(v)$ following the update of μ, we now provide two upper bounds.

Lemma 3.5 *If $\Psi(v) \leq \tau$ and $\beta = \dfrac{1}{\sqrt{1-\theta}}$. Then for all $p > 1$, $0 < \theta < 1$ and $v^+ = \dfrac{v}{\sqrt{1-\theta}}$ we have*

$$(a) \quad \Psi(v^+) \leq \frac{2p+1}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2.$$

$$(b) \quad \Psi(v^+) \leq \frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)}.$$

Proof: Due to Theorem 3.1 with $\Psi(v) \leq \tau$, $\beta = \dfrac{1}{\sqrt{1-\theta}} \geq 1$ and $\gamma\left(\dfrac{\Psi(v)}{n}\right) \geq 1$ $\left(\text{ i.e., } \beta\gamma\left(\dfrac{\Psi(v)}{n}\right) \geq 1\right)$ we can prove this lemma. For (a), we use (3.18) to obtain

$$\begin{aligned}\Psi(v^+) \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi(v)}{n}\right)\right) &\leq n\frac{2p+1}{4}\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi(v)}{n}\right)-1\right)^2\\ &\leq n\frac{2p+1}{4}\left(\frac{1+\sqrt{2\frac{\Psi(v)}{n}}}{\sqrt{1-\theta}}-1\right)^2\\ &\leq n\frac{2p+1}{4}\left(\frac{1+\sqrt{2\frac{\tau}{n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2\\ &\leq n\frac{2p+1}{4}\left(\frac{\theta+\sqrt{2\frac{\tau}{n}}}{\sqrt{1-\theta}}\right)^2=\frac{2p+1}{4(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2\end{aligned}$$

where the third inequality follows from the second part of (3.19) and the last inequality is determined by fact $1-\sqrt{1-\theta}=\dfrac{\theta}{1+\sqrt{1-\theta}}\leq\theta$. We obtain (b) From Corollary 3.1 as follows

$$\begin{aligned}\Psi(v^+) \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi(v)}{n}\right)\right) &\leq n\left(\frac{\left(\frac{1}{\sqrt{1-\theta}}\gamma\left(\frac{\Psi(v)}{n}\right)\right)^2-1}{2}\right)\\ &= \frac{n}{2}\left(\frac{1}{1-\theta}\gamma\left(\frac{\Psi(v)}{n}\right)^2-1\right)\\ &\leq \frac{n}{2(1-\theta)}\left(\gamma\left(\frac{\Psi(v)}{n}\right)^2+\theta-1\right)\\ &\leq \frac{n}{2(1-\theta)}\left(\left(1+\sqrt{2\frac{\Psi(v)}{n}}\right)^2+\theta-1\right)=\frac{2\tau+n\theta+2\sqrt{2n\tau}}{2(1-\theta)}\end{aligned}$$

which the desired result. □

Notation 1 *For all $p>1$, we denote*

$$\overline{\Psi}_0=\frac{2p+1}{4(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 \tag{3.22}$$

$$\widehat{\Psi}_0=\frac{2\tau+n\theta+2\sqrt{2n\tau}}{2(1-\theta)} \tag{3.23}$$

where $\overline{\Psi}_0$ and $\widehat{\Psi}_0$ are the upper bounds of $\Psi(v)$ for small-update and large-update, respectively.

4. Analysis of the algorithm

In this part, we study the complexity of the algorithm given in Algorithm 1. However, it must first determine the value of α and express the decrease in proximity function during an inner iteration. We note that during an inner iteration μ fixed.

4.1. Value for α

By using Algorithm 1, we get a new iteration (y_-^+, y_+^+) after each feasible step. When we denote the value of a step size as α, we have

$$\begin{aligned}y_-^+ &= y_- + \alpha\Delta y_- = y_-\left(e+\alpha\frac{\Delta y_-}{y_-}\right)=y_-\left(e+\alpha\frac{d_{y_-}}{v}\right)=\frac{y_-}{v}\left(v+\alpha d_{y_-}\right)\\ y_+^+ &= y_+ + \alpha\Delta y_+ = y_+\left(e+\alpha\frac{\Delta y_+}{y_+}\right)=y_+\left(e+\alpha\frac{d_{y_+}}{v}\right)=\frac{y_+}{v}\left(v+\alpha d_{y_+}\right)\end{aligned}$$

then we have

$$v^+ = \sqrt{\frac{y_-^+ y_+^+}{\mu}} = \sqrt{(v+\alpha d_{y_-})(v+\alpha d_{y_+})}.$$

From (3.16) and the definition of the proximity after a feasible step, it is clear that

$$\Psi(v^+) = \Psi\left(\sqrt{(v+\alpha d_{y_-})(v+\alpha d_{y_+})}\right) \leq \frac{1}{2}\left(\Psi(v+\alpha d_{y_-}) + \Psi(v+\alpha d_{y_+})\right).$$

For $\alpha > 0$ we consider the following function

$$f(\alpha) = \Psi(v^+) - \Psi(v) \quad \text{and} \quad f_1(\alpha) = \frac{1}{2}\left(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\right) - \Psi(v)$$

we can easily see that these two functions check

$$f(\alpha) \leq f_1(\alpha) \quad \text{and} \quad f(0) = f_1(0) = 0.$$

The following are the two successive derivatives of $f_1(\alpha)$ with respect to α

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{'}\left(v_i + \alpha(d_{y_-})_i\right)(d_{y_-})_i + \psi^{'}\left(v_i + \alpha(d_{y_+})_i\right)(d_{y_+})_i\right)$$

using (3.8) and (3.9), we obtain

$$f_1^{'}(0) = \frac{1}{2}\left\langle \nabla\Psi(v), (d_{y_-} + d_{y-+})\right\rangle = -\frac{1}{2}\|\nabla\Psi(v)\|^2 = -2\delta(v)^2$$

and

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{''}\left(v_i + \alpha(d_{y_-})_i\right)(d_{y_-})_i^2 + \psi^{''}\left(v_i + \alpha(d_{y_+})_i\right)(d_{y_+})_i^2\right).$$

Notation 2 *For the remainder of the paper, we will use* $v_{min} = \min_{i\in\{1,\ldots,n\}} v_i$, $\delta = \delta(v)$ *and* $\Psi = \Psi(v)$.

The following lemma establishes a relationship between δ and Ψ.

Lemma 4.1 *Let δ be defined as in (3.9). Then we have* $\delta \geq \sqrt{\frac{\Psi}{2}}$.

Proof: Due to (3.9) and (3.17), we have $\Psi = \sum_{i=1}^{n}\psi(v_i) \leq \sum_{i=1}^{n}\frac{\left(\psi^{'}(v_i)\right)^2}{2} = \frac{1}{2}\|\nabla\Psi\|^2 = 2\delta^2 \Rightarrow \sqrt{2}\delta \geq \sqrt{\Psi}$. □

Throughout the study, we will use the assumption that $\Psi \geq \tau \geq 1$ and by using Lemma 4.1, we obtain $\sqrt{2}\delta \geq \sqrt{\tau} \geq 1$. Now is the time to calculate the step size's value. For our objectives, we establish the findings listed below.

Lemma 4.2 *(Lemma 4.3, [4]) Let ρ be as defined in Lemma 3.4. Then, the largest step size α of the worst case is given by* $\overline{\alpha} := \frac{\rho(\delta) - \rho(2\delta)}{2\delta}$.

Lemma 4.3 *(Lemma 4.4, [4]) Let ρ and $\overline{\alpha}$ be as defined in Lemma 4.2. Then* $\overline{\alpha} \geq \frac{1}{\psi^{''}(\rho(2\delta))}$.

Lemma 4.4 *Let ρ and $\overline{\alpha}$ be as defined in Lemma 4.2. Then, we have* $\overline{\alpha} \geq \frac{1}{1 + p\,(8\delta+2)^{\frac{p+1}{p}}}, \forall p > 1$.

Proof: To facilitate the calculation for all $t \in]0,1]$, we put $t = \rho(2\delta)$ in (3.12). Then, we obtain

$$\psi^{''}(\rho(2\delta)) = 1 + \frac{1}{2}\left((p-1)e^{(p-1)(1-\rho(2\delta))} + \frac{p}{(\rho(2\delta))^{p+1}}\right) \leq 1 + \frac{p}{2}\left(e^{(p-1)(1-\rho(2\delta))} + \frac{1}{(\rho(2\delta))^{p+1}}\right)$$

Due to (3.20) and the last implication in (3.21), the terms can be reduced as follows

$$(\rho(2\delta))^{-(p+1)} = t^{-(p+1)} \leq (8\delta+2)^{\frac{p+1}{p}} \quad \text{and} \quad e^{(p-1)(1-\rho(2\delta))} = e^{(p-1)(1-t)} \leq 8\delta+2$$

substitution gives

$$\psi''(\rho(2\delta)) \leq 1 + \frac{p}{2}\left((8\delta+2) + (8\delta+2)^{\frac{p+1}{p}}\right) \leq 1 + p\,(8\delta+2)^{\frac{p+1}{p}}$$

using Lemma 4.3, we obtain

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{1 + p\,(8\delta+2)^{\frac{p+1}{p}}}$$

which completes the proof. □

For our algorithm, we define the default step size $\alpha^\star$ ($\alpha^\star \leq \overline{\alpha}$) as follows

$$\alpha^\star = \frac{1}{1 + p\,(8\delta+2)^{\frac{p+1}{p}}}. \tag{4.1}$$

4.2. Decrease of Ψ

Now, we use the default step size $\alpha^\star$, which is specified in (4.1), to explain how the proximity function Ψ will decrease over an inner iteration.

Theorem 4.1 *(Theorem 4.6, [4]) With $\alpha^\star$ being the default step size as given by (4.1), one has*

$$f(\alpha^\star) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))}.$$

Theorem 4.2 *Let $\alpha^\star$ be a default step size as given in (4.1), we have $f(\alpha^\star) \leq -\dfrac{\Psi^{\frac{p-1}{2p}}}{168p}$.*

Proof: By using Theorem 4.1 with (4.1) and $\sqrt{2}\delta \geq \sqrt{\Psi}$, it follows that

$$\begin{aligned} f(\alpha^\star) \leq \frac{-\delta^2}{\psi''(\rho(2\delta))} \leq \frac{-\delta^2}{1 + p\,(8\delta+2)^{\frac{p+1}{p}}} &\leq \frac{-\delta^2}{\sqrt{2}\delta^{\frac{p+1}{p}} + p\left(8\delta + 2\sqrt{2}\delta\right)^{\frac{p+1}{p}}} \\ &\leq \frac{-\delta^{2-\frac{p+1}{p}}}{\sqrt{2} + p\left(8+2\sqrt{2}\right)^{\frac{p+1}{p}}} \leq \frac{-\Psi^{\frac{p-1}{2p}}}{168p} \end{aligned}$$

which completes the proof. □

4.3. Complexity analysis

We conclude this section with a theorem that estimates the total number of iterations of our algorithm (Algorithm 1). This means counting how many inner iterations are required to obtain the situation $\Psi \leq \tau$. Whereas Ψ_k signifies subsequent values in the same outer iteration and $k = 1, \ldots, K$, K denotes the total number of inner iterations in the outer iteration, where Ψ_0 represents the value Ψ following theμ-update.

Lemma 4.5 *(Proposition 1.3.2, [24]) Let a sequence $t_k > 0$, $k = 0, \ldots, K$ that verifies*

$$t_{k+1} \leq t_k - \lambda t_k^{1-\nu} \quad \textit{with} \quad \lambda > 0, \quad 0 < \nu \leq 1 \quad \textit{and} \quad k = 0, \ldots, K$$

then $K \leq \dfrac{t_0^\nu}{\lambda\nu}$.

Using the definition of $f(\alpha)$ and Theorem 4.2 for $t_k = \Psi_k$, we have $\nu = \frac{p+1}{2p}$ and $\lambda = \frac{1}{168p}$. From Lemma 4.5, we can easily deduce that the upper bound of the total number of inner iterations is given by $K \leq 336p\Psi_0^{\frac{p+1}{2p}}$. The above explanation implies the following theorem, which gives an upper bound for the total number of iterations.

Theorem 4.3 *If $\tau \geq 1$, the total number of iteration is bounded above by $336p\Psi_0^{\frac{p+1}{2p}} \frac{\log \frac{n}{\varepsilon}}{\theta}$.*

Proof: The number of outer iterations is bounded above by $\frac{1}{\theta} \log \frac{n}{\varepsilon}$ (see [27]). By multiplying the number of outer iterations by the number of inner iterations, we deduce the upper bound for the total number of iterations as follows

$$K \frac{\log \frac{n}{\varepsilon}}{\theta} \leq 336p\Psi_0^{\frac{p+1}{2p}} \frac{\log \frac{n}{\varepsilon}}{\theta}$$

which completes the proof. □

Remark 4.1 To calculate the complexity bound for small-update methods, we will use $\tau = O(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. To get the complexity bounds for large-update method, we will use $\tau = O(n)$ and $\theta = \Theta(1)$.

Using Remark 4.1, we get $\overline{\Psi}_0 = O(p)$ and $O\left(p^{\frac{3p+1}{2p}} \sqrt{n} \log \frac{n}{\varepsilon}\right)$ iteration complexity for small-update method. $\widehat{\Psi}_0 = O(n)$ and $O\left(pn^{\frac{p+1}{2p}} \log \frac{n}{\varepsilon}\right)$ iteration complexity for large-update method where $\overline{\Psi}_0$ and $\widehat{\Psi}_0$ are defined as in (3.22) and (3.23), respectively.

Remark 4.2 If we substitute any constant value for p, the iteration complexity of the small-update method becomes $O(\sqrt{n} \log \frac{n}{\varepsilon})$. If $p = \frac{\log n}{2}$, we have $O\left(\sqrt{n} \log n \log \frac{n}{\varepsilon}\right)$ iteration complexity for the large-update method. These are the best-known complexity results.

5. Numerical results

In this section, we present some numerical results on some absolute value equations to confirm the effectiveness of our proposed function, where the experiments manipulation in the **Dev-Cpp 5.11 TDM-GCC 4.9.2 Setup** and execution on a PC. Throughout the algorithm, we assume that the accuracy parameter is 10^{-6}, the threshold parameter is $\sqrt{n}$, the barrier update is $0.15, 0.3, 0.6$ and 0.95, the practical step size is as follows $\alpha_{pra} < \min\left(\left|\frac{(y_-)_1}{(\Delta y_-)_1}\right|, \ldots, \left|\frac{(y_-)_n}{(\Delta y_-)_n}\right|, \left|\frac{(y_+)_1}{(\Delta y_+)_1}\right|, \ldots, \left|\frac{(y_+)_n}{(\Delta y_+)_n}\right|, 1\right)$ and the barrier parameter p is varied. Finally, we use **Iter** and **CPU** to represent the number of iterations and the time produced by our algorithm, respectively.

Problem 1 *Consider the AVE problem, where A and b are given by*

$$A = \begin{cases} 2n & if \quad i = j \\ n & if \quad i = j+1 \\ n & if \quad i = j-1 \\ 1 & if \quad i > j+1 \\ 1 & if \quad i < j-1 \end{cases} \quad and \quad b = \begin{pmatrix} 4n-3 \\ 5n-4 \\ \vdots \\ 5n-4 \\ 4n-3 \end{pmatrix}$$

for instance, if $n = 4$, we get

$$\begin{pmatrix} 8 & 4 & 1 & 1 \\ 4 & 8 & 4 & 1 \\ 1 & 4 & 8 & 4 \\ 1 & 1 & 4 & 8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} - \begin{pmatrix} |y_1| \\ |y_2| \\ |y_3| \\ |y_4| \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \\ 16 \\ 13 \end{pmatrix}$$

since $\sigma_{min}(A) = 1.171572 > 1$, *the AVE has a unique solution. Therefore, matrix* M *and vector* q *of the corresponding linear complementarity problem are given by*

$$M = \begin{pmatrix} 1.507936 & -0.412698 & 0.253968 & -0.158730 \\ -0.412698 & 1.793651 & -0.539682 & 0.253968 \\ 0.253968 & -0.539682 & 1.793651 & -0.412699 \\ -0.158730 & 0.253968 & -0.412698 & 1.507936 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We can easily check that the problem is strictly monotone since all dominant principal minors of the corresponding matrix are strictly positive and the strictly feasible starting point is given by

$$y_-^0 = (1,1,1,1)^T \quad \text{and} \quad y_+^0 = (2.190476, 2.095239, 2.095238, 2.190476)^T$$

the unique solution of the corresponding LCP is

$$y_-^* = (0,0,0,0)^T, \quad y_+^* = (1,1,1,1)^T$$

and the unique solution of the AVE

$$y^* = (1,1,1,1)^T.$$

The numerical results for our function with different values of θ *and* p *are as follows*

θ		0.15	0.3	0.6	0.95
$p = 1.1$	$\Psi(v^*)$	0.097122	0.567118	0.340311	0.315004
	Iter	107	52	24	11
	CPU (s)	0.63	0.53	0.38	0.28
$p = 2$	$\Psi(v^*)$	0.439155	0.033525	0.002828	0.237164
	Iter	110	52	25	12
	CPU (s)	0.66	0.67	0.73	0.55
$p = 4$	$\Psi(v^*)$	0.349496	0.055230	0.174536	1.894199
	Iter	112	58	27	18
	CPU (s)	0.91	0.82	0.73	0.68

Table 1: Numerical results of Problem 1 with different values of θ and p .

Problem 2 *(Problem* 1*, [3]) Consider that the general form of the absolute value equation GAVE is given as follow*

$$A = \begin{pmatrix} 8 & 0 & -1 & 1 & -20 \\ 1 & 1 & 1 & 4 & 25 \\ 1 & -5 & 0 & 8 & -10 \\ 0 & 8 & 1 & -6 & 1 \\ 3 & 5 & -3 & 0 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} -1.5 & 0 & 1.5 & 0.5 & 0.1 \\ 0 & 0.25 & 1 & 0 & 0.5 \\ 1 & 0.6 & 1 & 0.4 & 0.5 \\ 0 & 0.3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$$

The matrix A *and vector* b *of the corresponding absolute value equation are given by*

$$A = \begin{pmatrix} 5.951812 & -13.357438 & -1.441769 & 8.096392 & -56.164700 \\ -41.445786 & 112.610451 & 20.080326 & -77.108437 & 478.393616 \\ -8.951807 & 8.357422 & 4.441766 & -8.096379 & 46.164635 \\ 21.385540 & -50.140556 & -11.465863 & 37.228912 & -190.682709 \\ 36.626507 & -75.020065 & -20.923693 & 46.746983 & -381.526062 \end{pmatrix},$$

$$b = \begin{pmatrix} -0.461847 & 2.811247 & -0.538153 & -1.305220 & -2.329317 \end{pmatrix}^T$$

where the AVE has a unique solution since $\sigma_{min}(A) = 1.028468 > 1$ *and the matrix* M *and vector* q *of the corresponding linear complementarity problem are given by*

$$M = \begin{pmatrix} 1.790225 & 0.150306 & -0.314468 & 0.081853 & -0.006957 \\ -0.770773 & 0.792289 & -0.323584 & -0.285811 & -0.041988 \\ 0.985142 & -0.071720 & 0.657360 & -0.262341 & -0.145061 \\ -0.872206 & -0.245790 & -0.270383 & 0.832090 & -0.128345 \\ 0.066348 & 0.028801 & 0.019050 & 0.057718 & 0.994603 \end{pmatrix},$$
$$q = \begin{pmatrix} 0.067914 & 0.209847 & 0.103878 & 0.187635 & -0.011314 \end{pmatrix}^T.$$

The strictly feasible starting point is given by

$$y_-^0 = (1, 2, 1, 2, 1)^T, \quad y_+^0 = (2.001031, 0.088659, 0.933196, 0.089300, 1.241725)^T$$

the corresponding LCP and AVE have a unique solution

$$y_-^* = (0, 0, 0, 0, 0.011380)^T,\ y_+^* = (0.067836, 0.209369, 0.102228, 0.186174, 0.000005)^T,$$
$$y^* = (0.067836, 0.209369, 0.102228, 0.186174, -0.011376)^T,$$

respectively. The numerical results for our function with different values of θ *and* p *are as follows*

θ		0.15	0.3	0.6	0.95
$p = 1.1$	$\Psi(v^*)$	2.175327	2.206915	0.332701	1.165553
	Iter	110	57	32	16
	CPU (s)	0.87	0.75	0.66	0.63
$p = 2$	$\Psi(v^*)$	1.835603	0.858590	1.514876	0.020459
	Iter	111	59	33	17
	CPU (s)	0.93	0.83	0.79	0.77
$p = 4$	$\Psi(v^*)$	1.265605	0.687506	1.034920	1.213340
	Iter	113	59	31	22
	CPU (s)	1.00	0.94	0.86	0.83

Table 2: Numerical results of Problem 2 with different values of θ and p .

6. Conclusion

In this paper, we propose a primal-dual interior point algorithm for absolute value equation based on a new parametric kernel function. We proved that the proposed kernel function has the best known complexity bound for large-update methods, namely, $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ with the special choice of its parameter p and we confirmed this using some numerical experiments. Future research might focus on other methods to solve the *AVE* as a smoothing-type algorithm and compare them to see which one is more effective.

References

1. M.Achache, *Complexity analysis of an interior point algorithm for the semidefinite optimization based on a new kernel function with a double barrier term.* Acta Mathematica Sinica, English Series, 31 (3) : 543-556, (2015).
2. M. Achache and N. Hazzam, *Solving absolute value equations via linear complementarity and interior-point methods.* Journal of Nonlinear Functional Analysis. Article. ID 39: 1-10 (2018).
3. M. Achache and N. Tabchouche, *A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems*, Optimization Letters. 1-19: (2018).
4. Y.Q.Bai, M.El Ghami, C. Roos, *A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization,* SIAM J. Optim, 101-128 (2004).
5. Y.Q.Bai, M.El Ghami, C. Roos, *A new efficient large-update primal-dual interior-point method based on a finite barrier*, SIAM journal on optimization. 13 (3) : 766-782, (2003).

6. N.Boudjellal, H.Roumili & D.J.Benterki, *A primal-dual interior point algorithm for convex quadratic programming based on a new parametric kernel function.* A Journal of Mathematical Programming and Operations Research, (2020).

7. B.Bounibane , El.a.Djeffal, *Kernel function-based interior-point algorithms for linear optimisation,* Int. J. Mathematical Modelling and Numerical Optimisation, Vol. 9, No. 2, pp 158-177 (2019).

8. L.Caccetta, B.Qu, G.Zhou, *A globally and quadratically convergent method for absolute value equations.* Comput. Optim. Appl. 48 (2011), 45–58.

9. S.J.Chung, *NP-completeness of the linear complementarity problem.* J. Optim. Theory Appl. 60, 393–399 (1989).

10. R.W.Cottle, G.Dantzig, *Complementary pivot theory of mathematical programming.* Linear Algebra and its Applications, 103-125 (1968).

11. R.W.Cottle, J.S.Pang, R.E.Stone, *The linear complementarity problem,* Academic Press, New York, (1992).

12. El.A.Djeffal, M.Laouar, *A primal-dual interior-point method based on a new kernel function for linear complementarity problem,* Asian-European, (2018).

13. J-M.Feng, S-Y.Liu, *A Three-Step Iterative Method for Solving Absolute Value Equations.* (2020).

14. R.A.Horn and C.R.Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, (1985).

15. S.-L.Hu, Z.-H.Huang, *A note on absolute value equations.* Optim. Lett. 4, 417–424 (2010).

16. X.Jiang, Y.Zhang, *A smoothing-type algorithm for absolute value equations.* J. Ind. Manag. Optim. 9, 789–798 (2013).

17. M.Kojima, M.Shindoh , S.Hara, *Interior point methods for the monotone semidefinite linear complementarity in symmetric matrices.* SIAM J Optim. (1997); 7:86–125.

18. O.L.Mangasarian, *Absolute value programming.* Comput. Optim Appl. 36, 43-53 (2007).

19. O.L.Mangasarian, *Absolute value equation solution via concave minimization.* Optim. Lett. 1, 3–5 (2007).

20. O.L.Mangasarian, *A generalized Newton method for absolute value equations.* Optim. Let. 3, 101–108 (2009).

21. O.L.Mangasarian, R.R.Meyer, *Absolute value equations.* Linear Algebra Appl. 419, 359–367 (2006).

22. M.A.Noor, J.Iqbal, K.I.Noor, E.Al-Said, *On an iterative method for solving absolute value equations.* Optim. Lett.6(5), 1027–1033 (2012).

23. J.Peng, C.Roos & T.Terlaky, *Self-regular functions and new search directions for linear and semidefinite optimization.* Mathematical Programming, vol.93 : 129-171, (2002).

24. J. Peng, C. Roos, and T. Terlaky, *Self-Regularity A New Paradigm for Primal-Dual Interior-Point Algorithms,* Princeton University Press, Princeton, NJ, (2002).

25. J.Rohn, *A theorem of the alternatives for the equation* $Ax + B|x| = b$*,* Linear Multilinear Algebra, 421-426 52 (2004).

26. J.Rohn, V.Hooshyarbarkhsh, R.Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditions for unique solvability.* Optim. Lett. 8(1), 35–44 (2014).

27. C.Roos, T.Terlaky, J.-Ph.Vial, *Theory and Algorithms for Linear Optimization.* An Interior-Point Approach, John Wiley & Sons, Chichester, UK, (1997), 2nd Ed, Springer, (2006).

28. L.Yong, S.Zhang, F.Deng, W.Xiong, *Feasible interior point method for absolute value equation,* Fourth International Conference on Information and Computing, (2011).

29. M.W.Zhang, *A large-update interior-point algorithm for convex quadratic semidefinite optimization based on a new kernel function.* Acta Mathematica Sinica, English Series, vol.28 : 2313-2321, (2012).

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