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Discussion on the Existence Problems of Fixed Circle in Metric Spaces

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ABSTRACT: This paper deals with the problems of the existence of a fixed circle for multivalued mappings. To prove the validity of the postulates, we develop innovative fixed circle theorems in metric spaces by using Caristi's technique and validate them using illustrated instances with geometric explanation. In fact, for multivalued mappings two different versions of the existence theorems on fixed circle are presented.

Key Words: Fixed circle, multivalued mapping, Hausdorff metric.

Contents

1. Introduction

The Caristi fixed point theorem [4] is considered one of the most beautiful extensions of the Banach contraction principle [3]. Its generalization to multivalued cases play important role in several branches of mathematics. Let $(\mathcal{M}, d)$ be a metric space and let $\mathcal{T}$ be a multivalued mapping on $\mathcal{M}$ into the family $\mathcal{CB}(\mathcal{M})$ of all non-empty closed bounded subsets of $\mathcal{M}$. Define $\mathcal{H}(\mathcal{E},\mathcal{F}) = \max\left\{\sup_{u\in\mathcal{E}} \mathcal{D}(u,\mathcal{F}), \sup_{v\in\mathcal{F}} \mathcal{D}(v,\mathcal{E})\right\}$ for all $\mathcal{E},\mathcal{F} \in \mathcal{CB}(\mathcal{M})$ and $D(u,\mathcal{F}) = \inf_{v\in\mathcal{F}} d(u,v)$. The $\mathcal{H}$ is called Hausdorff metric with respect to d. Define a circle and disc with center u_0 and radius ρ on metric space $(\mathcal{M}, d)$ as $\mathcal{C}_{u_0,\rho} = \{u \in \mathcal{M} : d(u_0, u) = \rho\}$ and $\mathcal{D}_{u_0,\rho} = \{u \in \mathcal{M} : d(u_0, u) \leq \rho\}$ respectively. In 1969, Nadler [6] generalized the well-known Banach contraction principle to multivalued mappings which became a great source of inspiration for researchers working in metric fixed point theory.

Theorem 1.1 *[6] Let $\mathcal{T}$ be a multivalued mapping on complete metric space $\mathcal{M}$ to the family $\mathcal{CB}(\mathcal{M})$. If there exists $k \in (0, 1)$ such that*

$$\mathcal{H}(\mathcal{T}u, \mathcal{T}v) \leq kd(u, v), \quad for\ all\ u, v \in \mathcal{M}.$$

Then multivalued mapping $\mathcal{T}$ has at least one fixed point in $\mathcal{M}$.

In 1989, Mizoguchi and Takahashi [7] proved the multivalued version of Caristi's fixed point theorem as stated follows.

Theorem 1.2 *[7] Let $(\mathcal{M}, d)$ be a complete metric space and $\mathcal{T}$ be a multivalued mapping of $\mathcal{M}$ to the family $\mathcal{N}(\mathcal{M})$ of non-empty subsets of $\mathcal{M}$ such that there exists $v \in \mathcal{T}u$ satisfying, for every $u \in \mathcal{M}$,*

$$d(u, v) \leq \psi(u) - \psi(v),$$

where $\psi : \mathcal{M} \to (-\infty, +\infty]$ be a proper, bounded below and lower semi-continuous mapping of $\mathcal{M}$. Then $\mathcal{T}$ has at least one fixed point in $\mathcal{M}$.

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It is known that the mapping may have more than one fixed point. If a mapping fixes all the points of a circle (disc), then such a circle (disc) is called a fixed circle (fixed disc). Instance, the identity map on $\mathbb{C}$ fixes any disc on $\mathbb{C}$ while the map $z \to \bar{z}^{-1}$ fixes the unit circle only. Let us illustrates one of the general forms of a fixed circle. Let r be a fixed positive real number and $\mathcal{M} = \mathbb{C}$ be a complex metric space with the metric $d(z, \tilde{z}) = |z - \tilde{z}|$ for all $z, \tilde{z} \in \mathbb{C}$. Define $f_1 : \mathbb{C} \to \mathbb{C}$ as

$$f_1 z = \begin{cases} w_0 + \dfrac{r^2}{\bar{z} - \bar{w_0}} & \text{if } z \neq w_0, \\ w_0 & \text{if } z = w_0, \end{cases}$$

for all $z, w_0 \in \mathbb{C}$, where $\bar{z}$ and $\bar{w_0}$ are the conjugate of z and w_0 respectively. Then the circle $\mathcal{C}_{w_0,r}$ is fixed by the mapping f_1. It is worth to note that there are some mappings which map a circle to itself, but do not fix all the points of the circle. For instance, suppose that $f_2 : \mathbb{C} \to \mathbb{C}$ given as

$$f_2 z = \begin{cases} w_0 + \dfrac{r^2}{z - w_0} & \text{if } z \neq w_0, \\ w_0 & \text{if } z = w_0. \end{cases}$$

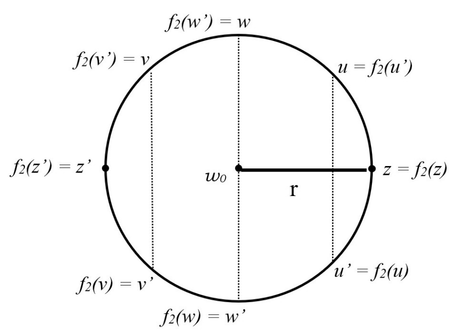

Figure 1: f_2 fixes only two points z and z' on the circle $\mathcal{C}_{w_0,r}$

Then $f_2(\mathcal{C}_{w_0,r}) = \mathcal{C}_{w_0,r}$, but all points of the circle $\mathcal{C}_{w_0,r}$ are not fixed by the mapping f_2, see Figure 1. In case $\mathcal{M} = \mathbb{R}$, f_2 also fixes the circle $\mathcal{C}_{w_0,r}$ because $\bar{z} = z$. Now consider the mappings which can fix only three points of the circle $\mathcal{C}_{0,r}$ on $\mathbb{C}$. Let $f_3, f_4, : \mathbb{C} \to \mathbb{C}$ are defined as

$$f_3 z = \begin{cases} \dfrac{r^3}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases} \quad \text{and} \quad f_4 z = \begin{cases} \dfrac{-r^3}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

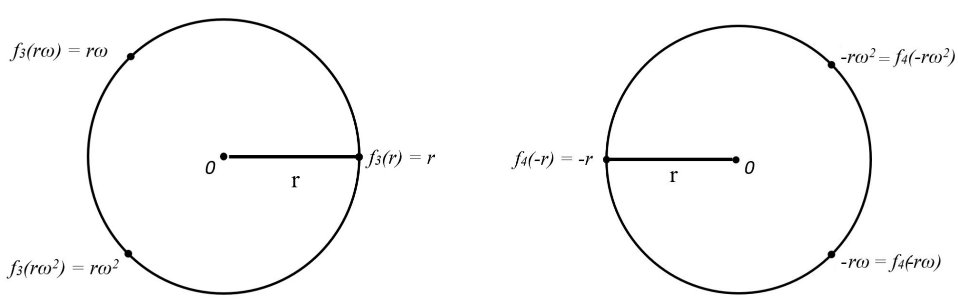

Figure 2: f_3 and f_4 fix the three points of $\mathcal{C}_{0,r}$

Then f_3 and f_4 fix only three points of $\mathcal{C}_{0,r}$, but not the circle $\mathcal{C}_{0,r}$, see Figure 2, where $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

Various types of fixed circle and fixed disc theorems have been developed by using different approaches for self-mappings in metric spaces and generalized metric spaces. In [10], Taş and Özgür, obtained some fixed circle theorems by using the Caristi's inequality on metric spaces. It has been further generalized by Özgür et al. [11] and Tomar et al. [15] in S-metric spaces and partial metric spaces, respectively. In [14], Taş et al. defined Ćirić type, Hardy-Rogers type, Reich type and Chatterjea type F_c-contractions and using it they obtained some fixed circle and fixed disc results in metric spaces. Özgür [9], also proved some fixed circle and fixed disc results via simulation functions. The readers can also see some more results in [1,2,8,12] and the references therein. In this paper we investigate some fixed circle theorems using the Caristi's technique on a metric space. In fact, we provide two different versions of existence theorems for fixed circle for multivalued mappings. Also, we introduce the conditions which ensure the uniqueness of fixed circle. Consequently the results are applied to discontinuous multivalued mappings.

2. Main results

It is known that a multivalued mapping $\mathcal{T}$ has a fixed point u, if $u \in \mathcal{T}u$. The fixed circle of a multivalued mapping are defined as follows.

Definition 2.1 *[13] Let $(\mathcal{M}, d)$ be a metric space, $\mathcal{C}_{u_0,\rho}$ be the circle on $\mathcal{M}$. Consider a multivalued mapping $\mathcal{T}$ from $\mathcal{M}$ to $\mathcal{N}(\mathcal{M})$ then the circle $\mathcal{C}_{u_0,\rho}$ is said to be fixed circle of a multivalued mapping $\mathcal{T}$ if $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$.*

Example 2.1 Let $\mathcal{M} = [0, 2]$ with $d(u, v) = |u - v|$ and $f : \mathcal{M} \to \mathcal{M}$ given as

$$f(u) = \begin{cases} \frac{1}{2} - \frac{u}{2} & \text{if} \quad 0 \le u \le 1, \\ \frac{u}{2} - \frac{1}{2} & \text{if} \quad 1 \le u \le 2. \end{cases}$$

Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ by

$$\mathcal{T}u = \begin{cases} \{f(u)\} & \text{if } u \in \mathcal{C}_{1,1}, \\ \{f(u)\} \cup \{\frac{3}{2}\} & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{T}$ has fixed points $\frac{1}{3}$ and $\frac{3}{2}$, but $\mathcal{T}$ doesn't fix the circle $\mathcal{C}_{1,1}$.

Now, we investigate the sufficient conditions for existence of fixed circle for multivalued mappings in two different versions.

Version I In this version, we give existence theorems for a fixed circle via a multivalued mapping with Hausdorff metric.

Theorem 2.1 *Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be the circle on $\mathcal{M}$. If there exists a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ satisfying*

$(\mathcal{C}_1)$ $\mathcal{H}(\{u\}, \mathcal{T}u) \le d(u_0, u) - \mathcal{H}(\{u_0\}, \mathcal{T}u)$,

$(\mathcal{C}_2)$ $\mathcal{H}(\{u_0\}, \mathcal{T}u) \ge \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let $u \in \mathcal{C}_{u_0,\rho}$ be any arbitrary point. To show $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. Using $(\mathcal{C}_1)$, we have

$$\mathcal{H}(\{u\}, \mathcal{T}u) \le d(u_0, u) - \mathcal{H}(\{u_0\}, \mathcal{T}u) = \rho - \mathcal{H}(\{u_0\}, \mathcal{T}u). \tag{2.1}$$

By the condition $(\mathcal{C}_2)$, we have two cases either $\mathcal{H}(\{u_0\}, \mathcal{T}u) > \rho$ or it is equal to ρ. If $\mathcal{H}(\{u_0\}, \mathcal{T}u) > \rho$, then by (2.1), we have $\mathcal{H}(\{u\}, \mathcal{T}u) < 0$ which is a contradiction. Therefore we must have $\mathcal{H}(\{u_0\}, \mathcal{T}u) = \rho$. Again by using (2.1), we obtain

$$\mathcal{H}(\{u\}, \mathcal{T}u) \le \rho - \mathcal{H}(\{u_0\}, \mathcal{T}u) = \rho - \rho = 0. \tag{2.2}$$

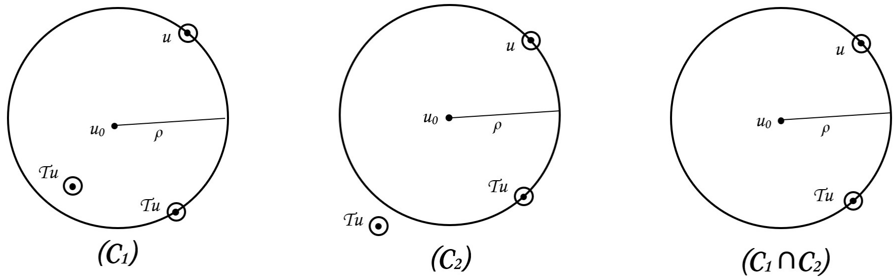

Figure 3: The geometric interpretation of the conditions in Theorem 2.1

Therefore, $\mathcal{H}(\{u\}.\mathcal{T}u) = 0$ if and only if $\max\{\sup_{a\in\{u\}} d(a,\mathcal{T}u), \sup_{b\in\mathcal{T}u} d(b,\{u\})\} = 0$. Equivalently, $d(a,\mathcal{T}u) = 0$, $\forall\ a \in \{u\}$ and $d(b,\{u\}) = 0$, $\forall\ b \in \mathcal{T}u$. Since $\{u\}$ and $\mathcal{T}u$ are closed in $\mathcal{M}$, $d(a,\mathcal{T}u) = 0 \Leftrightarrow a \in \mathcal{T}u$ and $d(b,\{u\}) = 0 \Leftrightarrow b \in \{u\}$. Consequently, $\mathcal{T}u = \{u\}$ for each $u \in \mathcal{C}_{u_0,\rho}$, that is $\mathcal{T}$ fixes the circle $\mathcal{C}_{u_0,\rho}$. □

Remark 2.1 Clearly, the condition $(\mathcal{C}_1)$ ensure that $\mathcal{T}u$ never contained any point which is exterior of the circle $\mathcal{C}_{u_0,\rho}$, for each $u \in \mathcal{C}_{u_0,\rho}$ and $(\mathcal{C}_2)$ ensure that $\mathcal{T}u$ is not contained any point which is interior of the circle $\mathcal{C}_{u_0,\rho}$, for each $u \in \mathcal{C}_{u_0,\rho}$. Consequently, $\mathcal{T}u = \{u\}$, for each $u \in \mathcal{C}_{u_0,\rho}$ and also $\mathcal{T}(\mathcal{C}_{u_0,\rho}) \subset \mathcal{C}_{u_0,\rho}$. In the cases whenever the circle has only one point (see, Example 2.10) we obtain $\mathcal{T}(\mathcal{C}_{u_0,\rho}) = \mathcal{C}_{u_0,\rho}$.

Remark 2.2 Notice that converse statement of Theorem 2.1 is not true (see, Example 2.11) $\mathcal{T}$ fixes the circle $\mathcal{C}_{\frac{1}{2},\frac{1}{2}}$. But $\mathcal{T}$ doesn't satisfy condition $(\mathcal{C}_1)$.

The following example illustrates Theorem 2.1.

Example 2.2 Let $\mathcal{M} = \{1 + \frac{1}{2^n}; n = 1,2,3,4\} \cup \{0, \frac{1}{4}, \frac{1}{2}\}$ be a metric space with usual metric $d(u,v) = |u - v|$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ given as

$$\mathcal{T}u = \begin{cases} \{u\} & \text{if } u \in \mathcal{C}_{\frac{1}{4},\frac{1}{4}}, \\ \{1 + \frac{1}{2^{n+1}}\} & \text{if } u \in \{1 + \frac{1}{2^n}; n = 1,2,3\} \\ \{\frac{3}{2}\} & \text{if } u \in \{1 + \frac{1}{2^4}, \frac{1}{4}\}. \end{cases}$$

for all $u \in \mathcal{M}$. Then we can check that $\mathcal{T}$ satisfied condition $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. Clearly $\mathcal{C}_{\frac{1}{4},\frac{1}{4}}$, is a fixed circle of $\mathcal{T}$.

Example 2.3 Let $\mathcal{M} = \mathbb{R}$ be a metric space with usual metric $d(u,v) = |u - v|$ and $\mathcal{C}_{0,\frac{1}{2}}$ be a circle on $\mathcal{M}$. Define a multivalued mapping $\mathcal{T} : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ given as

$$\mathcal{T}u = \begin{cases} [\frac{1}{4}, \frac{1}{3}] & \text{if} \quad u \in \mathcal{C}_{0,\frac{1}{2}}, \\ \{\frac{1}{2}\} & \text{if} \quad \text{otherwise}. \end{cases}$$

for all $u \in \mathcal{M}$. Clearly the mapping $\mathcal{T}$ not satisfies conditions $(\mathcal{C}_2)$ and $(\mathcal{C}_1)$ of Theorem 2.1. So, $\mathcal{T}$ doesn't fix any points of the circle $\mathcal{C}_{0,\frac{1}{2}}$.

Now, we present some examples to show that Theorem 2.1 fail out if the one of the condition is not satisfied.

Example 2.4 Let $\mathcal{M} = \mathbb{R}$ be a metric space with usual metric and $\mathcal{C}_{2,2}$ be a circle on $\mathcal{M}$. Define a multivalued mapping $\mathcal{T} : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ given as

$$\mathcal{T}u = \begin{cases} \{2\} & \text{if } u \in \mathcal{C}_{2,2}, \\ \{1,3\} & \text{otherwise}. \end{cases}$$

for all $u \in \mathcal{M}$. Then the mapping $\mathcal{T}$ satisfies the condition $(\mathcal{C}_1)$ of Theorem 2.1 but doesn't satisfy the condition $(\mathcal{C}_2)$. Clearly, $\mathcal{T}$ doesn't fix any points of the circle $\mathcal{C}_{2,2}$.

Example 2.5 Let $\mathcal{M} = (-2, 2)$ be a metric space with usual metric $d(u, v) = |u - v|$ and $\mathcal{C}_{1,\frac{1}{4}}$ be a circle on $\mathcal{M}$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ given as

$$\mathcal{T}u = \left\{1, \frac{5}{4}\right\} \ \forall\, u \in \mathcal{M}.$$

Then the multivalued mapping $\mathcal{T}$ satisfies $(\mathcal{C}_2)$ of Theorem 2.1 but not $(\mathcal{C}_1)$. Thus, the circle $\mathcal{C}_{1,\frac{1}{4}}$ is not fixed by the mapping $\mathcal{T}$.

The following example shows that the fixed circle for a multivalued mapping need not be unique.

Example 2.6 Let $\mathcal{M} = \mathbb{R}$ be a metric space with usual metric $d(u, v) = |u - v|$ and $\mathcal{C}_{0,2}$ be a circle on $\mathcal{M}$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ as

$$\mathcal{T}u = \begin{cases} \{\frac{4}{u}\} & \text{if } u \in \mathcal{C}_{0,2}, \\ \{4\} & \text{otherwise.} \end{cases}$$

Then clearly, we can check that the multivalued mapping $\mathcal{T}$ satisfied conditions of Theorem 2.1. Hence the circle $\mathcal{C}_{0,2}$ is fixed by multivalued mapping $\mathcal{T}$ and it is not only the circle which is fixed by $\mathcal{T}$ there are other circles $\mathcal{C}_{1,3}$ and $\mathcal{C}_{3,1}$ which are also fixed by $\mathcal{T}$.

The following theorem gives another solution for fixed circle problem for multivalued mapping.

Theorem 2.2 *Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be the circle on $\mathcal{M}$. If there exists a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ satisfying*

$(\mathcal{C}_1)$ $\mathcal{H}(\{u\}, \mathcal{T}u) \le d(u_0, u) + \mathcal{H}(\{u_0\}, \mathcal{T}u) - 2\rho$,

$(\mathcal{C}_2)$ $\mathcal{H}(\mathcal{T}u, \{u_0\}) \le \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let $u \in \mathcal{C}_{u_0,\rho}$ be any arbitrary point. To show $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. Using $(\mathcal{C}_1)$, we have

$$\mathcal{H}(\{u\}, \mathcal{T}u) \le d(u_0, u) + \mathcal{H}(\{u_0\}, \mathcal{T}u) - 2\rho = \mathcal{H}(\{u_0\}, \mathcal{T}u) - \rho. \tag{2.3}$$

By the condition $(\mathcal{C}_2)$, we have two cases either $\mathcal{H}(\{u_0\}, \mathcal{T}u) < \rho$ or it is equal to ρ. If $\mathcal{H}(\{u_0\}, \mathcal{T}u) < \rho$, then by (2.3), we have $\mathcal{H}(\{u_0\}, \mathcal{T}u) < 0$ which is a contradiction. Therefore we must have $\mathcal{H}(\{u_0\}, \mathcal{T}u) = \rho$. Again by using (2.3), we obtain

$$\mathcal{H}(\{u\}, \mathcal{T}u) \le \mathcal{H}(\{u_0\}, \mathcal{T}u) - \rho = \rho - \rho = 0.$$

Therefore, $\mathcal{H}(\{u\}, \mathcal{T}u) = 0$. Remaining part is similar as proof of Theorem 2.1. □

Remark 2.3 We notice that condition $(\mathcal{C}_1)$ ensure that $\mathcal{T}u$ can never contained any point which is interior of the circle $\mathcal{C}_{u_0,\rho}$, for each $u \in \mathcal{C}_{u_0,\rho}$ and $(\mathcal{C}_2)$ ensure that $\mathcal{T}u$ can not be contained any point which is in exterior of the circle $\mathcal{C}_{u_0,\rho}$, for each $u \in \mathcal{C}_{u_0,\rho}$. Consequently, $\mathcal{T}u = \{u\}$, for each $u \in \mathcal{C}_{u_0,\rho}$ and also we obtain $\mathcal{T}(\mathcal{C}_{u_0,\rho}) \subset \mathcal{C}_{u_0,\rho}$. In the cases whenever the circle has only one point we obtain $\mathcal{T}(\mathcal{C}_{u_0,\rho}) = \mathcal{C}_{u_0,\rho}$.

We continue with an example which satisfy the requirements of Theorem 2.2 as follows:

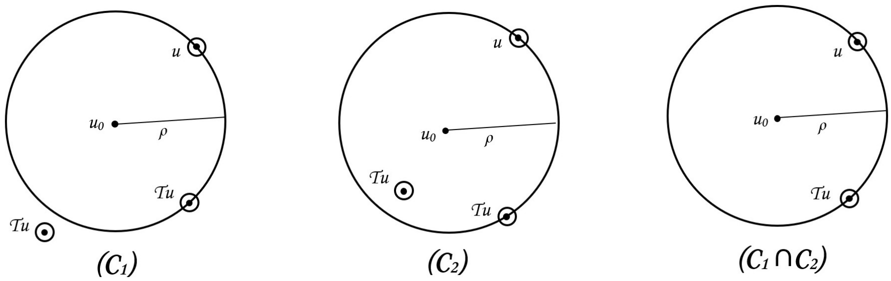

Figure 4: The geometric interpretation of the conditions in Theorem 2.2

Example 2.7 Let $\mathcal{M} = [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ be a metric space with metric $d(u,v) = \frac{|u-v|}{\sqrt{1+u^2}\sqrt{1+v^2}}$ and the circle $\mathcal{C}_{u_0,\frac{1}{2}} = \{u \in \mathcal{M} : d(u_0,u) = \frac{|u_0-u|}{\sqrt{1+u_0^2}\sqrt{1+u^2}} = \frac{1}{2}\} = \{u \in \mathcal{M} : \frac{|u|}{\sqrt{1+u^2}} = \frac{1}{2}\}$ where $u_0 = 0$.
So, $\mathcal{C}_{0,\frac{1}{2}} = \left\{u \in \mathcal{M} : u^2 = \frac{1}{3}\right\}$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ as

$$\mathcal{T}u = \begin{cases} \{u\} & \text{if } u \in \mathcal{C}_{0,\frac{1}{2}}, \\ \{0, \frac{1}{2}\} & \text{otherwise} \end{cases}$$

for all $u \in \mathcal{M}$. Clearly, $\mathcal{T}$ satisfied condition $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ of Theorem 2.2, so $\mathcal{T}$ fixes the circle $\mathcal{C}_{0,\frac{1}{2}}$.

Now, we present an example in order to show that Theorem 2.2 fail out, if the condition $(\mathcal{C}_1)$ is satisfied but the condition $(\mathcal{C}_2)$ is not.

Example 2.8 Let $(\mathbb{R}, d)$ be a usual metric space and $\mathcal{C}_{0,1}$ be a unit circle on $\mathbb{R}$. Define a multivalued mapping $\mathcal{T} : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ as

$$\mathcal{T}u = \begin{cases} \{-2, -3\} & \text{if } u = -1, \\ \{2, 3\} & \text{if } u = 1, \\ \{2\} & \text{otherwise} \end{cases}$$

for all $u \in \mathbb{R}$. Then the mapping $\mathcal{T}$ satisfies $(\mathcal{C}_1)$ but doesn't satisfy $(\mathcal{C}_2)$ of Theorem 2.2, Clearly, $\mathcal{T}$ doesn't fix the unit circle $\mathcal{C}_{0,1}$.

Example 2.9 Let $(\mathbb{R}, d)$ be a usual metric space and $\mathcal{C}_{2,1}$ be the circle on $\mathbb{R}$. Let us define the mapping $\mathcal{T} : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ as

$$\mathcal{T}u = \left[\frac{3}{2}, \frac{5}{2}\right], \quad \forall\, u \in \mathbb{R}.$$

Then the mapping $\mathcal{T}$ satisfies the condition $(\mathcal{C}_2)$ but doesn't satisfy the condition $(\mathcal{C}_1)$ of Theorem 2.2. Thus $\mathcal{T}$ doesn't fix the circle $\mathcal{C}_{2,1}$.

Theorem 2.3 *Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be the circle on $\mathcal{M}$. If there exists a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ satisfying*

$(\mathcal{C}_1)$ $\mathcal{H}(\{u\}, \mathcal{T}u) \leq d(u_0, u) - \mathcal{H}(\{u_0\}, \mathcal{T}u)$,

$(\mathcal{C}_2)$ $\beta\mathcal{H}(\{u\}, \mathcal{T}u) + \mathcal{H}(\{u_0\}, \mathcal{T}u) \geq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$, where $\beta \in [0, 1)$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let $u \in \mathcal{C}_{u_0,\rho}$ be any arbitrary point. To show $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. Assume that $u \notin \mathcal{T}u$, for some $u \in \mathcal{C}_{u_0,\rho}$, then $\mathcal{H}(\{u\}, \mathcal{T}u) > 0$. By using condition $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, we have

$$\begin{aligned}\mathcal{H}(\{u\}, \mathcal{T}u) &\leq d(u, u_0) - \mathcal{H}(\{u_0\}, \mathcal{T}u) = \rho - \mathcal{H}(\{u_0\}, \mathcal{T}u)\\ &\leq \beta\mathcal{H}(\{u\}, \mathcal{T}u) + \mathcal{H}(\{u_0\}, \mathcal{T}u) - \mathcal{H}(\{u_0\}\mathcal{T}u)\\ &= \beta\mathcal{H}(\{u\}, \mathcal{T}u),\end{aligned}$$

which is a contradiction. Therefore we have $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. □

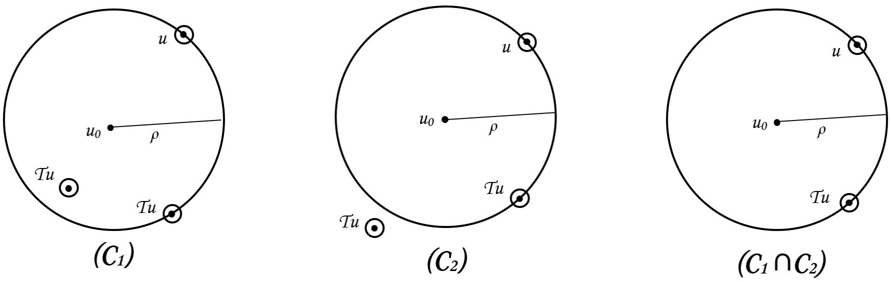

Figure 5: The geometric interpretation of the conditions in Theorem 2.3

Remark 2.4 Clearly, Theorem 2.1 implies the Theorem 2.3.

The following example illustrates Theorem 2.3.

Example 2.10 Let $\mathcal{M} = \{x \in \mathbb{R} : x > 0\}$ and mapping $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be defined as $d(u, v) = |\frac{1}{u} - \frac{1}{v}|$ for all $u, v \in \mathcal{M}$. Then $(\mathcal{M}, d)$ be a metric space. Consider the circle

$$\mathcal{C}_{2,1} = \left\{u \in \mathcal{M} : d(u, 2) = \left|\frac{1}{u} - \frac{1}{2}\right| = 1\right\} = \left\{\frac{2}{3}\right\}$$

and define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ given as

$$\mathcal{T}u = \begin{cases} \{\frac{2}{3}\} & \text{if} \quad u \in \mathcal{C}_{2,1}, \\ \{\frac{3}{5}\} & \text{if} \quad \text{otherwise} \end{cases}$$

for all $u \in \mathcal{M}$. Then we can check that the multivalued mapping $\mathcal{T}$ satisfies all the condition of Theorem 2.3. Thus the circle $\mathcal{C}_{2,1}$ is fixed by $\mathcal{T}$.

Remark 2.5 From the Examples 2.4 and 2.5, we can check the significance of conditions $(\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$ in Theorem 2.3.

Version II The following example shows that the multivalued mapping $\mathcal{T}$ fixes the circle, but doesn't satisfy condition $(\mathcal{C}_1)$ of the Theorems 2.1, 2.2 and 2.3.

Example 2.11 Let $\mathcal{M} = [0, 1] \cup \{\frac{3}{2}\}$ be a metric space with metric $d(u, v) = |u - v|$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ given as

$$\mathcal{T}u = \begin{cases} \{u, 1\} & \text{if } u \in \mathcal{C}_{\frac{1}{2},\frac{1}{2}}, \\ \{\frac{3}{2}\} & \text{otherwise.} \end{cases}$$

Further, we notice that $\mathcal{T}$ doesn't satisfy condition $(\mathcal{C}_1)$ of Theorem 2.1, 2.2 and 2.3. But the circle $\mathcal{C}_{\frac{1}{2},\frac{1}{2}}$ is fixed by $\mathcal{T}$.

Now, we present the following results which are more general existence results of fixed circle for a multivalued mapping.

Theorem 2.4 *Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be a circle on $\mathcal{M}$. If a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}u$ satisfying*

$(\mathcal{C}_1)$ $d(u, v) \leq d(u_0, u) - d(u_0, v)$,

$(\mathcal{C}_2)$ $d(u_0, v) \geq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let for each $u \in \mathcal{C}_{u_0,\rho}$ there is $v \in \mathcal{T}u$. Now by using $(\mathcal{C}_1)$, we have

$$d(u, v) \leq d(u_0, u) - d(u_0, v) = \rho - d(u_0, v). \tag{2.4}$$

By the condition $(\mathcal{C}_2)$, the point v may lies on the circle or may be exterior of the circle $\mathcal{C}_{u_0,\rho}$. Then we have two cases if v is an exterior point of circle $\mathcal{C}_{u_0,\rho}$, then we get a contradiction by the inequality (2.4). Therefore we must have $d(u_0, v) = \rho$. Using (2.4), we have

$$d(u, v) \leq \rho - d(u_0, v) = \rho - \rho = 0.$$

This implies

$$d(u, \mathcal{T}u) = \inf\{d(u, v) : v \in \mathcal{T}u\} = 0,$$

and since $\mathcal{T}u$ is closed, we must have, $u \in \mathcal{T}u$ for each $u \in \mathcal{C}_{u_0,\rho}$. Hence, the circle $\mathcal{C}_{u_0,\rho}$ is fixed circle of mapping $\mathcal{T}$. □

Following examples depicts the significance of condition $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ of Theorem 2.4 in the survival of fixed circle for multivalued mappings.

Example 2.12 Let $\mathcal{M} = \mathbb{R}$ be a metric space with metric $d(u, v) = |u - v|$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ given as

$$\mathcal{T}u = \begin{cases} [0, 1] & \text{if } u \in \mathcal{C}_{\frac{3}{2},\frac{1}{2}}, \\ \{\frac{1}{2}\} & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ satisfies condition $(\mathcal{C}_2)$ of Theorem 2.1, but doesn't satisfy condition $(\mathcal{C}_1)$ for $u = 2$. Therefore $\mathcal{T}$ doesn't fix the circle $\mathcal{C}_{\frac{3}{2},\frac{1}{2}}$.

Example 2.13 let $\mathcal{M} = \mathbb{R}$ be a metric space with metric $d(u, v) = |u - v|$ with unit circle $\mathcal{C}_{0,1}$. Define a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ given as

$$\mathcal{T}u = \{0\},$$

for all $u \in \mathcal{M}$. Then $\mathcal{T}$ satisfies condition $(\mathcal{C}_1)$ but doesn't satisfy $(\mathcal{C}_2)$. Hence the unit circle is not fixed by $\mathcal{T}$.

Let $\mathcal{T} : \mathcal{M} \to \mathcal{N}(\mathcal{M})$ be a multivalued mapping. For $\epsilon > 0$, we define a set $\mathcal{T}_\epsilon(u) \subset \mathcal{M}$ as

$$\mathcal{T}_\epsilon(u) = \{v \in \mathcal{T}u : d(u, v) \leq (1 + \epsilon)\mathcal{D}(u, \mathcal{T}u)\}, \quad for\ any\ u \in \mathcal{M}.$$

We derive the following corollary from Theorem 2.4, based on closed set $\mathcal{T}_\epsilon(u)$.

Corollary 2.1 *Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be a circle on $\mathcal{M}$. If a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}_\epsilon(u)$ satisfying*

$(\mathcal{C}_1)$ $\mathcal{D}(u, \mathcal{T}u) \leq \dfrac{d(u_0, u) - d(u_0, v)}{1 + \epsilon}$,

$(\mathcal{C}_2)$ $d(u_0, v) \geq \rho$, *for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.*

Proof: Let for each $u \in \mathcal{C}_{u_0,\rho}$ there is $v \in \mathcal{T}_\epsilon(u)$. Now by using $(\mathcal{C}_1)$, we have

$$(1+\epsilon)\mathcal{D}(u,\mathcal{T}u) \leq d(u_0,u) - d(u_0,v) = \rho - d(u_0,v). \tag{2.5}$$

Since $d(u,v) \leq (1+\epsilon)\mathcal{D}(u,\mathcal{T}u)$, we have

$$d(u,v) \leq \rho - d(u_0,v).$$

Hence, by Theorem 2.4, the circle $\mathcal{C}_{u_0,\rho}$ is fixed by $\mathcal{T}$. □

Theorem 2.5 *Let $(\mathcal{M},d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be a circle on $\mathcal{M}$. If a mapping $\mathcal{T}:\mathcal{M}\to\mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}u$ satisfying*

$(\mathcal{C}_1)$ $d(u,v) \leq d(u_0,u) + d(u_0,v) - 2\rho$,

$(\mathcal{C}_2)$ $d(u_0,v) \leq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by $\mathcal{T}$.

Proof: Let for each $u \in \mathcal{C}_{u_0,\rho}$ there is $v \in \mathcal{T}u$. Now by using $(\mathcal{C}_1)$, we have

$$d(u,v) \leq d(u_0,u) + d(u_0,v) - 2\rho = d(u_0,v) - \rho. \tag{2.6}$$

By the condition $(\mathcal{C}_2)$, the point v may lies on the circle or may be an interior of the circle $\mathcal{C}_{u_0,\rho}$. Then we have two cases, if v is an interior point of circle then we have a contradiction by the inequality (2.6). Therefore we must have $d(u_0,v) = \rho$. Using (2.6), we have

$$d(u,v) \leq d(u_0,v) - \rho = \rho - \rho = 0.$$

This implies

$$d(u,\mathcal{T}u) = \inf\{d(u,v) : v \in \mathcal{T}u\} = 0,$$

and since $\mathcal{T}u$ is closed, we must have $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. Hence, the circle $\mathcal{C}_{u_0,\rho}$ is fixed circle of mapping $\mathcal{T}$. □

The following example illustrates Theorem 2.5.

Example 2.14 Let $\mathcal{M} = \{0, 0.5, 1, 1.5, 2\}$ and $d:\mathcal{M}\times\mathcal{M}\to\mathbb{R}$ be a real valued function given by

$$d(u,v) = \begin{cases} |u| + |u-v| + |v| & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathcal{M},d)$ be a metric space. Take a circle $\mathcal{C}_{2,4} = \{u \in \mathcal{M} : d(2,u) = 4\} = \{0, 0.5, 1, 1.5\}$ and define a multivalued mapping $\mathcal{T}:\mathcal{M}\to\mathcal{C}(\mathcal{M})$ as

$$\mathcal{T}u = \begin{cases} \{u, 1.5\} & \text{if } u \in \mathcal{C}_{2,4}, \\ \{0, 2\} & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ satisfied conditions of the Theorem 2.5. Hence, $\mathcal{T}$ fixes the circle $\mathcal{C}_{2,4}$.

The following corollary is of Theorem 2.5.

Corollary 2.2 *Let $(\mathcal{M},d)$ be a metric space and $\mathcal{C}_{u_0,\rho}$ be a circle on $\mathcal{M}$. If a mapping $\mathcal{T}:\mathcal{M}\to\mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}_\epsilon(u)$ satisfying*

($\mathcal{C}_1$) $\mathcal{D}(u, \mathcal{T}u) \leq \dfrac{d(u_0, u) + d(u_0, v) - 2\rho}{1+\epsilon}$,

($\mathcal{C}_2$) $d(u_0, v) \leq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let for each $u \in \mathcal{C}_{u_0,\rho}$ there is $v \in \mathcal{T}_\epsilon(u)$. Now by using ($\mathcal{C}_1$), we have

$$(1+\epsilon)\mathcal{D}(u, \mathcal{T}u) \leq d(u_0, u) + d(u_0, v) - 2\rho = d(u_0, v) - \rho. \tag{2.7}$$

Since, $d(u, v) \leq (1+\epsilon)\mathcal{D}(u, \mathcal{T}u)$, we have

$$d(u, v) \leq d(u_0, v) - \rho.$$

Hence, by Theorem 2.5, the circle $\mathcal{C}_{u_0,\rho}$ is fixed by multivalued mapping $\mathcal{T}$. □

The following theorem is an extension of Theorem 2.4.

Theorem 2.6 *Let $\mathcal{C}_{u_0,\rho}$ be a circle on the metric space $(\mathcal{M}, d)$ with center u_0 and radius ρ. If a multivalued mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}u$ satisfying*

($\mathcal{C}_1$) $d(u, v) \leq d(u_0, u) - d(u_0, v)$,

($\mathcal{C}_2$) $\beta d(u, v) + d(u_0, v) \geq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$, where $\beta \in [0, 1)$. Then the circle $\mathcal{C}_{u_0,\rho}$ is fixed by the multivalued mapping $\mathcal{T}$.

Proof: Let $u \in \mathcal{C}_{u_0,\rho}$ be any arbitrary point. To show $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. Assume if $u \notin \mathcal{T}u$, for some $u \in \mathcal{C}_{u_0,\rho}$. So, $d(u, v) > 0$, for each $v \in \mathcal{T}u$. By using condition ($\mathcal{C}_1$) and ($\mathcal{C}_2$), we have

$$\begin{aligned} d(u, v) &\leq d(u, u_0) - d(u_0, u) = \rho - d(u_0, v) \\ &\leq \beta d(u, v) + d(v, u_0) - d(u_0, v) \\ &= \beta d(u, v), \end{aligned}$$

which is a contradiction for $\beta \in [0, 1)$. Therefore we have $u \in \mathcal{T}u$, for each $u \in \mathcal{C}_{u_0,\rho}$. □

Example 2.15 Let $\mathcal{M} = \mathbb{R}$ with the metric

$$d(u, v) = \begin{cases} \min\{|u| + |v|, |u-1| + |v-1|\} & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

Take a circle $\mathcal{C}_{1,2}$ on $\mathcal{M}$ and define a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ as

$$\mathcal{T}u = \begin{cases} \{u\} & \text{if } u \in \mathcal{C}_{1,2}, \\ [\frac{7}{2} \frac{9}{2}] & \text{otherwise.} \end{cases}$$

for all $u \in \mathcal{M}$. Clearly, the mapping $\mathcal{T}$ satisfies all the condition of Theorem 2.6. Thus, the circle $\mathcal{C}_{1,2}$ is fixed by $\mathcal{T}$.

Remark 2.6 From the Examples 2.12 and 2.13, we can check the significance of conditions ($\mathcal{C}_1$) and ($\mathcal{C}_2$) of Theorem 2.6.

Corollary 2.3 *Let $\mathcal{C}_{u_0,\rho}$ be a circle on the metric space $(\mathcal{M}, d)$ with center u_0 and radius ρ. If a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ such that there is $v \in \mathcal{T}_\epsilon(u)$ satisfying*

($\mathcal{C}_1$) $\mathcal{D}(u,\mathcal{T}u) \leq \dfrac{d(u_0,u)-d(u_0,v)}{1+\epsilon}$,

($\mathcal{C}_2$) $\beta d(u,\mathcal{T}u)+d(u_0,v) \geq \rho$,

for each point $u \in \mathcal{C}_{u_0,\rho}$, *where* $\beta \in [0,1)$. *Then the circle* $\mathcal{C}_{u_0,\rho}$ *is fixed by the multivalued mapping* $\mathcal{T}$.

To ensure the uniqueness of fixed circle of multivalued mapping $\mathcal{T}$, we can assume one of the following conditions in hypothesis of Theorem 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6.

($\tilde{\mathcal{C}_3}$) $\delta(\mathcal{T}u,\mathcal{T}v) \leq kd(u,v)$, for each $u \in \mathcal{C}_{u_0,\rho}$, $v \in \mathcal{C}_{u_0,\rho}$ or $\mathcal{M}$ and for some $k \in [0,1)$.

($\tilde{\mathcal{C}_3}'$) $\delta(\mathcal{T}u,\mathcal{T}v) \leq k\max\{d(u,v),\mathcal{D}(u,\mathcal{T}u),\mathcal{D}(\mathcal{T}v,v)\}$, for some $k \in [0,1)$ and for each $u \in \mathcal{C}_{u_0,\rho}$, $v \in \mathcal{C}_{u_0,\rho}$ or $\mathcal{M}$.

($\tilde{\mathcal{C}_3}''$) $\delta(\mathcal{T}u,\mathcal{T}v) \leq k\max\{d(u,v),\mathcal{D}(u,\mathcal{T}v),\mathcal{D}(u,\mathcal{T}u),\mathcal{D}(v,\mathcal{T}v),\mathcal{D}(v,\mathcal{T}u)\}$,

for each $u \in \mathcal{C}_{u_0,\rho}$, $v \in \mathcal{C}_{u_0,\rho}$ or $\mathcal{M}$, where $k \in [0,1)$ and $\delta(\mathcal{E},\mathcal{F}) = \sup\{d(u,v) : u \in \mathcal{E}, u \in \mathcal{F}\}$, for all non-empty bounded subsets $\mathcal{E},\mathcal{F} \in \mathcal{M}$.

Theorem 2.7 *Adding one of the condition* ($\tilde{\mathcal{C}_3}$), ($\tilde{\mathcal{C}_3}'$) *and* ($\tilde{\mathcal{C}_3}''$) *to the hypothesis of the Theorem 2.1 (resp., Theorem 2.2, 2.3, 2.4, 2.5 and 2.6), we obtain uniqueness of the fixed circle* $\mathcal{C}_{u_0,\rho}$ *for a multivalued mapping* $\mathcal{T}$.

Proof: Assume that $\mathcal{C}_{u_0,\rho_1}$ and $\mathcal{C}_{u_0,\rho_2}$ are two fixed circle of mapping $\mathcal{T}$. Let $p \in \mathcal{C}_{u_0,\rho_1}$ and $q \in \mathcal{C}_{u_0,\rho_2}$ be any points. By condition ($\tilde{\mathcal{C}_3}$), we obtain

$$d(p,q) \leq \delta(\mathcal{T}p,\mathcal{T}q) \leq kd(p,q).$$

Since $k \in [0,1)$, we have a contradiction. Thus $\mathcal{T}$ fixes only one circle. Similarly we can prove in view of condition ($\tilde{\mathcal{C}_3}'$) and ($\tilde{\mathcal{C}_3}''$). □

3. Application on discontinuous multivalued mappings

Let us recall the notation,

$$\mathcal{N}(p,u) = \max\left\{d(p,u),\mathcal{D}(p,\mathcal{T}p),\mathcal{D}(u,\mathcal{T}u),\frac{\mathcal{D}(p,\mathcal{T}u)+\mathcal{D}(\mathcal{T}p,u)}{2}\right\}.$$

In 1992, Hicks and Rhoades, give a proposition on continuity at fixed point of multivalued mapping, stated as:

Proposition 3.1 [5] *Let a mapping* $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ *with* u *be a fixed point of* $\mathcal{T}$. *Then* $\mathcal{T}$ *is continuous at its fixed point* u *if and only if* $\lim_n \mathcal{D}(v_n,\mathcal{T}v_n) = 0$, *whenever* $\{v_n\} \subset \mathcal{M}$ *with* $\lim_n v_n = u$.

Proposition 3.2 *Let* $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ *be a multivalued mapping on metric space* $(\mathcal{M},d)$ *and the circle* $\mathcal{C}_{u_0,\rho}$ *is fixed by* $\mathcal{T}$. *Then* $\lim_{p\to u} N(p,u) = 0$ *if and only if* $\mathcal{T}$ *is continuous at each* $u \in \mathcal{C}_{u_0,\rho}$.

Proof: Let a mapping $\mathcal{T}$ fixes the circle $\mathcal{C}_{u_0,\rho}$ and $u \in \mathcal{C}_{u_0,\rho}$, $\{p_n\} \subset \mathcal{M}$ such that $p_n \to u$. If $\lim_{p_n\to u} N(p_n,u) = 0$, that is

$$\lim_{p_n\to u} \max\left\{d(p_n,u),\mathcal{D}(p_n,\mathcal{T}p_n),\mathcal{D}(u,\mathcal{T}u),\frac{\mathcal{D}(p_n,\mathcal{T}u)+\mathcal{D}(\mathcal{T}p_n,u)}{2}\right\} = 0.$$

Therefore $\lim_{p_n\to u} \mathcal{D}(p_n,\mathcal{T}p_n) = 0$. Thus $\mathcal{T}$ is continuous at each $u \in \mathcal{C}_{u_0,\rho}$. Conversely, let the multivalued mapping $\mathcal{T}$ is continuous at each point u of its fixed circle $\mathcal{C}_{u_0,\rho}$, then $\mathcal{T}p_n \to \mathcal{T}u = u$ and $\lim_{p_n\to u} \mathcal{D}(p_n,\mathcal{T}p_n) = 0$. Hence we must have $\lim_{p_n\to u} N(p_n,u) = 0$. □

Corollary 3.1 *Let $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ be a multivalued mapping on metric space $(\mathcal{M}, d)$ and the circle $\mathcal{C}_{u_0,\rho}$ is fixed by $\mathcal{T}$. Then $\lim_{p\to u} \max\{d(p,u),\ \mathcal{D}(p,\mathcal{T}p), \mathcal{D}(\mathcal{T}u,u)\} = 0$ if and only if $\mathcal{T}$ is continuous at each $u \in \mathcal{C}_{u_0,\rho}$.*

Remark 3.1 Clearly, form the Proposition 3.2, $\lim_{p\to u} N(p,u) \neq 0$ if and only if $\mathcal{T}$ is discontinuous at each point $u \in \mathcal{C}_{u_0,\rho}$.

Proposition 3.3 *Let $\mathcal{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ be a multivalued mapping on a metric space $(\mathcal{M}, d)$ and the circle $\mathcal{C}_{u_0,\rho}$ is fixed by $\mathcal{T}$. Then*

$$\lim_{p\to u} \max\{d(p,u), \mathcal{D}(p,\mathcal{T}p), \mathcal{D}(\mathcal{T}u,u), \mathcal{D}(p,\mathcal{T}u), \mathcal{D}(\mathcal{T}p,u)\} = 0$$

if and only if $\mathcal{T}$ is continuous at each $u \in \mathcal{C}_{u_0,\rho}$.

Proof: Let $\mathcal{T}$ fixes the circle $\mathcal{C}_{u_0,\rho}$ and let $u \in \mathcal{C}_{u_0,\rho}$, $\{p_n\} \subset \mathcal{M}$ such that $p_n \to u$. If

$$\lim_{p_n\to u} \max\{d(p_n,u), \mathcal{D}(p_n,\mathcal{T}p_n), \mathcal{D}(\mathcal{T}u,u), \mathcal{D}(p_n,\mathcal{T}u), \mathcal{D}(\mathcal{T}p_n,u)\} = 0.$$

Therefore $\lim_{p_n\to u} \mathcal{D}(p_n,\mathcal{T}p_n) = 0$. Thus $\mathcal{T}$ is continuous at each $u \in \mathcal{C}_{u_0,\rho}$. Conversely, let the mapping $\mathcal{T}$ is continuous at each point u of its fixed circle $\mathcal{C}_{u_0,\rho}$, then $\mathcal{T}p_n \to \mathcal{T}u = u$ and $\lim_{p_n\to u} \mathcal{D}(p_n,\mathcal{T}p_n) = 0$. Hence, we must have

$$\lim_{p_n\to u} \max\{d(p_n,u), \mathcal{D}(p_n,\mathcal{T}p_n), \mathcal{D}(\mathcal{T}u,u), \mathcal{D}(p_n,\mathcal{T}u), \mathcal{D}(\mathcal{T}p_n,u)\} = 0.$$

□

Remark 3.2 Clearly, form the Proposition 3.3,

$$\lim_{p\to u} \max\{d(p,u), \mathcal{D}(p,\mathcal{T}p), \mathcal{D}(\mathcal{T}u,u), \mathcal{D}(p,\mathcal{T}u), \mathcal{D}(\mathcal{T}p,u)\} \neq 0$$

if and only if $\mathcal{T}$ is discontinuous at each point $u \in \mathcal{C}_{u_0,\rho}$.

Example 3.1 Let $\mathcal{M} = \mathbb{R}$ be a metric space with usual metric $d(u,v) = |u - v|$. Define a multivalued mapping $\mathcal{T}_1 : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ as

$$\mathcal{T}_1 u = \begin{cases} \{2\} & \text{if } -1 < u \leq 2, \\ \{2u-5\} & \text{if } 2 < u \leq 5, \\ [0,1] & \text{otherwise.} \end{cases}$$

For the circle $\mathcal{C}_{\frac{7}{2},\frac{3}{2}} = \{2,5\}$ the mapping $\mathcal{T}_1$ fulfils the conditions of Theorem 2.1. Hence the circle $\mathcal{C}_{\frac{7}{2},\frac{3}{2}}$ is fixed by mapping $\mathcal{T}_1$. Note that $\mathcal{T}_1$ is discontinuous at $u \in \{-1,2,5\}$ and satisfies Remarks 3.1 and 3.2.

Example 3.2 Let $\mathcal{M} = \mathbb{R}$ be a metric space with usual metric $d(u,v) = |u - v|$. Define a mapping $\mathcal{T}_2 : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ as

$$\mathcal{T}_2 u = \begin{cases} \{1\} & \text{if } 1 \leq u \leq 2, \\ \{2u-3\} & \text{if } 2 < u \leq 3, \\ [\frac{3}{2},\frac{5}{2}] & \text{otherwise.} \end{cases}$$

For the circle $\mathcal{C}_{2,1} = \{1,3\}$ mapping $\mathcal{T}_2$ fulfils the condition of Theorem 2.2. Hence the circle $\mathcal{C}_{2,1} = \{1,3\}$ is fixed by mapping $\mathcal{T}_2$. We can check that $\mathcal{T}_2$ is discontinuous at $u \in \{1,3\}$ and satisfies Remarks 3.1 and 3.2.

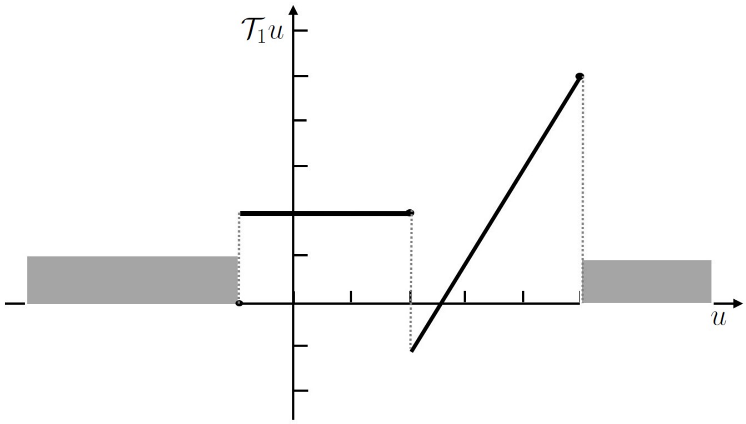

Figure 6: The graph of multivalued discontinuous mapping $\mathcal{T}_1$

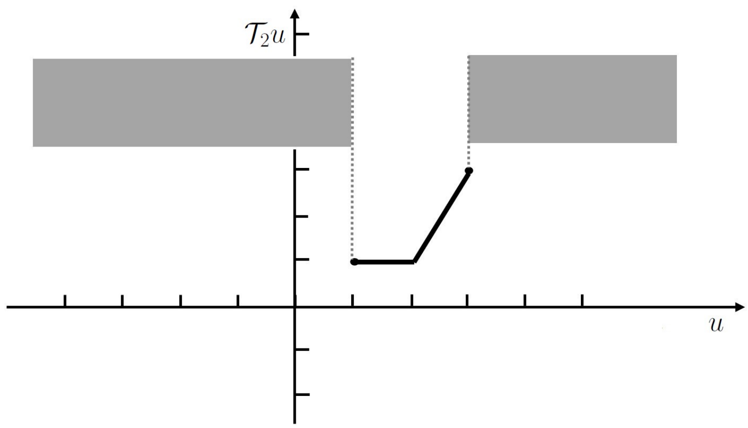

Figure 7: The graph of multivalued discontinuous mapping $\mathcal{T}_2$

4. Conclusion

From both versions of theorems, we observe that if multivalued mapping $\mathcal{T}$ satisfies conditions of Theorem 2.1, 2.2 and 2.3, for each point in the circle $\mathcal{C}_{u_0,\rho}$ of metric space $\mathcal{M}$, then it must satisfy conditions of the Theorem 2.4, 2.5 and 2.6 respectively. But from Examples 2.11 and 2.14 we can say that if $\mathcal{T}$ satisfies conditions of Theorem 2.4, 2.5 and 2.6, then $\mathcal{T}$ need not satisfy the conditions of Theorems 2.1, 2.2 and 2.3 for each point in the circle $\mathcal{C}_{u_0,\rho}$. Hence, Version-II is more general than Version-I.

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