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# Characterizations of Pseudo Quasi-Einstein Spacetimes in Gray's Decomposition

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ABSTRACT: In this study, we analyze pseudo quasi-Einstein spacetimes endowed with Gray's decomposition, as well as generalized Robertson-Walker spacetimes. For pseudo quasi-Einstein spacetimes, we study the form of the Ricci tensor in all the $O(n)$-invariant subspaces provided by Gray's decomposition of the gradient of the Ricci tensor. In all cases we obtain that the integral curves of the velocity vector field are geodesic under certain restrictions except one case. Also it is established that a pseudo quasi-Einstein generalized Robertson-Walker spacetime is a perfect fluid spacetime. Finally pseudo quasi-Einstein perfect fluid spacetimes have been investigated.

Key Words: Pseudo quasi-Einstein spacetimes, Gray's decompositions, geodesic, perfect fluid spacetimes, generalized Robertson-Walker spacetimes.

## Contents



## 1. Introduction

Lorentzian manifold is the subclass of a semi-Riemannian manifold. The index of the Lorentzian metric $g$ is 1. A Lorentzian manifold $M^n$ $(n \geq 4)$ admitting a globally timelike vector is physically known as spacetime. Several authors have explored spacetimes in various ways, such as [2,3,5,6,9,10,12,13,18,21] and many others.

A non-flat semi-Riemannian manifold $(M^n, g)$ $(n \geq 3)$ is called a *pseudo quasi-Einstein manifold* (denoted by $P(QE)_n$) [17] if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta \Pi(X) \Pi(Y) + \gamma D(X,Y), \tag{1.1}$$

where $\alpha$, $\beta$, $\gamma$ are non-zero smooth functions and $\Pi$ is a non-zero 1-form such that $g(X,\mu) = \Pi(X)$ for all vector fields $X$ and $\mu$ being a unit vector field called the generators of the manifold, $D$ is a symmetric $(0,2)$ tensor with vanishing trace and satisfying $D(X,\mu) = 0$ for all vector fields $X$. Also $\alpha$, $\beta$ and $\gamma$ are called the associated scalars; $\Pi$ is the associated 1-form of the manifold and $D$ is called the structure tensor of the manifold.

If the Ricci tensor satisfies (1.1), a Lorentzian manifold is referred to as pseudo quasi-Einstein spacetime. The existence of such a spacetime have been proved by [17]. In this case, the vector field $\mu$ related to the associated 1-form $\Pi$ is treated as a unit timelike vector field, that is, $\Pi(\mu) = g(\mu,\mu) = -1$.

The Weyl conformal curvature tensor of a Lorentzian manifold $(M^n, g)$ $(n \geq 4)$ is stated as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\left[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\right] + \frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{1.2}$$

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$\mathcal{Q}$ is the Ricci operator satisfying the relation $S(X,Y) = g(\mathcal{Q}X,Y)$ and $r$ being the scalar curvature. From the above definition it can be seen that

$$(\text{div}C)(X,Y)Z = \left(\frac{n-3}{n-2}\right)\Big[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} - \frac{1}{2(n-1)}\{g(Y,Z)\,dr(X) - g(X,Z)\,dr(Y)\}\Big]. \tag{1.3}$$

An $n$-dimensional ($n > 2$) Lorentzian manifold is referred to be a generalized Robertson-Walker (briefly, GRW) spacetime if the metric adopts the following local structure:

$$ds^2 = -(d\varepsilon)^2 + q^2(\varepsilon)\, g^*_{u_1u_2} dx^{u_1} dx^{u_2},$$

where $q$ is a $\varepsilon$-dependent function and $g^*_{u_1u_2} = g^*_{u_1u_2}(x^{u_3})$ are only functions of $x^{u_3}$ $(u_1, u_2, u_3 = 2, 3, \ldots, n)$. Thus a GRW spacetime can be represented as $-\mathcal{I} \times_{q^2} \bar{M}$, where $\bar{M}$ is a Riemannian manifold of dimension $(n-1)$. If $\bar{M}$ is of dimension 3 and possesses the constant sectional curvature, then the spacetime shrinks to a Robertson-Walker (briefly, RW) spacetime.
Lorentzian manifolds with the Ricci tensor

$$S(X,Y) = a_1 g(X,Y) + a_2 \Pi(X)\Pi(Y), \tag{1.4}$$

where $a_1 = \kappa\left(\frac{p-\sigma}{2-n}\right)$ and $a_2 = \kappa(p+\sigma)$ are scalars and $\mu$ is a unit timelike vector field corresponding to the 1-form $\Pi$, are called perfect fluid spacetimes. If in particular $a_1$ and $a_2$ are constants, then the geometers called quasi-Einstein spacetimes.
The energy momentum tensor $T$ represents the matter content of the spacetime, which is considered to be fluid. The energy momentum tensor for a perfect fluid spacetime resembles the shape [16]

$$T(X,Y) = pg(X,Y) + (p+\sigma)\Pi(X)\Pi(Y), \tag{1.5}$$

where $\sigma$ stands for energy density and $p$ stands for isotropic pressure. The velocity vector field $\mu$ is the metrically analogous unit timelike vector field to the non-zero 1-form $\Pi$.
The Einstein's field equations (briefly, EFE) without cosmological constant is as follows:

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{1.6}$$

$S$ stands for the Ricci tensor and $r$ stands for the scalar curvature, $\kappa$ is the gravitational constant. According to EFE, the geometry of spacetime is determined by matter, whereas the non-flat metric of spacetime governs matter motion. The above form (1.4) of the Ricci tensor is determined from Einstein's equation using (1.5).
We consider pseudo quasi-Einstein spacetimes in this work, which is a unique kind of spacetime. The Lorentzian setting supports the results obtained for pseudo quasi-Einstein manifolds. The following is the outline of the paper:
After preliminaries in section 3, we investigate each of the seven cases of Gray's decomposition of $P(QE)_n$. The analysis of $P(QE)_n$ with GRW spacetime is presented in section 4. In the last section, we focus at pseudo quasi-Einstein perfect fluid spacetimes.

## 2. Preliminaries

Considering a frame field and contracting $X$ and $Y$ in (1.1), yields

$$r = n\alpha - \beta, \tag{2.1}$$

where $r = \sum_{i=1}^{n} \varepsilon_i S(e_i, e_i)$ is the scalar curvature and $\varepsilon_i = g(e_i, e_i) = \pm 1$.
The covariant derivative of (2.1) and (1.1) gives us

$$dr(Z) = n\,d\alpha(Z) - d\beta(Z) \tag{2.2}$$

and

$$(\nabla_Z S)\,(X,Y) = d\alpha\,(Z)\,g\,(X,Y) + \beta\,[(\nabla_Z \Pi)\,(X)\,\Pi\,(Y) + \Pi\,(X)\,(\nabla_Z \Pi)\,(Y)] \\ + d\beta\,(Z)\,\Pi\,(X)\,\Pi\,(Y) + d\gamma\,(Z)\,D\,(X,Y) + \gamma\,(\nabla_Z D)\,(X,Y)\,. \tag{2.3}$$

Substituting $X = \mu$ in (1.1), we reveal

$$S\,(Y,\mu) = (\alpha - \beta)\,\Pi\,(Y)\,. \tag{2.4}$$

## 3. Gray's Decompositions

According to A. Gray [7], $\nabla S$ can be split into terms that are $O\,(n)$-invariant (for additional information, read [1,11]). In $O\,(n)$-invariant terms, $\nabla S$ can be represented as follows [15]:

$$(\nabla_X S)\,(Y,Z) = \hat{R}\,(X,Y)\,Z + \alpha_1\,(X)\,g\,(Y,Z) + \alpha_2\,(Y)\,g\,(X,Z) + \alpha_2\,(Z)\,g\,(X,Y)\,, \tag{3.1}$$

for all $X$, $Y$, $Z$, where

$$\alpha_1\,(X) = \frac{n}{(n-1)\,(n+2)}\nabla_X r, \quad \alpha_2\,(X) = \frac{(n-2)}{2\,(n-1)\,(n+2)}\nabla_X r$$

with $\hat{R}\,(X,Y)\,Z = \hat{R}\,(X,Z)\,Y$ is the tensor which has vanishing trace and represented as

$$\hat{R}\,(X,Y)\,Z = \frac{1}{3}\left[\hat{R}\,(X,Y)\,Z + \hat{R}\,(Y,Z)\,X + \hat{R}\,(Z,X)\,Y\right] \\ + \frac{1}{3}\left[\hat{R}\,(X,Y)\,Z - \hat{R}\,(Y,X)\,Z\right] + \frac{1}{3}\left[\hat{R}\,(X,Y)\,Z - \hat{R}\,(Z,X)\,Y\right]. \tag{3.2}$$

The decompositions (3.1) and (3.2) yield $O\,(n)$-invariant subspace, which is characterized by linear invariant equations in $\nabla S$.
The following equation can be used to determine the relationship between $\nabla S$ and div$C$:

$$(\mathrm{div}C)\,(X,Y)\,Z = \left(\frac{n-3}{n-2}\right)\left[\hat{R}\,(X,Y)\,Z - \hat{R}\,(Y,X)\,Z\right]. \tag{3.3}$$

The subspaces in Gray's decomposition are as described in the following:

(i) $\nabla S = 0$ characterizes the subspaces that are **trivial**.

(ii) $\hat{R}\,(X,Y)\,Z = 0$ characterizes the **subspace** $\mathcal{J}$, that is,

$$(\nabla_X S)\,(Y,Z) = \alpha_1\,(X)\,g\,(Y,Z) + \alpha_2\,(Y)\,g\,(X,Z) + \alpha_2\,(Z)\,g\,(X,Y)\,, \tag{3.4}$$

where $\alpha_1$, $\alpha_2$ are 1-forms. Manifolds satisfying this requirement (3.4) are called *Sinyukov manifolds* [19].

(iii) The **subspace** $\mathcal{J}'$ (referred to as the subspace $\mathcal{A}$) is defined as follows:

$$(\nabla_X S)\,(Y,Z) + (\nabla_Y S)\,(Z,X) + (\nabla_Z S)\,(X,Y) = 0, \tag{3.5}$$

which yields that the scalar curvature $r$ is constant. Also, the Ricci tensor is Killing [20] if equation (3.5) holds.

(iv) The Ricci tensor is of Codazzi type in **the subspaces** $\mathcal{B}$ **and** $\mathcal{B}'$, that is,

$$(\nabla_X S)\,(Y,Z) = (\nabla_Y S)\,(X,Z)\,. \tag{3.6}$$

(v) The cyclic condition for the Ricci tensor in **the subspace** $\mathcal{J} \oplus \mathcal{A}$ is

$$\begin{aligned}&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\&= \frac{2\,g(Y,Z)}{(n+2)}\,dr(X) + \frac{2\,g(Z,X)}{(n+2)}\,dr(Y) + \frac{2\,g(X,Y)}{(n+2)}\,dr(Z),\end{aligned} \tag{3.7}$$

that is, $S$ is conformal Killing [20].

(vi) The Ricci tensor fulfills the following Codazzi condition in **the subspace** $\mathcal{J} \oplus \mathcal{B}$

$$\nabla_X \left[S(Y,Z) - \frac{r}{2(n-1)} g(Y,Z)\right] = \nabla_Y \left[S(X,Z) - \frac{r}{2(n-1)} g(X,Z)\right], \tag{3.8}$$

which gives $\mathrm{div}C = 0$.

(vii) The scalar curvature is covariant constant in **the subspace** $\mathcal{A} \oplus \mathcal{B}$.

Consider each of these seven cases separately.

**Case (i):** $\nabla S = 0$, therefore from (2.3) we find

$$\begin{aligned}&d\alpha(Z)\,g(X,Y) + \beta\left[(\nabla_Z \Pi)(X)\,\Pi(Y) + \Pi(X)(\nabla_Z \Pi)(Y)\right]\\&+ d\beta(Z)\,\Pi(X)\,\Pi(Y) + d\gamma(Z)\,D(X,Y) + \gamma(\nabla_Z D)(X,Y) = 0.\end{aligned} \tag{3.9}$$

Setting $X = Y = \mu$ in (3.9), we infer that

$$d\alpha(Z) = d\beta(Z). \tag{3.10}$$

Also $\nabla S = 0$ implies that the scalar curvature $r$ is constant.
Hence, from (2.2) and (3.10) we conclude that

$$d\alpha(Z) = 0 = d\beta(Z). \tag{3.11}$$

Using (3.11) in (3.9) infers

$$\beta\left[(\nabla_Z \Pi)(X)\,\Pi(Y) + \Pi(X)(\nabla_Z \Pi)(Y)\right] + d\gamma(Z)\,D(X,Y) + \gamma(\nabla_Z D)(X,Y) = 0. \tag{3.12}$$

Switching $X$ and $Z$ in (3.12), we obtain

$$\beta\left[(\nabla_X \Pi)(Z)\,\Pi(Y) + \Pi(Z)(\nabla_X \Pi)(Y)\right] + d\gamma(X)\,D(Z,Y) + \gamma(\nabla_X D)(Z,Y) = 0. \tag{3.13}$$

Subtracting (3.13) from (3.12) and using the structure tensor $D$ is of Codazzi type, we reveal

$$\begin{aligned}&\beta\left[(\nabla_Z \Pi)(X)\,\Pi(Y) + \Pi(X)(\nabla_Z \Pi)(Y) - (\nabla_X \Pi)(Z)\,\Pi(Y) - \Pi(Z)(\nabla_X \Pi)(Y)\right]\\&+ d\gamma(Z)\,D(X,Y) - d\gamma(X)\,D(Z,Y) = 0.\end{aligned} \tag{3.14}$$

Substituting $Y = \mu$ in (3.14) gives

$$(\nabla_X \Pi)(Z) - (\nabla_Z \Pi)(X) = 0, \tag{3.15}$$

that is, $d\Pi(X,Z) = 0$. This shows that the associated 1-form $\Pi$ is closed. Hence the velocity vector field $\mu$ is irrotational. Consequently, the vorticity of the velocity vector field is zero everywhere.
Replacing $Z$ by $\mu$ in (3.15), we arrive at

$$(\nabla_\mu \Pi)(X) = 0, \quad \text{that is,} \quad g(X, \nabla_\mu \mu) = 0,$$

for all $X$.
Thus we can say that:

**Theorem 3.1** *If a $P(QE)_n$ spacetime is included in the trivial subspace, then the vorticity of the velocity vector field is zero and the integral curves of the velocity vector field $\mu$ are geodesics, provided the structure tensor $D$ is of Codazzi type.*

**Case (ii):** The Ricci tensor satisfies the relation $\hat{R}(X,Y)Z = 0$ in the subspace $\mathcal{J}$ and hence from the relation (3.3) we obtain $\mathrm{div}C = 0$. So we arrive

$$\frac{1}{2(n-1)}\left[dr(X)g(Y,Z) - dr(Z)g(X,Y)\right] = (\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y). \tag{3.16}$$

Equations (2.2), (2.3) and (3.16) together yield

$$\begin{aligned}&\frac{1}{2(n-1)}\left[\{nd\alpha(X) - d\beta(X)\}g(Y,Z) - \{nd\alpha(Z) - d\beta(Z)\}g(X,Y)\right]\\ &= d\alpha(X)g(Y,Z) + d\beta(X)\Pi(Y)\Pi(Z) + \beta[(\nabla_X\Pi)(Y)\Pi(Z)\\ &+\Pi(Y)(\nabla_X\Pi)(Z)] + d\gamma(X)D(Y,Z) + \gamma(\nabla_X D)(Y,Z)\\ &- d\alpha(Z)g(X,Y) - d\beta(Z)\Pi(X)\Pi(Y) - \beta[(\nabla_Z\Pi)(X)\Pi(Y)\\ &+\Pi(X)(\nabla_Z\Pi)(Y)] - d\gamma(Z)D(X,Y) - \gamma(\nabla_Z D)(X,Y).\end{aligned} \tag{3.17}$$

If the associated scalars $\alpha$, $\beta$ are constants and the structure tensor $D$ is of Codazzi type, then from (3.17) we observe that

$$\begin{aligned}&\beta[(\nabla_X\Pi)(Y)\Pi(Z) + \Pi(Y)(\nabla_X\Pi)(Z)] + d\gamma(X)D(Y,Z)\\ &- \beta[(\nabla_Z\Pi)(X)\Pi(Y) + \Pi(X)(\nabla_Z\Pi)(Y)] - d\gamma(Z)D(X,Y) = 0.\end{aligned} \tag{3.18}$$

Setting $Y = \mu$ in (3.18), we reach

$$(\nabla_Z\Pi)(X) - (\nabla_X\Pi)(Z) = 0, \tag{3.19}$$

that is, $d\Pi(Z,X) = 0$. This shows that the associated 1-form $\Pi$ is closed.
The associated 1-form $\Pi$ is closed entails that the velocity vector field $\mu$ is irrotational. Therefore, the velocity vector field has zero vorticity.
Again putting $Z = \mu$ in (3.19), we get

$$(\nabla_\mu\Pi)(X) = 0, \quad \text{that is,} \quad g(X, \nabla_\mu\mu) = 0,$$

for all $X$.
As a result, we may say the following:

**Theorem 3.2** *If a $P(QE)_n$ spacetime belongs to the subspace $\mathcal{J}$, then the velocity vector field is vorticity free and the integral curves of the velocity vector field $\mu$ are geodesics, provided the associated scalars $\alpha$, $\beta$ are constants and the structure tensor $D$ is of Codazzi type.*

**Case (iii):** If $P(QE)_n$ is included in the subspace $\mathcal{A}$, then

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{3.20}$$

from which it follows that the scalar curvature $r$ is constant and hence (2.2) gives us

$$nd\alpha(Z) = d\beta(Z). \tag{3.21}$$

Using (2.3) and (3.21) in (3.20) reveals that

$$\begin{aligned}&d\alpha(X)g(Y,Z) + nd\alpha(X)\Pi(Y)\Pi(Z) + \beta[(\nabla_X\Pi)(Y)\Pi(Z) + \Pi(Y)(\nabla_X\Pi)(Z)]\\ &+ d\gamma(X)D(Y,Z) + \gamma(\nabla_X D)(Y,Z) + d\alpha(Y)g(Z,X) + nd\alpha(Y)\Pi(Z)\Pi(X)\\ &+ \beta[(\nabla_Y\Pi)(Z)\Pi(X) + \Pi(Z)(\nabla_Y\Pi)(X)] + d\gamma(Y)D(Z,X) + \gamma(\nabla_Y D)(Z,X)\\ &+ d\alpha(Z)g(X,Y) + nd\alpha(Z)\Pi(X)\Pi(Y) + \beta[(\nabla_Z\Pi)(X)\Pi(Y) + \Pi(X)(\nabla_Z\Pi)(Y)]\\ &+ d\gamma(Z)D(X,Y) + \gamma(\nabla_Z D)(X,Y) = 0.\end{aligned} \tag{3.22}$$

If the associated scalar $\alpha$ is constant and the structure tensor $D$ is cyclic parallel, then from (3.22) we acquire

$$\beta\left[(\nabla_X\Pi)(Y)\Pi(Z)+\Pi(Y)(\nabla_X\Pi)(Z)\right]+\beta\left[(\nabla_Y\Pi)(Z)\Pi(X)+\Pi(Z)(\nabla_Y\Pi)(X)\right]+d\gamma(X)D(Y,Z)$$
$$+\beta\left[(\nabla_Z\Pi)(X)\Pi(Y)+\Pi(X)(\nabla_Z\Pi)(Y)\right]+d\gamma(Y)D(Z,X)+d\gamma(Z)D(X,Y)=0. \tag{3.23}$$

Replacing $Y$ and $Z$ by $\mu$ in (3.23), we can derive

$$(\nabla_\mu\Pi)(X)=0, \quad \text{that is,} \quad g(X,\nabla_\mu\mu)=0,$$

for all $X$.
Consequently we arrive at the following theorem:

**Theorem 3.3** *If a $P(QE)_n$ spacetime is included in the subspace $\mathcal{A}$, then the integral curves of the velocity vector field $\mu$ are geodesics, provided the associated scalar $\alpha$ is constant and the structure tensor $D$ is cyclic parallel.*

**Case (iv):** If $P(QE)_n$ is included in $\mathcal{B}$ and $\mathcal{B}'$, then

$$(\nabla_X S)(Y,Z)=(\nabla_Z S)(X,Y), \tag{3.24}$$

from which it follows that the scalar curvature $r$ is constant and hence (2.2) implies

$$n d\alpha(Z)=d\beta(Z). \tag{3.25}$$

Hence in view of (2.3), (3.24) and (3.25) we acquire

$$d\alpha(X)g(Y,Z)+nd\alpha(X)\Pi(Y)\Pi(Z)+\beta\left[(\nabla_X\Pi)(Y)\Pi(Z)+\Pi(Y)(\nabla_X\Pi)(Z)\right]$$
$$+d\gamma(X)D(Y,Z)+\gamma(\nabla_X D)(Y,Z)-d\gamma(Z)D(X,Y)-\gamma(\nabla_Z D)(X,Y)$$
$$=d\alpha(Z)g(X,Y)+nd\alpha(Z)\Pi(X)\Pi(Y)+\beta\left[(\nabla_Z\Pi)(X)\Pi(Y)+\Pi(X)(\nabla_Z\Pi)(Y)\right]. \tag{3.26}$$

If the associated scalar $\alpha$ is constant and the structure tensor $D$ is of Codazzi type, then from (3.26), we have

$$\beta\left[(\nabla_X\Pi)(Y)\Pi(Z)+\Pi(Y)(\nabla_X\Pi)(Z)\right]+d\gamma(X)D(Y,Z)$$
$$=\beta\left[(\nabla_Z\Pi)(X)\Pi(Y)+\Pi(X)(\nabla_Z\Pi)(Y)\right]+d\gamma(Z)D(X,Y). \tag{3.27}$$

Setting $Y=\mu$ in (3.27), we reach

$$(\nabla_X\Pi)(Z)-(\nabla_Z\Pi)(X)=0, \tag{3.28}$$

that is, $d\Pi(X,Z)=0$. This shows that the associated 1-form $\Pi$ is closed. Hence the velocity vector field $\mu$ is irrotational. Consequently, the vorticity of the velocity vector field is zero everywhere.
Again putting $Z=\mu$ in (3.28), we arrive

$$(\nabla_\mu\Pi)(X)=0, \quad \text{that is,} \quad g(X,\nabla_\mu\mu)=0,$$

for all $X$.
Hence we can state the following theorem:

**Theorem 3.4** *If a $P(QE)_n$ spacetime is included in the class $\mathcal{B}$ and $\mathcal{B}'$, then the vorticity of the velocity vector field is zero and the integral curves of the velocity vector field $\mu$ are geodesics, provided the associated scalar $\alpha$ is constant and the structure tensor $D$ is of Codazzi type.*

**Case (v):** In this subspace, the Ricci tensor satisfies the equation (3.7). Mantica et al. [15] show that the subspaces $\mathcal{J}\oplus\mathcal{A}$ and $\mathcal{J}$ are equivalent. In this circumstances, we reach div$C=0$. Consequently, the result is the same as in Theorem 3.2.

**Case (vi):** Let $P\left(QE\right)_n$ belong to $\mathcal{J}\oplus\mathcal{B}$. In this case, we obtain $\text{div}C=0$. So in this case, also we get the same outcome of Theorem 3.2.

**Case (vii):** The scalar curvature $r$ is covariant constant in the subspace $\mathcal{A}\oplus\mathcal{B}$, that is, $dr\left(Z\right)=0$, for all $Z$.
Using $dr=0$ in (1.3), we reach

$$\left(\text{div}C\right)\left(X,Y\right)Z=\left(\frac{n-3}{n-2}\right)\left[\left(\nabla_X S\right)\left(Y,Z\right)-\left(\nabla_Y S\right)\left(X,Z\right)\right]. \tag{3.29}$$

Therefore we conclude the following:

**Theorem 3.5** *If a $P\left(QE\right)_n$ spacetime belongs to the subspace $\mathcal{A}\oplus\mathcal{B}$, then the divergence of the conformal curvature tensor vanishes if and only if the Ricci tensor is of Codazzi type.*

## 4. Pseudo quasi-Einstein GRW spacetimes

In this section, we consider the pseudo quasi-Einstein spacetime is a GRW spacetime and then look at the features of $P\left(QE\right)_n$ GRW spacetimes.
In [14], the authors established that a Lorentzian manifold of dimension $n\geq 3$ admits a unit timelike torse forming vector field if and only if it is a GRW spacetime:

$$\left(\nabla_X\Pi\right)\left(Y\right)=\psi\left[g\left(X,Y\right)+\Pi\left(X\right)\Pi\left(Y\right)\right] \tag{4.1}$$

and

$$S\left(X,\mu\right)=\lambda g\left(X,\mu\right), \tag{4.2}$$

for some smooth functions $\psi$ $(\neq 0)$ and $\lambda$ on $M$.
Now,

$$\left(\nabla_X S\right)\left(Y,\mu\right)=XS\left(Y,\mu\right)-S\left(\nabla_X Y,\mu\right)-S\left(Y,\nabla_X\mu\right). \tag{4.3}$$

Using (4.1) and (4.2) in (4.3), we arrive

$$\left(\nabla_X S\right)\left(Y,\mu\right)=\left(X\lambda\right)\Pi\left(Y\right)+\lambda\psi g\left(X,Y\right)-\psi S\left(X,Y\right), \tag{4.4}$$

where $\left(X\lambda\right)=g\left(X,\text{grad}\,\lambda\right)$.
Taking the covariant derivative of (2.4), we reach

$$\left(\nabla_X S\right)\left(Y,\mu\right)=\left[d\alpha\left(X\right)-d\beta\left(X\right)\right]\Pi\left(Y\right)+\left(\alpha-\beta\right)\left(\nabla_X\Pi\right)\left(Y\right). \tag{4.5}$$

Equations (4.1) and (4.5) turn into

$$\left(\nabla_X S\right)\left(Y,\mu\right)=\left[d\alpha\left(X\right)-d\beta\left(X\right)\right]\Pi\left(Y\right)+\psi\left(\alpha-\beta\right)\left[g\left(X,Y\right)+\Pi\left(X\right)\Pi\left(Y\right)\right]. \tag{4.6}$$

Comparing the equations (4.4) and (4.6), we find

$$\left[d\alpha\left(X\right)-d\beta\left(X\right)\right]\Pi\left(Y\right)+\psi\left(\alpha-\beta\right)\left[g\left(X,Y\right)+\Pi\left(X\right)\Pi\left(Y\right)\right]=\left(X\lambda\right)\Pi\left(Y\right)+\lambda\psi g\left(X,Y\right)-\psi S\left(X,Y\right). \tag{4.7}$$

Setting $Y=\mu$ in (4.7) and using (4.2), we turn up

$$\left(X\lambda\right)=d\alpha\left(X\right)-d\beta\left(X\right). \tag{4.8}$$

Now, contraction of (4.7) gives

$$\left(\mu\lambda\right)+n\lambda\psi-\psi r=d\alpha\left(\mu\right)-d\beta\left(\mu\right)+\psi\left(\alpha-\beta\right)\left(n-1\right). \tag{4.9}$$

Equations (2.1), (4.8) and (4.9) together yield

$$\lambda=\left[\left(\frac{2n-1}{n}\right)\alpha-\beta\right]. \tag{4.10}$$

From (4.2) and (4.10), we observe that

$$S(X,\mu)=\left[\left(\frac{2n-1}{n}\right)\alpha-\beta\right]g(X,\mu). \tag{4.11}$$

This means that $\mu$ is an eigenvector corresponding to the eigenvalue $\left[\left(\frac{2n-1}{n}\right)\alpha-\beta\right]$ of $S$.
In view of equations (4.7), (4.8) and (4.10), we acquire

$$S(X,Y)=\alpha\left(\frac{n-1}{n}\right)g(X,Y)-(\alpha-\beta)\Pi(X)\Pi(Y), \tag{4.12}$$

which is a perfect fluid spacetime.
Hence we arrive to the following result:

**Theorem 4.1** *A* $P(QE)_n$ GRW *spacetime is a perfect fluid spacetime.*

According to EFE without cosmological constant, the Ricci tensor becomes

$$S(X,Y)=\kappa\left(\frac{p-\sigma}{2-n}\right)g(X,Y)+\kappa(p+\sigma)\Pi(X)\Pi(Y). \tag{4.13}$$

In contrast to equation (4.12) we notice $\kappa\left(\frac{p-\sigma}{2-n}\right)=\alpha\left(\frac{n-1}{n}\right)$ and $\kappa(p+\sigma)=-(\alpha-\beta)$.
It is known that [14] a 4-dimensional perfect fluid GRW spacetime is a RW spacetime. Guilfoyle and Nolan [8] proved that a four-dimensional perfect fluid spacetime with $p+\sigma\neq 0$ is a Yang pure space if and only if it is a RW spacetime.
Since $\alpha\neq\beta$ in general, so we obtain $p+\sigma\neq 0$.
This leads to the following corollary:

**Corollary 4.1** *A* $P(QE)_4$ GRW *spacetime is a Yang pure space.*

If $\alpha=\beta$, then we infer that $p+\sigma=0$. This represents a dark matter era [4].
Hence, we conclude the result as:

**Corollary 4.2** *A* $P(QE)_n$ GRW *spacetime represents a dark matter era for* $\alpha=\beta$.

## 5. Pseudo quasi-Einstein perfect fluid spacetimes

In this section, we characterize pseudo quasi-Einstein perfect fluid spacetimes.
Equations (1.1), (1.5) and (1.6) together yield

$$\left(\alpha-\frac{r}{2}\right)g(X,Y)+\beta\Pi(X)\Pi(Y)+\gamma D(X,Y)=\kappa\left[pg(X,Y)+(p+\sigma)\Pi(X)\Pi(Y)\right]. \tag{5.1}$$

Replacing $Y$ by $\mu$ in (5.1) entails that

$$\sigma=-\frac{1}{\kappa}\left[\alpha-\beta-\frac{r}{2}\right]. \tag{5.2}$$

Using (2.1) in (5.2) we find

$$\sigma=\frac{1}{\kappa}\left[\left(\frac{n-2}{2}\right)\alpha+\frac{\beta}{2}\right]. \tag{5.3}$$

Contracting the equation (5.1), we acquire

$$n\left(\alpha-\frac{r}{2}\right)-\beta=(n-1)\kappa p-\kappa\sigma. \tag{5.4}$$

In light of (2.1), (5.3) and (5.4) we get

$$p = \frac{1}{\kappa}\left[-\left(\frac{n-2}{2}\right)\alpha + \frac{\beta}{2}\right]. \tag{5.5}$$

Now equations (5.3) and (5.5) together yield

$$p + \sigma = \frac{\beta}{\kappa}. \tag{5.6}$$

Since in a $P(QE)_n$ spacetime $\beta \neq 0$, consequently $p + \sigma \neq 0$.
Hence we write:

**Theorem 5.1** *Every $P(QE)_n$ perfect fluid spacetime cannot admit dark matter era.*

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