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On the $\mathcal{H}_\psi$-quiver metric and the Banach contraction principle

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ABSTRACT: Paper aims to unify in a one-and-only theory several generalizations of the notion of metric as well as its probabilistic versions. We establish that the following well-known metrics are very special cases namely, Ultrametric, b-Metric, Probabilistic b-Metric, Branciari Metric and Probabilistic Generalized Metric. Our approach pushed us to generalize and study the *Banach contraction principle* in the framework of this unification.

Key Words: Ultrametric space, b-Metric space, Probabilistic b-metric space, b-Menger space, Quiver space, Poset, Triangle function, t-norms, t-conorms, Order embedding, Branciari metric space, Probabilistic Generalized Metric space, Ordered field, Banach contraction principle.

Contents

1. Introduction and preliminaries

Let E be a nonempty set. The mapping d defined from $E \times E$ into the set of all non-negative real numbers $\mathbb{R}_+$, is called a *metric* on E, if, the following axioms are satisfied,

- $\forall x, y \in E$, $d(x, y) = 0$, if and only if $x = y$, (*the separation*)
- $\forall x, y \in E$, $d(x, y) = d(y, x)$, (*the symmetry*)
- $\forall x, y, z \in E$, $\quad d(x, y) \leq d(x, z) + d(z, y)$. (*the triangle inequality*)

Then, the ordered pair (E, d) is the so called a *metric* space [14]. In 1944 *Marc Krasner* introduced the notion of *ultrametric* space [13] in which the triangle inequality is strengthened to :

$$\forall x, y, z \in E,\ d(x, y) \leq sup\{d(x, z), d(z, y)\}.$$

It follows that, every ulrametric space is a metric space, indeed,

$$\forall x, y, z \in E,\ d(x, y) \leq sup\{d(x, z), d(z, y)\} \leq d(x, z) + d(z, y).$$

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The idea of the notion of *b-metric* space was initiated from the work of *Bourbaki* [23] and *Bakhtin*. *Czerwik* gave an axiom which was weaker than the triangle inequality and formelly defined a b-metric space [22]. Indeed, the b-metric space is a space in which the triangle inequality is replaced by :

$$\forall x, y, z \in E, \quad d(x, y) \leq s[d(x, z) + d(z, y)],$$

for a given real number s in $[1, \infty)$. It should be noted that the class of b-metric spaces is effectively larger than the class of metric spaces since a b-metric is a metric when $s = 1$.

In 2000 *Branciari* introduced a new metric by substituting the triangle inequality with the *quadrilateral inequality*.

Definition 1.1 *[21] Let E be a nonempty set and let $d : E \times E \to \mathbb{R}_+$, then the pair (E, d) is called the Branciari metric space (or generalized metric space) if its differs from a metric space only in the triangle inequality such that, for all $x, y \in E$ and for all distinct $u, v \in E$ each of them distinct from x and y,*

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y). \textit{ (the quadrilateral inequality)}$$

Then, the map d is called a Branciari metric (or generalized metric).

In 1942 *Karl Menger* introduced the notion of a *statistical* metric space [13] as natural generalisation of metric space in which the distance $d(x, y)$ between two points x and y is replaced by a distribution function f_{xy}. The value $f_{xy}(t)$ of this function at every real number t can be interpreted as the probability that the distance between x and y is less than t. For the historical details, as well as for the motivations behind the introduction of *Probabilistic metric spaces*, the reader should refer to the book by *Schweizer* and *Sklar* [1].

More precisely, let $\mathbb{R}$ be the set of all real numbers, and let :

$$\Delta^+ = \{f : \mathbb{R} \longrightarrow [0, 1] \mid f \text{ is nondecreasing, left-continuous, and } f(0) = 0\}$$

The axiomatic characterization of a probabilistic metric space is quite similar to that of a metric space. In such a space the range of the distance function is the set Δ^+ rather than the set of non-negative real numbers and a suitable *semigroup* operation defined on Δ^+ replaces the operation of addition in the *triangle inequality*. The set Δ^+ has a *natural partial order*, namely, $f \leq h$ if and only if $f(t) \leq h(t)$, for every t. The greatest element in Δ^+ with respect to this order is the *Heavyside* distribution function defined by:

$$\varepsilon_\infty(t) = \begin{cases} 0 & \text{if } t \text{ is negative (or null)} \\ 1 & \text{if } t \text{ is positive} \end{cases}$$

Remark 1.1 Since any function f in Δ^+ is equal to zero on $]-\infty, 0]$, we can consider the set Δ^+ consisting of non-decreasing functions f defined on $[0, \infty[$ that satisfy $f(0) = 0$ which are left-continous on $]0, \infty[$. Notice that we can extend these functions to $[0, \infty]$ by adding the condition $f(\infty) = 1$. Such functions are called a *distance distributive functions*.

Definition 1.2 *A commutative, associative and nondecreasing mapping T from $[0, 1] \times [0, 1]$ into $[0, 1]$ is called a t-norm if and only if,*

$$T(a, 1) = a \quad \textit{for all} \quad a \in [0, 1].$$

Example 1.1 We mention the three typical examples of t-norms as follows: for all $a, b \in [0, 1]$,

- $T_P(a, b) = ab$,
- $T_{Min}(a, b) = Min(a, b)$,
- $T_L(a, b) = Max(a + b - 1, 0)$.

Definition 1.3 *A probabilistic semi-metric space (briefly, a semi-PM space) is an ordered pair* $(E, \mathcal{F})$ *where* E *is a set, and* $\mathcal{F}$ *is a mapping from* $E \times E$ *into* Δ^+ *such that for all pairs of points* x *and* y *in* E*:*

1. $\mathcal{F}(x, y) = \varepsilon_\infty$ *if and only if* $x = y$*,*
2. $\mathcal{F}(x, y) = \mathcal{F}(y, x)$*.*

The function $\mathcal{F}(x, y)$ is usually denoted by f_{xy}, and $f_{xy}(t)$, its value at a real number t, is interpreted as the probability that the distance between x and y is less than t.

Definition 1.4 *A two place function* τ *mapping* $\Delta^+ \times \Delta^+$ *into* Δ^+ *is called a triangle function if, for all* $f, h,$ *and* l *in* Δ^+*,*

1. $\tau(f, \varepsilon_\infty) = f$*, (boundry condition)*
2. $\tau(f, h) \geq \tau(f, l)$ *whenever* $h \geq l$*, (monotonicity)*
3. $\tau(f, h) = \tau(h, f)$*, (commutativity)*
3. $\tau(f, \tau(h, l)) = \tau(\tau(f, h), l)$*. (associativity)*

For any argument t, the value of the distribution function $\tau(f, h)$ at t is denoted by $\tau(f, h)(t)$. A t-norm T is called *continuous* if it's continuous as a function, in the usual interval topology on $[0, 1] \times [0, 1]$, (Similarly for left- and right-continuity). Therefore, if, T is left-continuous t-norm, then the operation τ_T from $\Delta^+ \times \Delta^+$ into Δ^+ defined by :

$$\tau_T(t) = sup\{T(f(u), h(v)) \mid u + v = t\}$$

is a triangle function [5].

Definition 1.5 *[1] Let* $(E, \mathcal{F})$ *be a semi-PM space, and let* τ *be a triangle function. Then* $(E, \mathcal{F}, \tau)$ *is a probabilistic metric space under* τ *if, for every triple of points* $x, y,$ *and* z *in* E*:*

$$f_{xy} \geq \tau(f_{xz}, f_{zy}). \quad (\textit{the probabilistic triangle inequality})$$

If, this is the case, then we say that $(E, \mathcal{F}, \tau)$ *is a probabilistic metric space.*

Definition 1.6 *[1] Let* $(E, \mathcal{F}, \tau)$ *be a probabilistic metric space and* $\tau = \tau_T$ *for a t-norm* T*. Then* $(E, \mathcal{F}, \tau)$ *is the so called Menger space.*

Definition 1.7 *[18] Let* $(E, \mathcal{F})$ *be a semi-PM space,* τ *is a triangle function and let* $s \geq 1$ *be a real number. Then a quadruple* $(E, \mathcal{F}, \tau, s)$ *is called a probabilistic b-metric space if, for every triple of points* x, y, z *in* E *and for all* $t > 0$ *:*

$$f_{xy}(st) \geq \tau(f_{xz}, f_{zy})(t).$$

If, $\tau = \tau_T$ *for some t-norm* T*, then* $(E, \mathcal{F}, \tau, s)$ *is called a b-Menger space.*

Definition 1.8 *[16] Let* $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that,* $\delta(t) < t$ *for* $t > 0$*, and* ρ *be a selfmap of a probabilistic b-metric space* $(E, \mathcal{F}, \tau, s)$*. We say that* ρ *is a* δ*-probabilistic contraction if,*

$$\forall x, y \in E \ \textit{ and } \ t > 0, \ f_{\rho(x), \rho(y)}(\delta(t)) \geq f_{x,y}(st).$$

Definition 1.9 *[17] Let* $(E, \mathcal{F})$ *be a semi-PM space, and let* τ *be a triangle function. Then* $(E, \mathcal{F}, \tau)$ *is called a probabilistic generalized metric space if, for all* $x, y \in E$ *and for all distinct points* $u, v \in E$*, each of them distinct from* x *and* y*,*

$$f_{xy} \geq \tau(f_{xu}, \tau(f_{uv}, f_{vy})). \quad (\textit{the probabilistic quadrilateral inequality}).$$

In order theory, an order embedding is a special kind of *monotone function*, which provides a way to include one partially ordered set into another, indeed,

Definition 1.10 *[9] Let $(G,\leq)$ and $(G^{'},\leq^{'})$ be two posets, a function $\phi: G \to G^{'}$ is an order embedding if ϕ is both order-preserving and order-reflecting, i.e,*

$$\forall x,y \in G, \quad x \leq y \Leftrightarrow \phi(x) \leq^{'} \phi(y).$$

An order embedding is necessarily *injective*, indeed, for all x,y in G,

$$\begin{aligned}\phi(x)=\phi(y) \quad &\Leftrightarrow \quad \phi(x) \leq^{'} \phi(y) \quad \textit{and} \quad \phi(y) \leq^{'} \phi(x)\\ &\Leftrightarrow \quad x \leq y \quad \textit{and} \quad y \leq x\\ &\Leftrightarrow \quad x=y.\end{aligned}$$

Then, an *order isomorphism* can be characterized as a surjective order embedding. As a consequence, any order embedding ϕ restricts to an isomorphism between its domain G and its image $Rang(\phi)$. Since ϕ is an injective mapping then,

$$\forall x,y \in G, \ x < y \Leftrightarrow \phi(x) <^{'} \phi(y).$$

Another characterization of order isomorphisms is that they are exactly *the monotone bijections* that have a monotone inverse. If ϕ is an order isomorphism, then so is its inverse function. if ϕ is an order isomorphism from $(G,\leq)$ to $(G^{'},\leq^{'})$ and ψ is an order isomorphism from $(G^{'},\leq^{'})$ to $(G^{''},\leq^{''})$, then the function composition of $\psi\circ\phi$ is itself an order isomorphism, from $(G,\leq)$ to $(G^{''},\leq^{''})$. When an additional algebraic structure is imposed on the partially ordered sets $(G,\leq)$ and $(G^{'},\leq^{'})$, a function from G to $G^{'}$ must satisfy additional properties to be regarded as an isomorphism.

Definition 1.11 *An order isomorphism ϕ from a poset G to itself is called an order automorphism.*

Definition 1.12 *[9] A well-order relation $\leq$ on a set G is a total order on G with the property that every nonempty subset of G has a least element in this ordering. The set G together with the well-order relation is then called a well-ordered set.*

2. Main results

The purpose of this paper is to unify all these generalizations even for probabilistic versions. Indeed, our approach is based on the *quiver spaces* [10] which is a recent generalization of metric spaces. In such spaces, the range of the metric is a given nonempty partially ordered magma $(G,\leq,*)$ which contains the least element e_0. By generalizing the triangle inequality for a quiver space we establish that the following metrics are very special cases namely, Ultrametric, b-Metric, Probabilistic b-Metric, Branciari Metric and Probabilistic Generalized Metric. Our approach led us to generalize and study the *Banach contraction principle* in the framework of this theory. Throughout this work, we adopt the following notation :

- $\mathbb{N}$ is the set of all non-negative integers,
- $\mathbb{Z}$ is the set of all integers,
- $\mathbb{Q}$ is the set of all rational numbers,
- i_G : is the *identity* transformation of the set G,
- $Rang(\phi)$: is the range of the map ϕ,
- $A - B$: is the set of elements in A but not in B.
- ln : is the logarithm function.
- exp : is the exponential function.

For reasons and motivations mainly related to particle physics namely, the phenomenon of *quantum entanglement* [2], we have recently introduced the notion of the *quiver space* [10] as a generalization of metric space, indeed,

Definition 2.1 *Let E be a nonempty set and $(G, \leq, *)$ be a nonempty poset magma which contains the least element e_0. Let $\mathcal{F}$ be a mapping from $E \times E$ into G such that, $(x, y) \mapsto \mathcal{F}(x, y) = f_{x,y}$, then $(E, \mathcal{F})$ is called a quiver space on $(G, \leq, *)$, if the following three axioms are satisfied :*

- $\forall x, y \in E, \quad f_{x,y} = e_0$ *if and only if* $x = y$,
- $\forall x, y \in E, \quad f_{x,y} = f_{y,x}$,
- $\forall x, y, z \in E, \quad f_{x,y} \leq f_{x,z} * f_{z,y}$.

Having introduced the necessary terms, we now turn to our main topic. Let $(G, \leq, *)$ be a nonempty poset magma which contains the least element e_0 and let $\mathcal{F}$ be a mapping defined from $E \times E$ into $(G, \leq, *)$, such that, $\forall x, y \in E$, $\mathcal{F}(x, y) = f_{x,y}$. We consider that, $(G^{'}, \preceq)$ is a nonempty poset, ϕ is a map from G to $G^{'}$, and let $\mathcal{F}_\phi$ be the mapping defined by :

$$\mathcal{F}_\phi : E \times E \to (G^{'}, \preceq)$$

$$(x, y) \mapsto \mathcal{F}_\phi(x, y) = (\phi \circ \mathcal{F})(x, y) = \phi(f_{x,y}) = c_{x,y}$$

Proposition 2.1 *Let $(E, \mathcal{F})$ be a quiver space on $(G, \leq, *)$ with the least element e_0, and let ϕ be an order embedding from $(G, \leq)$ into $(G^{'}, \preceq)$, such that, $\forall x, y \in G$, $\phi(x * y) \preceq \phi(x) \perp \phi(y)$, where $\perp$ is an operation on Rang(ϕ). Therefore, $(E, \mathcal{F}_\phi)$ is a quiver space on (Rang$(\phi), \preceq, \perp)$ with the least element $\phi(e_0)$.*

Proof: Let $(E, \mathcal{F})$ be a quiver space on $(G, \leq, *)$, suppose that, ϕ is an order embedding from $(G, \leq)$ into $(G^{'}, \preceq)$, therefore,

- Let $x^{'}$ be in *Rang*(ϕ), then, there exists a unique $x \in G$ such that, $\phi(x) = x^{'}$. Since ϕ is an order embedding from $(G, \leq)$ into $(G^{'}, \preceq)$, then, $e_0 \leq x \Leftrightarrow \phi(e_0) \preceq \phi(x) = x^{'}$, it follows that, $\phi(e_0)$ is the least element of (*Rang*$(\phi), \preceq)$,
- Since ϕ is injective, then,

$$\forall x, y \in E,\ c_{x,y} = \phi(e_0) \Leftrightarrow \phi(f_{x,y}) = \phi(e_0) \Leftrightarrow f_{x,y} = e_0 \Leftrightarrow x = y,$$

- $\forall x, y \in E$, $c_{x,y} = c_{y,x} \Leftrightarrow \phi(f_{x,y}) = \phi(f_{y,x}) \Leftrightarrow f_{x,y} = f_{y,x}$,
- For all x, y, z in E,

$$\begin{aligned} f_{x,y} \leq f_{x,z} * f_{z,y} \quad &\Leftrightarrow \quad \phi(f_{x,y}) \preceq \phi(f_{x,z} * f_{z,y}) \\ &\Rightarrow \quad \phi(f_{x,y}) \preceq \phi(f_{x,z}) \perp \phi(f_{z,y}) \\ &\Leftrightarrow \quad c_{x,y} \preceq c_{x,z} \perp c_{z,y}. \end{aligned}$$

Since $\perp$ it's an operation on *Rang*(ϕ), it follows that, $(E, \mathcal{F}_\phi)$ is a quiver space on (*Rang*$(\phi), \preceq, \perp)$ with the least element $\phi(e_0)$. □

Corollary 2.1 *Let $(E, \mathcal{F})$ be a quiver space on $(G, \leq, *)$ with the least element e_0, ϕ is an order embedding from $(G, \leq)$ into $(G^{'}, \preceq)$, and let $\perp$ be the operation on Rang(ϕ), defined by, for all x, y in Rang(ϕ),*

$$x \perp y = \phi(\phi^{-1}(x) * \phi^{-1}(y)),$$

where ϕ^{-1} is the inverse of ϕ from Rang(ϕ) into G, therefore, the following are equivalent,

(i) $(E, \mathcal{F})$ *is a quiver space on* $(G, \leq, *)$ *with the least element* e_0.

(ii) $(E, \mathcal{F}_\phi)$ *is a quiver space on* (*Rang*$(\phi), \preceq, \perp)$ *with the least element* $\phi(e_0)$.

Remark 2.1 Since ϕ is an injective mapping, then the previous operation $\perp$ on $Rang(\phi)$ is well-defined. In the sequal, the operation $\perp$ will play an important role for the proofs of most theorems.

Now, we will introduce a new space as a generalization of the quiver space which will be the main object of our study.

Definition 2.2 *Let ψ be an order embedding from $(G, \leq)$ to itself then, $(E, \mathcal{F})$ is called a 0_ψ quiver space on $(G, \leq, *)$ with the least element e_0, if, the following axioms are satisfied :*

- $\forall\, x, y \in E, \quad f_{x,y} = e_0$ *if and only if* $x = y$, *(the generalized separation)*
- $\forall\, x, y \in E, \quad f_{x,y} = f_{y,x}$, *(the symmetry)*
- $\forall\, x, y, z \in E, \quad \psi(f_{x,y}) \leq f_{x,z} * f_{z,y}$. *(the generalized triangle inequality)*

Then, the function $\mathcal{F}$ is called the $\mathcal{H}_\psi$-quiver metric on E.

Remark 2.2 Let $(E, \mathcal{F})$ be a quiver space on $(G, \leq, *)$ with the least element e_0, then, $(E, \mathcal{F})$ is a 0_{i_G} quiver space on $(G, \leq, *)$ with the least element e_0.

Note that, we can define several classes of a 0_ψ quiver space according to the choice of the actions of ψ in the generalized triangle inequality, indeed,

Definition 2.3

- *$(E, \mathcal{F})$ is called a 1_ψ quiver space on $(G, \leq, *)$ with the least element e_0, if its differs from a 0_ψ quiver space only in the third property such that,*

$$\forall\, x, y, z \in E, \quad f_{x,y} \leq \psi(f_{x,z} * f_{z,y}),$$

- *$(E, \mathcal{F})$ is called a 2_ψ quiver space on $(G, \leq, *)$ with the least element e_0, if its differs from a 0_ψ quiver space only in the third property such that,*

$$\forall\, x, y, z \in E, \quad f_{x,y} \leq \psi(f_{x,z}) * \psi(f_{z,y}),$$

- *$(E, \mathcal{F})$ is called a 3_ψ quiver space on $(G, \leq, *)$ with the least element e_0, if its differs from a 0_ψ quiver space only in the third property such that,*

$$\forall\, x, y, z \in E, \quad \psi(f_{x,y}) \leq \psi(f_{x,z}) * \psi(f_{z,y}).$$

Proposition 2.2 *Let ψ be an order automorphism of $(G, \leq)$, where ψ^{-1} is the inverse of ψ. Then, the following are equivalent,*

(i) $(E, \mathcal{F})$ *is a* 0_ψ *quiver space on* $(G, \leq, *)$ *with the least element* e_0.

(ii) $(E, \mathcal{F})$ *is a* $1_{\psi^{-1}}$ *quiver space on* $(G, \leq, *)$ *with the least element* e_0.

If, in addition, ψ is an automorphism of $(G, *)$ which means for all x, y in G, $\psi(x * y) = \psi(x) * \psi(y)$, it follows that,

- *$(E, \mathcal{F})$ is a 1_ψ quiver space on $(G, \leq, *)$ with the least element e_0, it's equivalent to $(E, \mathcal{F})$ is a 2_ψ quiver space on $(G, \leq, *)$ with the least element e_0,*
- *$(E, \mathcal{F})$ is a 3_ψ quiver space on $(G, \leq, *)$ with the least element e_0, it's equivalent to $(E, \mathcal{F})$ is a quiver space on $(G, \leq, *)$ with the least element e_0.*

Proof: Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$ with the least element e_0, since ψ is an order automorphism of $(G, \leq)$, then so is its inverse function ψ^{-1}, therefore, for all x, y and $z \in E$,

$$\psi(f_{x,y}) \leq f_{x,z} * f_{z,y} \Leftrightarrow f_{x,y} \leq \psi^{-1}(f_{x,z} * f_{z,y}).$$

If, ψ is an automorphism of $(G,*)$ it follows that, for all x,y and $z\in E$,

$$f_{x,y}\leq \psi(f_{x,z}*f_{z,y})=\psi(f_{x,z})*\psi(f_{z,y})$$

And for all x,y and $z\in E$,

$$\psi(f_{x,y})\leq \psi(f_{x,z})*\psi(f_{z,y})=\psi(f_{x,z}*f_{z,y})\Leftrightarrow f_{x,y}\leq f_{x,z}*f_{z,y}.$$

□

Definition 2.4 *Let ψ be an order embedding from $(G,\leq)$ to itself then, $(E,\mathcal{F})$ is called a generalized 0_ψ quiver space on $(G,\leq,*)$ with the least element e_0, if the following axioms are satisfied:*

1. *$*$ is an associative operation,*
2. *$\forall\, x,y\in E,\quad f_{x,y}=e_0$, if and only if, $x=y$,*
3. *$\forall\, x,y\in E,\quad f_{x,y}=f_{y,x}$,*
4. *$\forall\, x,y\in E$ and for all distinct points $u,v\in E$, each of them distinct from x and y,*

$$\psi(f_{xy})\leq f_{xu}*f_{uv}*f_{vy}.$$

Then, the function $\mathcal{F}$ is called the generalized $\mathcal{H}_\psi$-quiver metric on E.

When there is no ambiguity, we will say that $(E,\mathcal{F})$ is a 0_ψ quiver space on $(G,\leq,*)$ without mentioning the least element e_0.

Remark 2.3 Note that, We can also define several classes of the generalized 0_ψ quiver space, for instance, let ψ be an order embedding from $(G,\leq)$ to itself then, $(E,\mathcal{F})$ is called a *generalized* 1_ψ quiver space on $(G,\leq,*)$, if it differs from a generalized 0_ψ quiver space only in the four property such that, $\forall\, x,y\in E$ and for all *distinct* points $u,v\in E$, each of them distinct from x and y,

$$f_{xy}\leq \psi(f_{xu}*f_{uv}*f_{vy}).$$

Now, let T be a given t-norm on $[0,1]$. The equivalent of *the complementary t-conorm* of T [1], for the notion of a triangle function on Δ^+, is given by the notion of the *reverse* operation that we have recently introduced to define a new kind of geometry called *the geometry of operations* [12]. Then, we recall some definitions and results related to the reverse operation that we will use in the sequal.

Definition 2.5 *[1] A commutative, associative and nondecreasing mapping S from $[0,1]\times[0,1]$ into $[0,1]$ is called a t-conorm if and only if,*

$$S(a,0)=a\quad \text{for all}\quad a\in[0,1].$$

The t-conorms are dual to t-norms under the order-reversing operation that assigns $1-x$ to x on $[0,1]$. Given a t-norm T, the complementary t-conorm S is defined by: $S(x,y)=1-T(1-x,1-y)$, $\forall x,y\in[0,1]$. The t-conorms are used to represent logical disjunction in fuzzy logic and union in fuzzy set theory.

Definition 2.6 *[12] Let $(G,\leq)$ be a poset which contains the greatest element e_∞, and τ is an operation on G, then, τ is called a triangle operation on G if, for all f,h, and n in G,*

1. *$f\tau e_\infty=f$,*
2. *$f\tau h\leq f\tau n$ whenever $h\leq n$,*
3. *$(f\tau h)\tau n=f\tau(h\tau n)$,*
4. *$f\tau g=g\tau f$.*

Definition 2.7 *[12] Let $(G^{'}, \leq^{'})$ be a poset which contains the least element e_0, and ν is an operation on $G^{'}$, then, ν is called a co-triangle operation on $G^{'}$ if, is commutative, associative, non-decreasing operation and*

$$\forall f^{'} \in G^{'}, \quad f^{'} \nu e_0 = f^{'}.$$

Remark 2.4 Let $(G, \leq)$ be a poset which contains the greatest element e_∞, then, if we define the opposite order of $\leq$ on G, by,

$$\forall x, y \in G, \quad x \leq^{opp} y \Leftrightarrow y \leq x$$

Therefore $(G, \leq^{opp})$ contains the least element e_0^{opp} which is equal to e_∞, it follows that, every triangle (resp co-triangle) operation τ on $(G, \leq)$ is a co-triangle (resp triangle) operation on $(G, \leq^{opp}$).

Let (G, τ) be a nonempty magma, and θ is an injective mapping from G into a given nonempty set $G^{'}$, then, θ restricts to a bijective mapping between its domain G and its image $Rang(\theta)$, the *reverse* operation [12] of τ on $Rang(\theta)$ under θ is the operation $\mathcal{H}_\theta(\tau)$ defined by:

$$\mathcal{H}_\theta(\tau) : Rang(\theta) \times Rang(\theta) \to Rang(\theta)$$

$$(x, y) \mapsto x\mathcal{H}_\theta(\tau)y = \theta(\theta^{-1}(x)\tau\theta^{-1}(y))$$

Theorem 2.1 *[12] Let G be a non empty set, τ is a given operation on G, and θ is an injective mapping from G into a given non empty set $G^{'}$, therefore, there exists a unique operation $\mathcal{H}_\theta(\tau)$ on $Rang(\theta)$ that makes θ a homomorphism from (G, τ) to $(Rang(\theta), \mathcal{H}_\theta(\tau))$.*

Remark 2.5 [12] Since $\forall x, y \in G$, $\theta(x\tau y) = \theta(x)\mathcal{H}_\theta(\tau)\theta(y)$, it follows that, $\forall x, y \in Rang(\theta)$, $\theta^{-1}(x\mathcal{H}_\theta(\tau)y) = \theta^{-1}(x)\tau\theta^{-1}(x)$, therefore, $\mathcal{H}_{\theta^{-1}}(\mathcal{H}_\theta(\tau)) = \tau$.

Theorem 2.2 *[12] Let G be a nonempty set, τ, ν are two operations on G, and let θ be an injective mapping from G into $G^{'}$, therefore,*

- *(G, τ, ν) is a field $\Leftrightarrow$ $(Rang(\theta), \mathcal{H}_\theta(\tau), \mathcal{H}_\theta(\nu))$ is a field,*
- *If, θ is bijective from G to $G^{'}$, therefore, (G, τ) is isomorphic to $(G^{'}, \mathcal{H}_\theta(\tau))$.*

Example 2.1 (The reverse operation) Let $\mathcal{P}(G)$ be the power set of a nonempty set G, θ is the involutory function from $\mathcal{P}(G)$ to itself, such that, $\forall A \in \mathcal{P}(G)$, $\theta(A) = \overline{A}$ where $\overline{A}$ is the complement set of A. Therefore,

$$\forall A, B \in \mathcal{P}(G),\ A \subseteq B \Leftrightarrow \theta(A) \subseteq^{opp} \theta(B).$$

Then, θ is an order isomorphism from $(\mathcal{P}(G), \subseteq)$ into $(\mathcal{P}(G), \subseteq^{opp})$, and since $(\mathcal{P}(G), \Delta, \cap)$ is a ring [5], where Δ is the symmetric difference, therefore, $(\mathcal{P}(G), \mathcal{H}_\theta(\Delta), \mathcal{H}_\theta(\cap))$ is also a *ring*, such that,

$$\begin{aligned} \forall A, B \in \mathcal{P}(G), \quad A\mathcal{H}_\theta(\Delta)B &= \theta(\theta^{-1}(A)\Delta\theta^{-1}(B)) \\ &= \theta(\overline{A}\Delta\overline{B}) \\ &= \theta(A\Delta B) \\ &= \overline{A\Delta B}. \end{aligned}$$

$$\begin{aligned} \forall A, B \in \mathcal{P}(G), \quad A\mathcal{H}_\theta(\cap)B &= \theta(\theta^{-1}(A)\Delta\theta^{-1}(B)) \\ &= \theta(\overline{A} \cap \overline{B}) \\ &= \theta(A \cup B) \\ &= \overline{A \cup B}. \end{aligned}$$

Since $\theta^2 = i_{\mathcal{P}(G)}$, therefore, $\mathcal{H}_{\theta^2}(\Delta) = \Delta$ and $\mathcal{H}_{\theta^2}(\cap) = \cap$. Notice that, $\mathcal{H}_\theta(\cup) = \cap$, indeed,

$$\begin{aligned} \forall A, B \in \mathcal{P}(G), \quad A\mathcal{H}_\theta(\cup)B &= \theta(\theta^{-1}(A) \cup \theta^{-1}(B)) \\ &= \theta(\overline{A} \cup \overline{B}) \\ &= \theta(\overline{A \cap B}) \\ &= A \cap B. \end{aligned}$$

Then, $\mathcal{H}_\theta(\cup) = \cap$.

Remark 2.6 Suppose that, G is a nonempty set, $+$ is an operation on G, and let θ be an injective mapping from G into $G^{'}$, therefore, if, $(G, +)$ is a group, it follows that, $(Rang(\theta), \mathcal{H}_\theta(+))$ is also a group, where the inverse of $x \in G$ under $+$, is denoted by, $-x$. Thus, the inverse y^{-1} of an element $y \in Rang(\theta)$ under $\mathcal{H}_\theta(\tau)$, is given by, $y^{-1} = \theta(-\theta^{-1}(y))$.

Theorem 2.3 *[12] Let $(G, \leq)$ be a poset which contains the least element e_0, τ is an operation on G, θ is an injective mapping from G into a given nonempty set $G^{'}$, and $\mathcal{H}_\theta(\tau)$ is the reverse operation of τ on $Rang(\theta)$ under θ, and let $\leq_\theta^{'}$ be a binary relation on $Rang(\theta)$ such that,*

$$\forall x^{'}, y^{'} \in Rang(\theta),\ x^{'} \leq_\theta^{'} y^{'} \Leftrightarrow \theta^{-1}(x^{'}) \leq \theta^{-1}(y^{'}).$$

Therefore,

1. *$(Rang(\theta), \leq_\theta^{'})$ is a poset, and $\theta(e_0)$ is the least element of $Rang(\theta)$,*
2. *τ and $\mathcal{H}_\theta(\tau)$ have the same monotonicity,*

Corollary 2.2 *[12] Let $(G, \leq)$ be a nonempty poset, and τ is a triangle operation on G, for all θ an injective mapping from G into a non empty set $G^{'}$, if we define the binary relation $\leq_\theta^{'}$ on $Rang(\theta)$ by:*

$$\forall x^{'}, y^{'} \in Rang(\theta),\ x^{'} \leq_\theta^{'} y^{'} \Leftrightarrow \theta^{-1}(x^{'}) \leq^{opp} \theta^{-1}(y^{'})$$

Therefore, $\mathcal{H}_\theta(\tau)$ is a co-triangle operation on $(Rang(\theta), \leq_\theta^{'})$. The operation $\mathcal{H}_\theta(\tau)$ is called the complementary co-triangle operation on $Rang(\theta)$ under θ of the triangle operation τ on G.

Corollary 2.3 *[12]*

- *A triangle function is a triangle operation on Δ^+,*
- *A triangular norm is a triangle operation on $[0, 1]$,*
- *A triangular conorm is a co-triangle operation on $[0, 1]$.*

Now, we establish some preliminaries results of the order embedding. Let $(G, \leq, *)$ be a nonempty poset magma, we define the following,

$\mathcal{E}mb_i(G, \leq)$: the set of all order embedding ψ from $(G, \leq)$ to itself, where

$$\forall x \in G, \quad \psi(x) \leq x.$$

$\mathcal{E}mb_i(G, *)$: the set of all order embedding from $(G, \leq)$ to itself, where

$$\forall x, y \in G,\ \psi(x * y) \leq \psi(x) * \psi(y).$$

$\mathcal{E}mb(G, *)$: the set of all order embedding from $(G, \leq)$ to itself, such that,

$$\forall x, y \in G,\ \psi(x * y) = \psi(x) * \psi(y).$$

And let,

$$\mathcal{E}mb_i(G, \leq, *) = \mathcal{E}mb_i(G, \leq) \cap \mathcal{E}mb(G, *)$$

Remark 2.7

- Observe that i_G belongs to $\mathcal{E}mb_i(G, \leq, *)$,
- If, ψ is an order automorphism of $(G, \leq)$, then we use the notation $\mathcal{A}ut_i$ instead of $\mathcal{E}mb_i$,
- If, ψ is an order embedding of $(G, \leq)$ such that, $\forall x \in G$, $x \leq \psi(x)$, then we will use the notation $\mathcal{E}mb_s(G, \leq)$, idem for $\mathcal{E}mb_s(G, *)$ and $\mathcal{E}mb_s(G, \leq, *)$, it follows that,

$$\mathcal{E}mb(G, *) = \mathcal{E}mb_i(G, *) \cap \mathcal{E}mb_s(G, *).$$

- We define ψ^n as the n-th iterate of ψ, where $n \in \mathbb{N}$, by : $\psi^0 = i_G$ and $\psi^{n+1} = \psi \circ \psi^n$,
- Let ψ_1, ψ_2 be two order embedding from $(G, \leq)$ into $(G', \leq')$, then, $\psi_1 \leq \psi_2$ (resp. $\psi_1 <' \psi_2$) means that, $\forall x \in G$, $\psi_1(x) \leq' \psi_2(x)$ (resp $\forall x \in G$, $\psi_1(x) <' \psi_2(x)$).

Proposition 2.3 *Let $(G, \leq, *)$ be a nonempty partially ordered magma, ψ is a mapping defined from G to itself, therefore,*

- $(\mathcal{E}mb_i(G, \leq), \circ)$ *is a magma.*
- *For all $\psi \in \mathcal{E}mb_i(G, \leq)$, and for all $n \in \mathbb{N}^*$,* $\quad \psi^n \leq \psi$,
- *For all $\psi \in \mathcal{E}mb_i(G, *)$, and for all $n \in \mathbb{N}$,* $\quad \psi^n \in \mathcal{E}mb_i(G, *)$.

Proof:

- Let ψ, ϕ in $\mathcal{E}mb_i(G, \leq)$. For all $x \in G$,

$$\phi(x) \leq x \Leftrightarrow \psi(\phi(x)) \leq \psi(x) \Leftrightarrow (\psi \circ \phi)(x) \leq x.$$

- By induction, for $n = 1$, $\psi^1 = \psi \leq \psi$. Suppose that the property hold for $n \in \mathbb{N}^*$, therefore, since $\psi \in \mathcal{E}mb_i(G, \leq)$, then, for all x in G, $\psi^{n+1}(x) = \psi^n(\psi(x)) \leq \psi(\psi(x)) \leq \psi(x)$, it follows that, for all $n \in \mathbb{N}^*$, $\psi^n \leq \psi$.
- By induction, for $n = 0$, $\psi^0 = i_G \in \mathcal{E}mb_i(G, *)$, and for $n = 1$, $\psi^1 = \psi$ in $\mathcal{E}mb_i(G, *)$. Suppose that the property hold for $n \in \mathbb{N}$, then, for all x, y in G, $\psi^n(x * y) \leq \psi^n(x) * \psi^n(y)$, since $\psi \in \mathcal{E}mb_i(G, *)$, it follows that, $\forall x, y \in G, \psi^{n+1}(x * y) = \psi(\psi^n(x * y)) \leq \psi(\psi^n(x) * \psi^n(y))$, then, $\forall x, y \in G,\ \psi^{n+1}(x * y) \leq \psi^{n+1}(x) * \psi^{n+1}(y)$. Therefore, ψ^{n+1} in $\mathcal{E}mb_i(G, *)$.

□

Theorem 2.4 *Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$ with the least element e_0, and let ϕ be an order embedding from $(G, \leq)$ to itself. Suppose that, ψ and ϕ commutates, it follows that, $(E, \mathcal{F}_\phi)$ is a 0_{ψ_r} quiver space on $(Rang(\phi), \leq, \mathcal{H}_\phi(*))$ with the least element $\phi(e_0)$ where ψ_r is the restriction of ψ on $Rang(\phi)$. In addition if, $\phi \in \mathcal{E}mb_i(G, \leq)$, then, $(E, \mathcal{F})$ is a $0_{\psi\circ\phi}$ quiver space on $(G, \leq, *)$.*

Proof: Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$, and ϕ is an order embedding from $(G, \leq)$ to itself.

- According to The proposition 2.1 then,
  - e_0 is the least element of $(G, \leq)$ its equivalent to $\phi(e_0)$ is the least element of $(Rang(\phi), \leq)$,
  - $\forall x, y \in E,\ \mathcal{F}_\phi(x, y) = \phi(e_0) \Leftrightarrow x = y \Leftrightarrow f_{x,y} = e_0$,
  - $\forall x, y \in E,\ \mathcal{F}_\phi(x, y) = \mathcal{F}_\phi(y, x) \Leftrightarrow f_{x,y} = f_{y,x}$,

For all x, y, z in E,

$$\begin{aligned}\psi(f_{x,y}) \leq f_{x,z} * f_{z,y} &\Leftrightarrow \phi(\psi(f_{x,y})) \leq \phi(f_{x,z} * f_{z,y})\\ &\Rightarrow \psi_r(\phi(f_{x,y})) \leq \phi(f_{x,z})\mathcal{H}_\phi(*)\phi(f_{z,y})\\ &\Leftrightarrow \psi_r(\mathcal{F}_\phi(x,y)) \leq \mathcal{F}_\phi(x,z)\mathcal{H}_\phi(*)\mathcal{F}_\phi(z,y).\end{aligned}$$

Since ψ is an order embedding from $(G, \leq)$ into itself, and

$$\psi_r(Rang(\theta)) = \psi(Rang(\theta)) = Rang(\psi \circ \phi) = \theta(Rang(\psi)) \subseteq Rang(\theta)$$

then, ψ_r is an order embedding from $(Rang(\theta), \leq)$ into itself, therefore, $(E, \mathcal{F}_\phi)$ is a 0_{ψ_r} quiver space on $(Rang(\phi), \leq, \mathcal{H}_\phi(*))$.

- Let $\phi \in \mathcal{E}mb_i(G, \leq)$, since $(E, \mathcal{F})$ is a 0_ψ quiver space on $(G, \leq, *)$, therefore, for all x, y and $z \in E$,

$$\begin{aligned}\psi(f_{x,y}) \leq f_{x,z} * f_{z,y} &\Leftrightarrow \phi(\psi(f_{x,y})) \leq \phi(f_{x,z} * f_{z,y})\\ &\Rightarrow (\psi \circ \phi)(f_{x,y}) \leq f_{x,z} * f_{z,y}\end{aligned}$$

Then, $(E, \mathcal{F})$ is a $0_{\psi\circ\phi}$ quiver space on $(G, \leq, *)$.

□

Corollary 2.4 *Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$, where ψ in $\mathcal{A}ut_i(G, \leq, *)$, it follows that,*

$$\forall p, q \in \mathbb{N}^*,\ (E, \mathcal{F}_{\psi^p}) \text{ is a } 0_{\psi^q} \text{ quiver space on } (G, \leq, *).$$

Proof:
Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$, where ψ is an order embedding from $(G, \leq)$ to itself. Since for all p, q in $\mathbb{N}^*$, ψ^p commutates with ψ^q, therefore, according to the Theorem 2.4 and by induction its easy to see that,

- If, $\psi \in \mathcal{E}mb_i(G, \leq)$, it follows that, $\forall n \in \mathbb{N}^*$, $(E, \mathcal{F})$ is a 0_{ψ^n} quiver space on $(G, \leq, *)$,
- If, $\psi \in \mathcal{E}mb_i(G, *)$, which means $\mathcal{H}_\phi(*) = *$, then, according to the Theorem 2.4, it follows that, $\forall n \in \mathbb{N}^*$, $(E, \mathcal{F}_{\psi^n})$ is a 0_ψ quiver space on $(Rang(\psi^n), \leq, *)$.

Therefore, since $\psi \in \mathcal{A}ut_i(G, \leq, *)$, it follows that, for all $p, q \in \mathbb{N}^*$,

$$(E, \mathcal{F}_{\psi^p}) \text{ is a } 0_{\psi^q} \text{ quiver space on } (G, \leq, *).$$

□

Remark 2.8

Notice that, for all ψ an order automorphism of $(G, \leq)$,

- $\psi \in \mathcal{A}ut_i(G, \leq) \Leftrightarrow \psi^{-1} \in \mathcal{A}ut_s(G, \leq)$,
- $\psi \in \mathcal{A}ut_i(G, *) \Leftrightarrow \psi^{-1} \in \mathcal{A}ut_s(G, *)$.

Let $\mathcal{E}mb_\leq(G)$ be the set of all order embedding from a nonempty poset $(G, \leq)$ to itself. Let $*$ be a given operation on G, we define the binary relation $[*]$ on $\mathcal{E}mb_\leq(G)$ by, for all $\psi, \varphi \in \mathcal{E}mb_\leq(G)$, $\psi = \varphi[*] \Leftrightarrow$ there exists θ an order automorphism of $(G, \leq)$ such that, $\mathcal{H}_\theta(*) = *$ and $\varphi = \theta \circ \psi \circ \theta^{-1}$.

Proposition 2.4 *The binary relation $[*]$ defined previously on $\mathcal{E}mb_\leq(G)$ is an equivalence relation.*

Proof: Let $*$ be an operation on a nonempty poset $(G, \leq)$. Therefore,

- For all ψ in $\mathcal{E}mb_{\leq}(G)$, $\psi = \psi[*]$, indeed, $\mathcal{H}_{i_G}(*) = *$ and $\psi = i_G \circ \psi \circ i_G^{-1}$.
- Let ψ, φ in $\mathcal{E}mb_{\leq}(G)$ such that, $\psi = \varphi[*]$, then, there exists θ an order automorphism of $(G, \leq)$ such that, $\mathcal{H}_\theta(*) = *$ and $\varphi = \theta \circ \psi \circ \theta^{-1}$, according to the proposition 2.3 it follows that,
$$\mathcal{H}_{\theta^{-1}}(*) = \mathcal{H}_{\theta^{-1}}(\mathcal{H}_\theta(*)) = \mathcal{H}_{i_G}(*) = * \text{ and } \varphi = \theta^{-1} \circ \psi \circ \theta.$$
Therefore, $\varphi = \psi[*]$.
- Suppose that, $\psi = \varphi[*]$ and $\varphi = \phi[*]$ then, there exists θ_1 and θ_2 two order automorphism of $(G, \leq)$ such that, $\varphi = \theta_1 \circ \psi \circ \theta_1^{-1}$ and $\phi = \theta_2 \circ \varphi \circ \theta_2^{-1}$ where $\mathcal{H}_{\theta_1}(*) = \mathcal{H}_{\theta_2}(*) = *$. It follows that,
$$\phi = (\theta_2 \circ \theta_1) \circ \psi \circ (\theta_2 \circ \theta_1)^{-1} \text{ and } \mathcal{H}_{\theta_2 \circ \theta_1}(*) = *.$$
Therefore, $\psi = \phi[*]$, then The binary relation $[*]$ defined on $\mathcal{E}mb_{\leq}(G)$ is an equivalence relation.

□

Remark 2.9 Let $(G, \leq)$ be a nonempty poset. Suppose that G is well-ordered, therefore for all $*$ an operation on G, the only possible class of space on $(G, \leq, *)$ where ψ is an automorphism of G is the quiver space, indeed, the only order automorphism of a given well-ordered set $(G, \leq)$ is the identity transformation of G [9].

3. Examples of a 0_ψ quiver spaces on a poset magma $(G, \leq, *)$

In this section we present some well-known examples of a 0_ψ quiver spaces. A nonempty set G equipped with a binary operation $*$, is a *monoid* [7], if it satisfies the following two axioms:

- For all x, y and z in G, the equation $(x * y) * z = x * (y * z)$ holds. *(associativity)*
- There exists an element e_0 in G such that for every element x in G, the equalities $e_0 * x = x$ and $x * e_0 = x$ hold. *(the identity element)*

Remark 3.1

- A monoid $(G, *)$ whose operation is *commutative* i.e for all x, y in G, $x * y = y * x$, is called a commutative monoid.
- A partial order $\leq$ of a nonempty poset $(G, \leq)$ is called compatible with a given operation $*$ on G, if, for all x, y, z and r in G,
$$x \leq y \text{ and } z \leq r \text{ implies } x * z \leq y * r.$$

In the sequal we consider that,

- $(G, \leq)$ is a nonempty poset which contains the least element e_0,
- $(G, *)$ is a commutative monoid with the identity element e_0,
- The partial order $\leq$ is compatible with the operation $*$.

3.1. Ultrametric metric spaces

Let $\leq$ be the standard less-than-or-equal relation on $\mathbb{R}$, therefore,

- $(\mathbb{R}_+, \leq)$ is a nonempty poset which contains the least element 0,
- $(\mathbb{R}_+, sup)$ is a commutative monoid with the identity element 0,
- $(\mathbb{R}_+, +)$ is a commutative monoid with the identity element 0,
- The partial order $\leq$ is compatible with both operations $+$ and sup.

Proposition 3.1 *Let (E,d) be an ultrametric space. Therefore, (E,d) is a $0_{i_{\mathbb{R}_+}}$ quiver space on $(\mathbb{R}_+, \leq, sup)$.*

Proof: Let (E,d) be an ultrametric space, then,

- $\forall x,y \in E$, $d(x,y)=0$, if and only if $x=y$,
- $\forall x,y \in E$, $d(x,y)=d(y,x)$,
- For all $x,y,z \in E$, $\quad i_{\mathbb{R}_+}(d(x,y)) = d(x,y) \leq sup\{d(x,z), d(z,y)\}$,
- For all x,y in $\mathbb{R}_+$, $\quad i_{\mathbb{R}_+}(sup(x,y)) = sup(x,y) = sup(i_{\mathbb{R}_+}(x), i_{\mathbb{R}_+}(y))$,
- for all x in $\mathbb{R}_+$, $\quad i_{\mathbb{R}_+}(x) = x \leq x$.

It follows that, $i_{\mathbb{R}_+} \in \mathcal{A}ut_i(\mathbb{R}_+, \leq, sup) = \mathcal{A}ut_s(\mathbb{R}_+, \leq, sup)$. □

3.2. b-metric spaces

Proposition 3.2 *Let (E,d) be a b-metric space for a given $s \geq 1$. Then, (E,d) is a 0_{ψ_s} quiver space on $(\mathbb{R}_+, \leq, +)$, such that, ψ_s is the order embedding of $(\mathbb{R}_+, \leq)$ defined by, for all t in $\mathbb{R}_+$, $\psi_s(t) = \frac{t}{s}$, where ψ_s in $\mathcal{A}ut_i(\mathbb{R}_+, \leq, +)$.*

Proof: Let (E,d) be a b-metric space for a given s in $[1,\infty)$, and ψ_s is defined by:

$$\psi_s : \mathbb{R}_+ \to \mathbb{R}_+$$

$$t \mapsto \psi_s(t) = \frac{t}{s}$$

- ψ_s is bijective from $\mathbb{R}_+$ to itself, indeed, $\psi_s \circ \varphi_s = \varphi_s \circ \psi_s = i_{\mathbb{R}_+}$ where $\forall t \in \mathbb{R}_+$, $\varphi_s(t) = st$, and for all $x,y \in \mathbb{R}_+$,
$$x \leq y \Leftrightarrow \frac{x}{s} \leq \frac{y}{s} \Leftrightarrow \psi_s(x) \leq \psi_s(y)$$
Then, ψ_s is an order automorphism of $(\mathbb{R}_+, \leq)$.
- For all x,y in $\mathbb{R}_+$,
$$\psi_s(x+y) = \frac{x+y}{s} = \frac{x}{s} + \frac{y}{s} = \psi_s(x) + \psi_s(y) \quad \text{and} \quad \psi_s(x) = \frac{x}{s} \leq x$$
Hence, $\psi_s \in \mathcal{A}ut_i(\mathbb{R}_+, \leq, +)$.

Since (E,d) is a b-metric space for s, then,

- $\forall x,y \in E, \quad d(x,y) = 0 \Leftrightarrow x = y$,
- $\forall x,y \in E, \quad d(x,y) = d(y,x)$
- For all $x,y,z \in E$,
$$\psi_s(d(x,y)) = \frac{d(x,y)}{s} \leq d(x,z) + d(z,y).$$

Therefore, (E,d) is a 0_{ψ_s} quiver space on $(\mathbb{R}_+, \leq, +)$ where $\psi_s \in \mathcal{A}ut_i(\mathbb{R}_+, \leq, +)$. □

Remark 3.2 For $s=1$, $\psi_s = i_{\mathbb{R}_+}$, it follows that, every metric space (E,d) is a $0_{i_{\mathbb{R}_+}}$ quiver space on $(\mathbb{R}_+, \leq, +)$.

3.3. A functional equation of commutativity

Motivate by the Theorem 2.4, we show some properties of monotone and strictly monotone functions that commutes with ψ_a where a in $\mathbb{R}_+^* - \{1\}$ and ψ_a is the linear function from $\mathbb{R}_+$ to itself such that, $\psi_a(t) = at$. Indeed, let ϕ be a mapping from $\mathbb{R}_+$ to itself that commutes with ψ_a. Therefore, ϕ commutes with ψ_a if and only if,

$$\text{For all } t \text{ in } \mathbb{R}_+,\ \phi(at) = a\phi(t),$$

It is clear that the linear functions are solutions of the previously functional equation. Let $a \in \mathbb{R}_+^* - \{1\}$, suppose that ϕ commutes with ψ_a. For $t = 0$, and since $a \neq 1$, it follows that, $\phi(0) = 0$. On the other hand, $\phi(1) = \frac{\phi(a)}{a}$ cannot be determined (which is normal, since we have seen that all linear functions are suitable). By induction, it's easy to see that, for all t in $\mathbb{R}_+$ and for all n in $\mathbb{N}$, $\phi(a^n t) = a^n\phi(t)$ and $\phi(\frac{t}{a^n}) = \frac{\phi(t)}{a^n}$, ($\phi$ commutes with ψ_a if and only if ϕ commutes with $\psi_{\frac{1}{a}}$). Let $\alpha_\phi(t) = \frac{\phi(t)}{t}$ for all t in $\mathbb{R}_+^*$, it follows that, for all $n \in \mathbb{N}$, and for all t in $\mathbb{R}_+^*$,

$$\alpha_\phi(t) = \alpha_\phi(a^n t) = \alpha_\phi(\frac{t}{a^n}).$$

Notice that, $\forall n \in \mathbb{N}^*,\ \alpha_\phi(1) = \phi(1) = \frac{\phi(a^n)}{a^n} = a^n\phi(\frac{1}{a^n})$. It follows that, for all n in $\mathbb{N}$, $\phi(a^n) = a^{n-1}\phi(a)$ and $\phi(\frac{1}{a^n}) = \frac{\phi(a)}{a^{n+1}}$. Suppose that ϕ is non-decreasing, and a in $(1, \infty)$,

- Let $t > 1$. If, there exists $p \in \mathbb{N}^*$ such that, $t = a^p$, then, $\alpha_\phi(t) = \alpha_\phi(1)$. Otherwise, there exists $p \in \mathbb{N}^*$ such that, t in $]a^p, a^{p+1}[$, it follows that,

$$\begin{aligned} a^p < t < a^{p+1} &\Rightarrow \phi(a^p) \le \phi(t) \le \phi(a^{p+1}) \\ &\Rightarrow a^p\phi(1) \le \phi(t) \le a^{p+1}\phi(1) \\ &\Rightarrow \frac{a^p\phi(1)}{a^{p+1}} \le \frac{\phi(t)}{t} \le \frac{a^{p+1}\phi(1)}{a^p} \\ &\Rightarrow \frac{\alpha_\phi(1)}{a} \le \alpha_\phi(t) \le a\alpha_\phi(1) \\ &\Rightarrow \frac{\phi(a)}{a^2} \le \alpha_\phi(t) \le \phi(a). \end{aligned}$$

  Therefore, for all $t \ge 1$, $\quad \frac{\phi(a)}{a^2}t \le \phi(t) \le \phi(a)t.$

- Let t in $]0, 1[$. If, there exists $m \in \mathbb{N}^*$ such that, $t = \frac{1}{a^m}$, then, $\alpha_\phi(t) = \alpha_\phi(1)$. Otherwise, there exists $m \in \mathbb{N}^*$ such that, t in $]\frac{1}{a^{m+1}}, \frac{1}{a^m}[$, in the similar way we obtain the previous results, which means, for all t in $\mathbb{R}_+^*$, $\frac{\alpha_\phi(1)}{a} \le \alpha_\phi(t) \le a\alpha_\phi(1)$.

  (i) If, $\phi(a) < a$, then, for all t in $\mathbb{R}_+^*$, $\frac{\phi(a)}{a^2} \le \alpha_\phi(t) < a$,
  (ii) If, $\phi(a) > a$, then, for all t in $\mathbb{R}_+^*$, $\frac{1}{a} < \alpha_\phi(t) \le \phi(a)$,
  (iii) If, $\phi(a) = a$, then, for all t in $\mathbb{R}_+^*$, $\frac{1}{a} \le \alpha_\phi(t) \le a$,
  (iv) ϕ is continuous in 0 and if, $\phi(a) > 0$ then, $\lim_{t\to+\infty} \phi(t) = +\infty$.

- It should be noted that, for all t_1, t_2 in $\mathbb{R}_+^*$, such that, $t_1 < t_2$ we have the following : $\frac{1}{a}\alpha_\phi(t_1) \le \alpha_\phi(t_2) \le a\alpha_\phi(t_1)$. Indeed, let $t_1 < t_2$ in $\mathbb{R}_+^*$, if, there exists $p \in \mathbb{N}^*$ such that, $t_2 = a^p t_1$, then, $\alpha_\phi(t_1) = \alpha_\phi(t_2)$. Otherwise, there exists $p \in \mathbb{N}^*$ such that, t_2 in $]a^p t_1, a^{p+1}t_1[$, or t_1 in $]\frac{t_2}{a^{p+1}}, \frac{t_2}{a^p}[$, and since ϕ is non-decreasing therefore, we obtain,

$$\frac{1}{a}\alpha_\phi(t_1) \le \alpha_\phi(t_2) \le a\alpha_\phi(t_1).$$

- For all n in $\mathbb{N}^*$ and for all t in $\mathbb{R}_+^*$,

$$\alpha_{\phi^n}(t) = \alpha_{\phi^{n-1}}(\phi(t))\alpha_\phi(t).$$

For example for $t = 1$, then, for all p in $\mathbb{N}$, $\alpha_{\phi^p}(1) = \frac{\phi^p(a)}{a}$. Since ϕ commutes with ψ_a, then, for all p in $\mathbb{N}$, ϕ^p commutes with ψ_a, it follows that,, for a given p in $\mathbb{N}$ we have, for all $n \in \mathbb{N}$, and for all t in $\mathbb{R}_+^*$,

$$\alpha_{\phi^p}(t) = \alpha_{\phi^p}(a^n t) = \alpha_{\phi^p}(\frac{t}{a^n}).$$

- If, there exits (x_0, y_0) in ${\mathbb{R}_+}^2$, such that, $\phi(x_0 + y_0) = \phi(x_0) + \phi(y_0)$, therefore, for all n in $\mathbb{N}$, the couples $(a^n x_0, a^n y_0)$ and $(\frac{x_0}{a^n}, \frac{y_0}{a^n})$ satisfies the same equation.

Example 3.1 1. Let a in $(1, \infty)$, ϕ is the function defined from $\mathbb{R}_+$ to itself by : $\phi(0) = 0$ and $\phi(t) = a^p$, where p is the unique integer such that, $t \in [a^p, a^{p+1}[$ if $t \geq 1$ and $\phi(t) = \frac{1}{a^p}$ with $t \in [\frac{1}{a^p}, \frac{1}{a^{p-1}}[$ if, $t < 1$. Indeed, for all $t \in [a^p, a^{p+1}[$, it follows that, $at \in [a^{p+1}, a^{p+2}[$, therefore $\phi(at) = a^{p+1} = a\phi(t)$ then, ϕ is a non-decreasing function which commutes with ψ_a.

2. Let $0 < k < a$, and ϕ_k is the function defined from $\mathbb{R}_+$ to itself by : $\phi_k(0) = 0$ and $\phi_k(t) = a^p + k(t - a^p)$, where p is the unique integer such that, $t \in [a^p, a^{p+1}[$ if $t \geq 1$ and $\phi_k(t) = \frac{1}{a^p} + k(t - \frac{1}{a^p})$ with $t \in [\frac{1}{a^p}, \frac{1}{a^{p-1}}[$ if, $t < 1$. Indeed, for all $t \in [a^p, a^{p+1}[$, it follows that, $at \in [a^{p+1}, a^{p+2}[$, therefore $\phi_k(at) = a^{p+1} + k(at - a^{p+1}) = a\phi_k(t)$, then, ϕ_k is a strictly non-decreasing function which commutes with ψ_a.

Remark 3.3 Notice that, if, there exists t_0 in $\mathbb{R}_+^*$ such that, $\alpha_\phi(t_0) < \alpha_\phi(1)$, therefore, there exists $p \in \mathbb{N}^*$ such that, t_0 in $]a^p, a^{p+1}[$, or t_0 in $]\frac{1}{a^{p+1}}, \frac{1}{a^p}[$, then, $0 < \frac{1}{a}\alpha_\phi(1) \leq \alpha_\phi(t_0)$ hence, $0 < \alpha_\phi(t_0)$, it follows that, $1 < \frac{\alpha_\phi(1)}{\alpha_\phi(t_0)} \leq a$, and $\frac{1}{a} \leq \frac{\alpha_\phi(t_0)}{\alpha_\phi(1)} < 1$. On the other hand, if, for example t_0 in $]a^p, a^{p+1}[$, then, for t_a in $\mathbb{R}_+^*$ such that, $\alpha_\phi(t_0)t_a = \phi(a^p) = a^p\phi(1)$, and since ϕ is non-decreasing, then, $\alpha_\phi(t_0)t_a \leq \phi(t_0) = \alpha_\phi(t_0)t_0$, it follows that,

$$a^p < t_a = \frac{\alpha_\phi(1)}{\alpha_\phi(t_0)} a^p \leq t_0 < a^{p+1}.$$

Definition 3.1 *Let (E, d) be a b-metric space for a given $s \in [1, \infty)$ and let ϕ_β be the order automorphism of $\mathbb{R}_+$ such that, $\phi_\beta(t) = \beta t$ where β in $\mathbb{R}_+^*$, therefore, (E, d_{ϕ_β}) is a 0_{ψ_s} quiver space on $(\mathbb{R}_+, \leq, +)$.*

Proof: For all β in $\mathbb{R}_+^*$ and for all t_1, t_2 in $\mathbb{R}_+$,

$$\begin{aligned} t_1 \mathcal{H}_{\phi_\beta}(+) t_2 &= \phi_\beta(\phi_\beta^{-1}(t_1) + \phi_\beta^{-1}(t_2)) \\ &= \phi_\beta(\frac{1}{\beta}t_1 + \frac{1}{\beta}t_2) \\ &= t_1 + t_2 \end{aligned}$$

Therefore, $\mathcal{H}_{\phi_\beta}(+) = +$. Since ϕ_β commutates with ψ_s then, according to the theorem 2.4, it follows that, (E, d_{ϕ_β}) is a 0_{ψ_s} quiver space on $(\mathbb{R}_+, \leq, +)$. □

3.4. The probabilistic b-metric space

Theorem 3.1 *Let θ be an order embedding from a nonempty poset $(G, \leq)$ into a nonempty poset $(G', \preceq)$. Therefore, the following are equivalent,*

(i) $(E, \mathcal{F})$ is a 0_ψ quiver space on $(G, \leq, \tau)$ with the least element e_0.

(ii) $(E, \mathcal{F}_\theta)$ is a 0_φ quiver space on $(Rang(\theta), \preceq, \mathcal{H}_\theta(\tau))$ with the least element $\theta(e_0)$.

Such that, $\varphi = \theta \circ \psi \circ \theta^{-1}$ is the order embedding from $(Rang(\theta), \preceq)$ to itself where θ^{-1} is the inverse of θ from $Rang(\theta)$ into G.

Proof: Let $(E,\mathcal{F})$ be a 0_ψ quiver space on $(G,\leq,\tau)$ with the least element e_0. Let θ be an order embedding from $(G,\leq)$ into a nonempty poset $(G^{'},\preceq)$ and let $\mathcal{F}_\theta$ be the mapping defined by :

$$\mathcal{F}_\theta : E\times E \to (G^{'},\preceq)$$

$$(x,y)\mapsto \mathcal{F}_\theta(x,y) = (\theta\circ\mathcal{F})(x,y) = \theta(f_{x,y}) = f^{'}_{x,y}$$

According to The proposition 2.1,

- e_0 is the least element of $(G,\leq)$ its equivalent to $\theta(e_0)$ is the least element of $(Rang(\theta),\preceq)$,
- $\forall x,y\in E,\ f^{'}_{x,y} = \theta(e_0) \Leftrightarrow \theta(f_{x,y}) = \theta(e_0) \Leftrightarrow f_{x,y} = e_0 \Leftrightarrow x = y.$
- $\forall x,y\in E,\ f^{'}_{x,y} = f^{'}_{y,x} \Leftrightarrow \theta(f_{x,y}) = \theta(f_{y,x}) \Leftrightarrow f_{x,y} = f_{y,x}.$

For all $x,y,z\in E$,

$$\begin{aligned}\psi(f_{x,y})\leq f_{x,z}\tau f_{z,y} &\Leftrightarrow \theta(\psi(f_{x,y}))\preceq\theta(f_{x,z}\tau f_{z,y})\\ &\Leftrightarrow (\theta\circ\psi\circ\theta^{-1})(f^{'}_{x,y})\preceq\theta(f_{x,z})\mathcal{H}_\theta(\tau)\theta(f_{z,y})\\ &\Leftrightarrow \varphi(f^{'}_{x,y})\preceq f^{'}_{x,z}\mathcal{H}_\theta(\tau)f^{'}_{z,y}.\end{aligned}$$

□

If, in addition, θ is surjective, then, $(E,\mathcal{F}_\theta)$ is 0_φ quiver space on $(G^{'},\preceq,\mathcal{H}_\theta(\tau))$ with the least element $\theta(e_0)$ such that, $\varphi = \theta\circ\psi\circ\theta^{-1}$ is the order embedding from $(G^{'},\preceq)$ to itself where θ^{-1} is the inverse of θ from $G^{'}$ into G. In addition, if, $G = G^{'}$ and $\preceq=\leq$, it follows that, $\varphi = \theta\circ\psi\circ\theta^{-1}$ is an order embedding of $(G,\leq)$.

Remark 3.4 According to the corollary 2.2, if, τ is a triangle operation (resp co-triangle operation) on G, then, for all θ an order embedding from $(G,\leq)$ into $(G,\leq^{opp})$, $\mathcal{H}_\theta(\tau)$ is a co-triangle operation (resp triangle operation) on $(Rang(\theta),\leq)$.

Corollary 3.1 *Let $(E,\mathcal{F})$ be a 0_ψ quiver space on $(G,\leq,*)$, if, there exists an θ an order automorphism of $(G,\leq)$ such that, θ commutates with ψ and $\mathcal{H}_\theta(*) = *$, therefore, $(E,\mathcal{F}_\theta)$ is a 0_ψ quiver space on $(G,\leq,*)$.*

Let $\leq$ be the natural partial order on the set of all functions f defined from $\mathbb{R}$ into the interval $[0,1]$, and let,

$$\Delta^- = \{f:\mathbb{R}\longrightarrow[0,1] \mid f \text{ is non-increasing, left-continuous, and } f(0)=1\}.$$

Remark 3.5 Since any function f in Δ^- is equal to 1 on $]-\infty,0]$, we can consider the set Δ^- consisting of non-increasing functions f defined on $[0,\infty[$ that satisfy $f(0)=1$ which are left-continous on $]0,\infty[$. We may also extend these functions to $[0,\infty]$ by adding the condition $f(\infty)=0$.

The set Δ^- has a *natural partial order*, namely, $f\leq h$ if and only if $f(t)\leq h(t)$, for every t. The least element in Δ^- with respect to this order is the function defined by:

$$\epsilon_0(t) = \begin{cases}1 & \text{si } t \text{ est non-positive}\\ 0 & \text{si } t \text{ est strictly non-negative}\end{cases}$$

- $(\Delta^+,\leq^{opp})$ is a nonempty poset which contains the least element ϵ_∞,
- $(\Delta^-,\leq)$ is a nonempty poset which contains the least element ϵ_0,
- (Δ^+,τ) is a commutative monoid with the identity element ϵ_∞,
- For all θ an order isomorphism from $(\Delta^+,\leq^{opp})$ into $(\Delta^-,\leq)$,

$$(\Delta^-,\mathcal{H}_\theta(\tau)) \text{ is a commutative monoid with the identity element } \epsilon_0,$$

- The partial order $\leq^{opp}$ is compatible with the operation τ,
- The partial order $\leq$ is compatible with the operation $\mathcal{H}_\theta(\tau)$,

Proposition 3.3 *Let $s \in [1, +\infty)$, and let τ be a triangle function on Δ^+. Therefore, every probabilistic b-metric space $(E, \mathcal{F}, \tau, s)$ its equivalent to the $(E, \mathcal{F})$ is a 0_{ψ_s} quiver space on $(\Delta^+, \leq^{opp}, \tau)$ with the least element ϵ_∞, such that, ψ_s is the order automorphism in $\mathcal{A}ut_i(\Delta^+, \leq^{opp})$ defined by,*

$$\text{For all } f \text{ in } \Delta^+, \text{ and for all } t \text{ in } \mathbb{R},\ \psi_s(f)(t) = f(st).$$

If, in addition $(E, \mathcal{F}, \tau, s)$ is a b-Menger space, then, ψ_s in $\mathcal{A}ut_i(\Delta^+, \leq^{opp}, \tau)$.

Proof: Let $(E, \mathcal{F}, \tau, s)$ be a probabilistic b-metric space, then, for every triple of points x, y, z in E,

$$\begin{array}{rcl} \forall t \in \mathbb{R},\ f_{xy}(st) \geq \tau(f_{xz}, f_{zy})(t) & \Leftrightarrow & \forall t \in \mathbb{R},\ f_{xy}(st) \leq^{opp} \tau(f_{xz}, f_{zy})(t) \\ & \Leftrightarrow & \psi_s(f_{x,y}) \leq^{opp} f_{xz}\tau f_{zy}. \end{array}$$

Where ψ_s is the order automorphism of $(\Delta^+, \leq^{opp})$ defined by:

$$\forall f \in \Delta^+,\ \psi_s(f)(t) = f(st), \quad \forall t \in \mathbb{R}$$

Indeed, let $s \in \mathbb{R}_+^*$, for all f, g in Δ^+,

$$\begin{array}{rcl} f \leq^{opp} g & \Leftrightarrow & g \leq f \\ & \Leftrightarrow & \forall t \in \mathbb{R},\ g(st) \leq f(st) \\ & \Leftrightarrow & \psi_s(g) \leq \psi_s(f) \\ & \Leftrightarrow & \psi_s(f) \leq^{opp} \psi_s(g). \end{array}$$

Since $\psi_{\frac{1}{s}} \circ \psi_s = \psi_s \circ \psi_{\frac{1}{s}} = i_{\Delta^+}$, then, ψ_s is bijective, and for all f in Δ^+, we have, for all $t \in \mathbb{R}$, $f(t) \leq f(st)$, then, $\psi_s(f) \leq^{opp} f$, which means that, ψ_s belongs to $\mathcal{A}ut_i(\Delta^+, \leq^{opp})$. According to the remark 2.4, $(\Delta^+, \leq^{opp})$ contains the least element e_0^{opp} which is equal to ϵ_∞, then, since $(E, \mathcal{F}, \tau, s)$ is a semi-PM space, it follows that, every probabilistic b-metric space $(E, \mathcal{F}, \tau, s)$ its equivalent to the $(E, \mathcal{F})$ is a 0_{ψ_s} quiver space on $(\Delta^+, \leq^{opp}, \tau)$ with the least element ϵ_∞, where ψ_s in $\mathcal{A}ut_i(\Delta^+, \leq^{opp})$. If, in addition $(E, \mathcal{F}, \tau, s)$ is a b-Menger space, with the t-norm T, then, for all f, g in Δ^+, and for all $t \in \mathbb{R}$,

$$\begin{array}{rcl} \{\psi_s(f)\tau_T\psi_s(f)\}(t) & = & sup\{T[\psi_s(f)(u), \psi_s(g)(v)] \mid u + v = t\} \\ & = & sup\{T[f(su), g(sv)] \mid u + v = t\} \\ & = & sup\{T[f(u^{'}), g(v^{'})] \mid u^{'} + v^{'} = st\} \\ & = & (f\tau_T g)(st) \\ & = & \psi_s(f\tau_T g)(t). \end{array}$$

Therefore, for all f, g in Δ^+, $\psi_s(f)\tau_T\psi_s(f) = \psi_s(f\tau_T g)$, then, $\psi_s \in \mathcal{A}ut_i(\Delta^+, \leq^{opp}, \tau)$. □

Theorem 3.2 *Let $s \in [1, +\infty)$, and τ is a triangle function on Δ^+. Therefore, every probabilistic b-metric space $(E, \mathcal{F}, \tau, s)$ is equivalent to the $(E, \mathcal{F}_\theta)$ is a 0_{φ_s} quiver space on $(\Delta^-, \leq, \mathcal{H}_\theta(\tau))$ with the least element ϵ_0, where $\mathcal{H}_\theta(\tau)$ is the complementary co-triangle operation on Δ^- under θ, of the triangle function τ seen as a triangle operation on Δ^+, such that,*

- *θ is the order isomorphism from Δ^+ into Δ^- defined by :*

$$\forall t \in \mathbb{R},\ \theta(f)(t) = 1 - f(t),$$

- *$\varphi_s = \theta \circ \psi_s \circ \theta^{-1}$ is the order automorphism in $\mathcal{A}ut_i(\Delta^-, \leq)$ defined by, for all $f \in \Delta^-$ and for all $t \in \mathbb{R}$, $\varphi_s(f)(t) = 1 - \psi_s(\theta^{-1}(f))(st)$.*

- *For all $f, g \in \Delta^-, \forall t \in \mathbb{R}$, $\mathcal{H}_\theta(\tau)(f,g)(t) = \tau(\theta^{-1}(f), \theta^{-1}(g))(t)$.*

If, in addition $(E, \mathcal{F}, \tau, s)$ is a b-menger space, then, ψ_s in $\mathcal{A}ut_i(\Delta^-, \leq, \mathcal{H}_\theta(\tau))$.

Proof: Let θ be the mapping defined from Δ^+ into Δ^- such that,

$$\forall t \in \mathbb{R}, \quad \theta(f)(t) = 1 - f(t).$$

Then, $\forall f, h \in \Delta^+, f \leq^{opp} h \Leftrightarrow \theta(f) \leq \theta(h)$, and since θ is bijective, it follows that, θ is an order isomorphism from $(\Delta^+, \leq^{opp})$ into $(\Delta^-, \leq)$. Let $(E, \mathcal{F}, \tau, s)$ be probabilistic b-metric space, according to The proposition 3.3, it follows that, $(E, \mathcal{F}, \tau, s)$ is 0_{ψ_s} quiver space on $(\Delta^+, \leq^{opp}, \tau)$ such that, ψ_s is the order automorphism in $\mathcal{A}ut_i(\Delta^+, \leq^{opp})$, defined by for all f in Δ^+, and for all t in $\mathbb{R}$, $\psi_s(f)(t) = f(st)$, and for all f in Δ^-, $\varphi_s(f) \leq f$, then, according to the theorem 3.1, we conclude that, $(E, \mathcal{F}_\theta)$ is a 0_{φ_s} quiver space on $(\Delta^-, \leq, \mathcal{H}_\theta(\tau))$, such that, $\varphi_s = \theta \circ \psi_s \circ \theta^{-1}$ is the order automorphism in $\mathcal{A}ut_i(\Delta^-, \leq)$, where

- For all $f, g \in \Delta^-$, $f\mathcal{H}_\theta(\tau)g = \theta(\theta^{-1}(f)\tau\theta^{-1}(g))$,
- For all $f \in \Delta^-$ and for all $t \in \mathbb{R}$, $\varphi_s(f)(t) = 1 - \psi_s(\theta^{-1}(f))(t)$.

□

3.5. The probabilistic generalized metric space

Let ψ be an order embedding from $(G, \leq)$ to itself and let $(E, \mathcal{F})$ be a generalized 0_ψ quiver space on $(G, \leq, *)$, θ is an order embedding from $(G, \leq)$ into a nonempty poset $(G^{'}, \preceq)$ and let $\mathcal{F}_\theta$ be the mapping defined by :

$$\mathcal{F}_\theta : E \times E \to (G^{'}, \preceq)$$

$$(x, y) \mapsto \mathcal{F}_\theta(x, y) = (\theta \circ \mathcal{F})(x, y) = \theta(f_{x,y}) = f^{'}_{x,y}$$

Since the operation $*$, is associative, then, according to the theorem 2.2, $\mathcal{H}_\theta(*)$ is an also associative, and for all $x, y \in E$ and for all *distinct* points $u, v \in E$, each of them distinct from x and y,

$$\begin{aligned} \theta(f_{x,u} * f_{u,v} * f_{v,y}) &= (f_{x,u} * f_{u,v})^{'}\mathcal{H}_\theta(*)f^{'}_{z,y} \\ &= \theta(f_{x,u} * f_{u,v})\mathcal{H}_\theta(*)f^{'}_{z,y}. \\ &= f^{'}_{x,u}\mathcal{H}_\theta(*)f^{'}_{u,v}\mathcal{H}_\theta(*)f^{'}_{z,y}. \end{aligned}$$

It follows that,

$$\begin{aligned} \psi(f_{x,y}) \leq f_{x,u} * f_{u,v} * f_{v,y} &\Leftrightarrow \theta(\psi(f_{x,y})) \preceq \theta(f_{x,u} * f_{u,v} * f_{v,y}) \\ &\Leftrightarrow (\theta \circ \psi \circ \theta^{-1})(f^{'}_{x,y}) \preceq f^{'}_{x,u}\mathcal{H}_\theta(*)f^{'}_{u,v}\mathcal{H}_\theta(*)f^{'}_{z,y} \\ &\Leftrightarrow \varphi(f^{'}_{x,y}) \preceq f^{'}_{x,u}\mathcal{H}_\theta(*)f^{'}_{u,v}\mathcal{H}_\theta(*)f^{'}_{z,y}. \end{aligned}$$

Corollary 3.2 *Let θ be an order embedding from $(G, \leq)$ into a nonempty poset $(G^{'}, \preceq)$. Therefore, every $(E, \mathcal{F})$ a generalized 0_ψ quiver space on $(G, \leq, \tau)$ with the least element e_0, its equivalent to $(E, \mathcal{F}_\theta)$ is a generalized 0_φ quiver space on $(Rang(\theta), \preceq, \mathcal{H}_\theta(\tau))$ with the least element $\theta(e_0)$, such that, $\varphi = \theta \circ \psi \circ \theta^{-1}$ is an order embedding from $(Rang(\theta), \preceq)$ to itself where θ^{-1} is the inverse of θ from $Rang(\theta)$ into G.*

Corollary 3.3 *Every generalized probabilistic metric space $(E, \mathcal{F}, \tau)$ is a generalized $0_{i_{\Delta^-}}$ quiver space on $(\Delta^-, \leq, \mathcal{H}_\theta(\tau))$ where θ is the order isomorphism from Δ^+ into Δ^-, such that, $\forall t \in \mathbb{R}$, $\theta(f)(t) = 1 - f(t)$.*

Corollary 3.4 *Every Branciari metric space (E, d) is a generalized $0_{i_{\mathbb{R}_+}}$ quiver space on $(\mathbb{R}_+, \leq, +)$.*

3.6. The cartesian product of a 0_ψ quiver spaces

Theorem 3.3 *Let $(E_1, \mathcal{F}^1)$ and $(E_2, \mathcal{F}^2)$ be two 0_ψ quiver spaces on $(G, \leq, \tau)$ where τ is a co-triangle operation on G and ψ is an automorphism of (G, τ). Therefore, the cartesian product $(E_1 \times E_2, \mathcal{F}^\times)$ is a 0_ψ quiver space on $(G, \leq, \tau)$ such that, $\mathcal{F}^\times$ is the mapping defined from $(E_1 \times E_2)^2$ into G by, for all $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$,*

$$\mathcal{F}^\times((x_1, y_1), (x_2, y_2)) = f^\times_{(x_1,y_1),(x_2,y_2)} = f^1_{x_1,x_2} \tau f^2_{y_1,y_2}.$$

Proof: Let $(E_1, \mathcal{F}^1)$ and $(E_2, \mathcal{F}^2)$ be two 0_ψ quiver spaces on $(G, \leq, \tau)$ where τ is a co-triangle operation on G and ψ is an automorphism of (G, τ). Let $\mathcal{F}^\times$ be the mapping from $(E_1 \times E_2)^2$ into G such that, for all $(x_1, y_1), (x_2, y_2)$ in $E_1 \times E_2$,

$$\mathcal{F}^\times((x_1, y_1), (x_2, y_2)) = f^\times_{(x_1,y_1),(x_2,y_2)} = f^1_{x_1,x_2} \tau f^2_{y_1,y_2}.$$

- Let (x_1, y_1) and (x_2, y_2) in $E_1 \times E_2$. Since τ is a co-triangle operation on G, it follows that, $f^1_{x_1,x_2} \leq f^1_{x_1,x_2} \tau f^2_{y_1,y_2}$ and idem for $f^2_{y_1,y_2}$. Therefore, since $(E_1, \mathcal{F}^1)$ and $(E_2, \mathcal{F}^2)$ are two 0_ψ quiver space on $(G, \leq, \tau)$, it follows that,

$$\begin{aligned} f^\times_{(x_1,y_1),(x_2,y_2)} = e_0 &\Leftrightarrow f^1_{x_1,x_2} \tau f^2_{y_1,y_2} = e_0 \\ &\Leftrightarrow f^1_{x_1,x_2} = e_0 \text{ and } f^2_{y_1,y_2} = e_0 \\ &\Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2 \\ &\Leftrightarrow (x_1, y_1) = (x_2, y_2). \end{aligned}$$

- For all $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$,

$$f^\times_{(x_1,y_1),(x_2,y_2)} = f^1_{x_1,x_2} \tau f^2_{y_1,y_2} = f^1_{x_2,x_1} \tau f^2_{y_2,y_1} = f^\times_{(x_2,y_2),(x_1,y_1)}.$$

- For all $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in $E_1 \times E_2$,

$$\begin{aligned} \psi(f^\times_{(x_1,y_1),(x_2,y_2)}) &= \psi(f^1_{x_1,x_2} \tau f^2_{y_1,y_2}) \\ &= \psi(f^1_{x_1,x_2}) \tau \psi(f^2_{y_1,y_2}) \\ &\leq (f^1_{x_1,x_3} \tau f^1_{x_3,x_2}) \tau (f^2_{y_1,y_3} \tau f^2_{y_3,y_2}) \\ &= (f^1_{x_1,x_3} \tau f^2_{y_1,y_3}) \tau (f^1_{x_3,x_2} \tau f^2_{y_1,y_3}) \\ &= f^\times_{(x_1,y_1),(x_3,y_3)} \tau f^\times_{(x_3,y_3),(x_2,y_2)}. \end{aligned}$$

Then, the cartesian product $(E_1 \times E_2, \mathcal{F}^\times)$ is a 0_ψ quiver space on $(G, \leq, \tau)$. □

4. The Banach contraction principle in a complete 0_ψ quiver space

In this section we prove the analogue of the Banach contraction principle in a complete 0_ψ quiver space on the positive cone $(P, \leq, *)$ of a cauchy-complete archimedean ordered field $(\mathbb{F}, *, \perp, <)$, where $\psi \in \mathcal{E}mb_i(P, \leq)$.

The Picard-Banach fixed point theorem stated by *Stefan Banach* in 1920, as part of his work on solving equations integrals, it is a very strong theorem which, under certain conditions, does not only give the existence fixed point, but also uniqueness and even an iterative method to determine it. After that, based on this finding, a large number of fixed point results have appeared in recent years. Generally speaking, there usually are two generalizations on them, one is from spaces, the other is from mappings. For example, let (E, d) be a metric space and $\rho : E \to E$, if there exists a $k \in [0, 1)$ such that,

$$d(\rho(x), \rho(y)) \leq kd(x, y) \text{ for every } x, y \in E$$

Then, ρ is the so called *k-contraction*. Every k-contraction $\rho : E \to E$ on a *complete* metric space (E, d) has one and only one *fixed* point. This is the well known *Banach* contraction principle.

4.1. Ordered fields

We recall some well-known definitions, and properties about the notion of an *ordered field* [4], [5] that we will use in the sequal and we prove the main result of the ordered fields.

Informally an ordered field is a field with an order relation $<$ that satisfies the usual rules of elementary algebra and arithmetic. The formal definition focuses on the subset of positive elements. The order relation $<$ is formally defined in terms of these positive elements. Indeed,

Definition 4.1 *An ordered field $(\mathbb{F}, *, \perp)$ is a field with a subset P such that,*

- *P is closed under addition and multiplication,*
- *For any element $x \in \mathbb{F}$ exactly one of the following occurs:*

$$x = 0_{\mathbb{F}},\ x \in P,\ x_{-1} \in P.$$

Where x_{-1} is the inverse of x under the operation $*$.

Remark 4.1 When we say that P is closed under addition and multiplication, we mean that if $x, y \in P$ then $x * y$ and $x \perp y$ are in P. The second condition, that exactly one of $x = 0_{\mathbb{F}},\ x \in P,\ x_{-1} \in P$ holds, is called the *Law of trichotomy.*

Definition 4.2 *Let $\mathbb{F}$ be an ordered field with a subset P. The elements in P are called the positive elements and P is called a positive cone.*

Definition 4.3 *Let $\mathbb{F}$ be an ordered field with a positive cone P. If, $x \in \mathbb{F}$ is such that $x_{-1} \in P$ then x is said to be negative.*

Theorem 4.1 *[4] $(\mathbb{F}, *, \perp)$ is an ordered field with a positive cone P, if and only if a relation $<$ can be defined on $\mathbb{F}$ satisfying,*

- *If, $x, y, z \in \mathbb{F}$, and $x < y$, then $x * z < y * z$,*
- *If, $x, y, z \in \mathbb{F}$, $0_{\mathbb{F}} < z$, and $x < y$, then $x \perp z < y \perp z$,*
- *For all $x, y \in \mathbb{F}$, exactly one of $x = y$, $x < y$, or $y < x$ holds,*
- *If $x, y, z \in \mathbb{F}$, $x < y$, and $y < z$, then $x < z$.*

Remark 4.2 Thus, given an ordered field $\mathbb{F}$, one may assume that there exists a relation $<$ defined on $\mathbb{F}$ satisfying the conditions listed in the theorem 4.1, indeed, let $(\mathbb{F}, *, \perp)$ be an ordered field with a positive cone P, therefore, we may define the relation $<$ on $\mathbb{F}$ by:

$$\forall x, y \in \mathbb{F},\ x < y \Leftrightarrow y * x_{-1} \text{ is a positive element.}$$

Then, we write $x < y$. The relation $<$ satisfies the conditions of the theorem 4.1. Conversely, suppose there is a relation $<$ that satisfies the conditions of the theorem 4.1, then, $P = \{x \mid x \in \mathbb{F} \text{ and } 0_{\mathbb{F}} < x\}$ is a positive cone of $\mathbb{F}$.

Proposition 4.1 *(The mixed transitivity) Let $x, y, z \in \mathbb{F}$. If $x < y$ and $y \leq z$, then $x < z$. Likewise, if $x \leq y$ and $y < z$ then $x < z$.*

Theorem 4.2 *[5] Let $(\mathbb{F}, *, \perp, <)$ be an ordered field, then, the relation $\leq$ satisfies the conditions of the theorem 4.1. In addition, $\leq$ is a total order on $\mathbb{F}$.*

Definition 4.4 *Let $(\mathbb{F}, *, \perp, <)$ be an ordered field. $\mathbb{F}$ is said to be archimedean, if, for each $x \in \mathbb{F}$, with $x > 0_{\mathbb{F}}$, there exists $n \in \mathbb{N}$ such that, $n.1_{\mathbb{F}} > x$.*

Definition 4.5 *The following function is called absolute value and can always be defined on an ordered field.*

$$|.| : (\mathbb{F}, *, \perp, <) \to P$$

$$|x| = \begin{cases} sup(x, x_{-1}) & if\ x \neq 0_{\mathbb{F}} \\ 0_{\mathbb{F}} & if\ x = 0_{\mathbb{F}} \end{cases}$$

Proposition 4.2 *[4] Let* $(\mathbb{F}, *, \perp, <)$ *be an ordered field. The function* $|.|$ *defined above is a norm, this means that it satisfies the following properties :*

- $\forall x \in \mathbb{F},\ |x| = 0_{\mathbb{F}} \Leftrightarrow x = 0_{\mathbb{F}}$,
- $\forall x, y \in \mathbb{F},\ |x \perp y| = |x| \perp |y|$,
- $\forall x, y \in \mathbb{F},\ |x * y| \leq |x| * |y|$.

Remark 4.3 Let $(\mathbb{F}, *, \perp, <)$ be an ordered field. We can also equip an ordered field $\mathbb{F}$ with a topological structure as follows. For each $x \in \mathbb{F}$ and $g > 0_{\mathbb{F}}$, the g-neighborhood of x is given by:

$$\mathcal{B}(x, g) = \{y \in \mathbb{F} \mid |y * x_{-1}| < g\}.$$

From which one can define open sets, closed sets, continuity, convergence, and another topological metric notions. In particular, every ordered field is a metric space.

Definition 4.6

- *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in an ordered field* $\mathbb{F}$ *is said to be converge to an element* x *of* $\mathbb{F}$*, if for every element* $g \in \mathbb{F}$ *where* $g > 0_{\mathbb{F}}$*, there exists an* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies* $|x * (x_n)_{-1}| \leq g$*. Then,* x *is said to be the limit of the sequence* $(x_n)_{n\in\mathbb{N}}$ *and one writes* $\lim x_n = x$.
- *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in an ordered field* $\mathbb{F}$ *is said to be Cauchy sequence in* $\mathbb{F}$*, if for every element* $g \in \mathbb{F}$ *where* $g > 0_{\mathbb{F}}$*, there exists an* $N \in \mathbb{N}$ *such that* $n, m \geq N$ *implies* $|x_n * (x_m)_{-1}| \leq g$.
- *An ordered field* $\mathbb{F}$ *is said to be Cauchy-complete if every cauchy sequence in* $\mathbb{F}$ *converges to an element of* $\mathbb{F}$.

Proposition 4.3 *[4] Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be sequences with values in an ordered field* $(\mathbb{F}, *, \perp, <)$*. If,* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *converge then the sequences* $(x_n * y_n)_{n\in\mathbb{N}}$*,* $(x_n \perp y_n)_{n\in\mathbb{N}}$ *and* $(c \perp x_n)_{n\in\mathbb{N}}$ *for a given* $c \in \mathbb{F}$ *converges and*

- $lim(x_n * y_n) = \lim x_n * \lim y_n$.
- $lim(x_n \perp y_n) = \lim x_n \perp \lim y_n$.
- *For all* $c \in \mathbb{F},\ lim(c \perp x_n) = c \perp \lim x_n$.

Theorem 4.3 *[20] Let* $(\mathbb{F}, *, \perp, <)$ *be an ordered field. The following conditions are equivalent,*

(i) $\mathbb{F}$ *is archimedean.*

(ii) The sequence $(h^n)_{n\in\mathbb{N}}$ *converges to* $0_{\mathbb{F}}$ *whenever* $|h| < 1_{\mathbb{F}}$.

(iii) The geometric series $1_{\mathbb{F}} * h * h^2 * \ldots$*, converges to* $(1_{\mathbb{F}} * h_{-1})^{-1}$ *whenever* $|h| < 1_{\mathbb{F}}$.

where $h^0 = 1_{\mathbb{F}}$*, and for all* $n \in \mathbb{N}^*,\ h^n = \underbrace{h \perp h \perp \ldots \perp h}_{n\ times}$.

Now, we prove the main result of the ordered fields, indeed,

Proposition 4.4 *Let* $(\mathbb{F}, \leq)$ *be a nonmepty poset equipped with two operations* $*$ *and* $\perp$*. Let* θ *be an injective mapping from* $\mathbb{F}$ *to a nonempty set* $\mathbb{F}'$*, then, the following are equivalent,* *(i)* $(\mathbb{F}, *, \perp, <)$ *is an archimedean ordered field.*

(ii) $(Rang(\theta), \mathcal{H}_\theta(*), \mathcal{H}_\theta(\perp), <_\theta^{'})$ *is an archimedean ordered field.*

If, in addition θ *is surjective, it follows that,*

$$(\mathbb{F}, *, \perp, <) \text{ is isomorphic to } (\mathbb{F}^{'}, \mathcal{H}_\theta(*), \mathcal{H}_\theta(\perp), <_\theta^{'}).$$

Proof: Let $(\mathbb{F}, \leq)$ be a nonmepty poset equipped with two operations $*$ and $\perp$. Let θ be an injective mapping from $\mathbb{F}$ into a nonempty set $\mathbb{F}^{'}$. According to the Theorems 2.1 and 2.2, it follows that, θ is an isomorphism from $(\mathbb{F}, *)$ and $(\mathbb{F}, \perp)$ into $(Rang(\theta), \mathcal{H}_\theta(*))$ and $(Rang(\theta), \mathcal{H}_\theta(\perp))$ respectively and θ is an order isomorphism from $(\mathbb{F}, \leq)$ into $(Rang(\theta), \leq_\theta^{'})$, then, $(\mathbb{F}, *, \perp, <)$ is an ordered field its equivalent to $(Rang(\theta), \mathcal{H}_\theta(*), \mathcal{H}_\theta(\perp), <_\theta^{'})$ is an ordered field. Therefore, if, θ is surjective, then, the ordered fields are isomorphic. □

Remark 4.4 Notice that, if, θ is an injective mapping from the ordered field $(\mathbb{F}, *, \perp, <)$, into a nonempty set $\mathbb{F}^{'}$, then, if, $(P, \leq, *)$ is the positive cone of $\mathbb{F}$, therefore, $(\theta(P), \leq_\theta^{'}, \mathcal{H}_\theta(*))$ is the positive cone of the ordred field $(Rang(\theta), \mathcal{H}_\theta(\tau), \mathcal{H}_\theta(\nu), <_\theta^{'})$.

Let $(\mathbb{R}, +, \cdot, <)$ be the cauchy-complete archimedean orderd field of real numbers,

Example 4.1 Let ψ_a be the order automorphism of $(\mathbb{R}, \leq)$ defined by, for all $t \in \mathbb{R}$, $\psi_a(t) = a \cdot t$ where $a \in \mathbb{R}_+^*$. Therefore, $\mathcal{H}_{\psi_a}(+) = +$, and for all $x, y \in \mathbb{R}$,

$$\begin{aligned} x\mathcal{H}_{\psi_a}(\cdot)y &= \psi_a(\psi_a^{-1}(x) \cdot \psi_a^{-1}(x)) \\ &= \psi_a(\frac{x}{a} \cdot \frac{y}{a}) \\ &= \psi_a(\frac{1}{a^2} \cdot (x \cdot y)) \\ &= \frac{1}{a} \cdot (x \cdot y) \\ &= \frac{x}{\sqrt{a}} \cdot \frac{y}{\sqrt{a}}, \end{aligned}$$

According to the proposition 4.4, then, $(\mathbb{R}, +, \mathcal{H}_{\psi_a}(\cdot), <)$ is an archimedean ordered field, with the positive cone $\mathbb{R}_+$.

4.2. The contraction of 0_ψ quiver space

Definition 4.7 *Let* $(\mathbb{F}, *, \perp, <)$ *be an ordered field equiped with the positive cone* P*. Let* $(C, \mathcal{C})$ *and* $(M, \mathcal{M})$ *be two* 0_{ψ_C} *and* 0_{ψ_M} *quiver spaces on* $(P, \leq, *)$ *respectively. Let* ρ *be a map from* C *into* M*. Then,* ρ *is called a* k_θ*-lipschitzienne from* $(C, \mathcal{C})$ *into* $(M, \mathcal{M})$*, if there exists* θ *an order isomorphism from* $(\mathbb{F}, \leq)$ *into* $(P^*, \leq)$*, where* P^* *the set of non-zero elements of* P*, such that,*

- *The operation* $\mathcal{H}_\theta(*)$ *is equal to the operation* $\perp$ *on* P^**,*
- *There exists* $k \in P$ *such that,* $\forall x, y \in C, \quad m_{\rho(x),\rho(y)} \leq k \perp \psi_C(c_{x,y})$.

With, for all $x, y \in C$, $\mathcal{C}(x, y) = c_{x,y}$ *and for all* $x, y \in M$, $\mathcal{M}(x, y) = m_{x,y}$.

Definition 4.8 *(The contraction) Let* $(\mathbb{F}, *, \perp, <)$ *be an ordered field equiped with the positive cone* P*. Let* $(E, \mathcal{F})$ *be a* 0_ψ *quiver space on* $(P, \leq, *)$ *and let* ρ *be a selfmap of* E*, then* ρ *is called the* k_θ*-contraction of* $(E, \mathcal{F})$ *if,* ρ *is a* k_θ*-lipschitzienne from* $(E, \mathcal{F})$ *to itself, with* $k \in [0_{\mathbb{F}}, \theta(0_{\mathbb{F}}))$.

Remark 4.5

- Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(P, \leq, *)$ a positive cone of an ordered field $(\mathbb{F}, *, \perp, <)$, and let ρ be k_θ-*contraction* of $(E, \mathcal{F})$ then,

$$\forall x, y \in E, \quad f_{\rho(x),\rho(y)} \leq k \perp \psi(f_{x,y}).$$

- It should be noted that, since θ is an isomorphism from $(\mathbb{F},*)$ into $(Rang(\theta),\mathcal{H}_\theta(*))$, it follows that, $(P^*,\mathcal{H}_\theta(*))$ is a commutative group with the identity element $\theta(0_\mathbb{F})$ and since $\mathcal{H}_\theta(*)=\perp$ on P^* which is closed under the operation $\perp$, and $0_\mathbb{F}<1_\mathbb{F}$ ($1_\mathbb{F}\in P^*$ for any ordered field $\mathbb{F}$) then, $\theta(0_\mathbb{F})=1_{P^*}=1_\mathbb{F}$ the identity element of $\perp$ on $\mathbb{F}$.

Proposition 4.5 *Let (E,d) be a metric space, and let ρ be a k-contraction of (E,d), therefore, ρ is $k_{\exp}$-contraction of (E,d).*

Proof: Let (E,d) be a metric space, and let ρ be a *k-contraction* of E. According to the remark 3.2, (E,d) is a $0_{i_{\mathbb{R}_+}}$ quiver space on $(\mathbb{R}_+,\leq,+)$. Since $(\mathbb{R},+,\cdot,\leq)$ is an ordered field with the positive cone $\mathbb{R}_+$ and the exponential function exp is an order isomorphism from $(\mathbb{R},\leq)$ into $(\mathbb{R}_+^*,\leq)$, such that,

$$\forall x,y\in\mathbb{R}_+^*,\quad x\mathcal{H}_{\exp}(+)y=\exp(\ln(x)+\ln(y))=x.y$$

Then, $\mathcal{H}_{\exp}(+)=\cdot$ on $\mathbb{R}_+^*$ where $k<\exp(0)=1$, and

$$\forall x,y\in E,\ d(\rho(x),\rho(y))\leq k\cdot d(x,y)=k\cdot i_{\mathbb{R}_+}(d(x,y)),$$

It follows that, ρ is $k_{\exp}$-contraction of the metric space (E,d). □

Proposition 4.6 *Let $(E,\mathcal{F})$ be a 0_{ψ_s} quiver space on $(\mathbb{R}_+,+,\leq)$ the positive cone of the ordered field of real numbers, where for all t in $\mathbb{R}_+$, $\psi_s(t)=\frac{t}{s}$ for a given $s\in[1,\infty)$. Let ρ be a selfmap of E, such that, there exists $k\in\mathbb{R}_+$, where for all x,y in E, $f_{\rho(x),\rho(y)}\leq k\cdot f_{x,y}$, with $s\cdot k<1$. Therefore, ρ is $(k_s)_{\exp_s}$-contraction of $(E,\mathcal{F})$, where $k_s=s\cdot k$ and $\exp_s$ is the order isomorphism from $(\mathbb{R},\leq)$ into $(\mathbb{R}_+^*,\leq)$ defined by,*

$$\textit{For all } t\in\mathbb{R},\ \exp_s(t)=\exp(\frac{t}{s}).$$

Proof: Let $(E,\mathcal{F})$ be a 0_{ψ_s} quiver space on $(\mathbb{R}_+,+,\leq)$, where for all t in $\mathbb{R}_+$, $\psi_s(t)=\frac{t}{s}$, for a given $s\in[1,\infty)$. Let $(\mathbb{R},+,\cdot,<)$ be the ordered field of real numbers, and $\exp_s$ is the order isomorphism from $(\mathbb{R},\leq)$ into $(\mathbb{R}_+^*,\leq)$, defined by, for all $t\in\mathbb{R}$, $\exp_s(t)=\exp(\frac{t}{s})$. Let ρ be a selfmap of E, such that, there exists $k\in\mathbb{R}_+$, and for all x,y in E, $f_{\rho(x),\rho(y)}\leq k\cdot f_{x,y}$, where $s\cdot k<1$. therefore,

$$\begin{aligned}x\mathcal{H}_{\exp_s}(+)y &= \exp_s(\exp_s^{-1}(x)+\exp_s^{-1}(y))\\ &= \exp_s(s\cdot\ln(x)+s\cdot\ln(y))\\ &= x\cdot y\end{aligned}$$

It follows that, $\mathcal{H}_{\exp_s}(+)=\cdot$, on $\mathbb{R}_+^*$, and for all x,y in E,

$$\begin{aligned}f_{\rho(x),\rho(y)}\leq k\cdot f_{x,y} &\Leftrightarrow f_{\rho(x),\rho(y)}\leq(s\cdot k)\cdot\frac{f_{x,y}}{s}\\ &\Leftrightarrow f_{\rho(x),\rho(y)}\leq k_s\mathcal{H}_{\exp_s}(+)\psi_s(f_{x,y}).\end{aligned}$$

Since $s\cdot k<1$, then, k_s in $[0,\exp_s(0))$, therefore, ρ is $(k_s)_{\exp_s}$-contraction of $(E,\mathcal{F})$. □

Theorem 4.4 *Let $(E,\mathcal{F})$ be a 0_ψ quiver space on the positive cone $(P,\leq,*)$ of an ordered field $(\mathbb{F},*,\perp,<)$. Let ρ be a k_θ-contraction of $(E,\mathcal{F})$, such that θ_p commutates with ψ, where θ_p is the restriction of θ on P. Therefore, ρ is $\theta(k)_{\theta_p}$-contraction of the $(E,\mathcal{F}_\theta)$ a 0_ψ quiver space on the positive cone $(\theta(P),\leq,\perp)$ of the ordered field $(P^*,\perp,\mathcal{H}_\theta(\perp),<)$.*

Proof: Let $(E,\mathcal{F})$ be a 0_ψ quiver space on the positive cone $(P,\leq,*)$ of an ordered field $(\mathbb{F},*,\perp,<)$. Let ρ be a k_θ-contraction of $(E,\mathcal{F})$, such that, θ commutates with ψ. Observe that, since θ is an order isomorphism from $(\mathbb{F}, \leq)$ into $(P^*, \leq)$, according to The proposition 4.4 and The remark 4.4, then, $(P^*, \perp, \mathcal{H}_\theta(\perp), <)$ is an ordered field with the positive cone $(\theta(P), \leq, \perp)$. For all $x, y \in E$,

$$\begin{aligned} f_{\rho(x),\rho(y)} \leq k \perp \psi(f_{x,y}) &\Leftrightarrow \theta_p(f_{\rho(x),\rho(y)}) \leq \theta_p(k \perp \psi(f_{x,y})) \\ &\Leftrightarrow \mathcal{F}_{\theta_p}(\rho(x), \rho(y)) \leq \theta_p(k)\mathcal{H}_{\theta_p}(\perp)\theta_p(\psi(f_{x,y})) \\ &\Leftrightarrow \mathcal{F}_{\theta_p}(\rho(x), \rho(y)) \leq \theta_p(k)\mathcal{H}_{\theta_p}(\perp)\psi(\theta_p(f_{x,y})) \\ &\Leftrightarrow \mathcal{F}_{\theta_p}(\rho(x), \rho(y)) \leq \theta_p(k)\mathcal{H}_{\theta_p}(\perp)\psi(\mathcal{F}_{\theta_p}(x, y)) \end{aligned}$$

Since θ_p commutates with ψ, according to the Theorem 2.4, then, $(E, \mathcal{F}_\theta)$ is a 0_ψ quiver space on $(\theta(P), \leq, \perp)$. Since k in $[0_\mathbb{F}, \theta(0_\mathbb{F}))$, it follows that, $\theta(k) \in [1_\mathbb{F}, \theta(1_\mathbb{F}))$ with $\theta_p(P^*) = \theta(P)^*$, then, ρ is $\theta(k)_{\theta_p}$-contraction of the $(E, \mathcal{F}_{\theta_p})$ a 0_ψ quiver space on the positive cone $(\theta(P), \leq, \perp)$ of the ordered field $(P^*, \perp, \mathcal{H}_\theta(\perp), <)$. □

Definition 4.9 *Let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(G, \leq, *)$, let ρ be a selfmap of E and φ is a selfmap of G, then, ρ is said to be,*

- *A right r_φ-function of $(E, \mathcal{F})$ if, $\forall x, y \in E$, $f_{\rho(x),\rho(y)} \leq (\varphi \circ \psi)(f_{x,y})$,*
- *A left l_φ-function of $(E, \mathcal{F})$ if, $\forall x, y \in E$, $\varphi(f_{\rho(x),\rho(y)}) \leq \psi(f_{x,y})$.*

Corollary 4.1 *Let $(E, \mathcal{F}, \tau, s)$ be a probabilistic b-metric space for a given $s \in [1, \infty)$. Therefore, every ρ a δ-probabilistic contraction of E, is the left l_ν-function of the $(E, \mathcal{F}_\theta)$ a 0_{φ_s} quiver space on $(\Delta^-, \leq, \mathcal{H}_\theta(\tau))$ where ν is an order embedding from $(\Delta^-, \leq)$ to itself defined by,*

$$\forall f \in \Delta^-, \forall t > 0,\ \nu(f)(t) = (\theta(f) \circ \delta)(t),$$

with, θ and φ_s are defined as in The theorem 3.2.

Proposition 4.7 *Let $(\mathbb{F}, *, \perp, <)$ be an ordered field equiped with the positive cone P and $(E, \mathcal{F})$ be a 0_ψ quiver space on $(P, \leq, *)$. Every ρ a h_θ-contraction of $(E, \mathcal{F})$, where $0_\mathbb{F} < h$, is the right r_ϕ-function of $(E, \mathcal{F})$ where ϕ in $\mathcal{E}mb_i(P, \leq)$, such that, $\forall p \in P$, $\phi(p) = h \perp p$.*

Proof: Let $(\mathbb{F}, *, \perp, <)$ be an ordered field equiped with the positive cone P and $(E, \mathcal{F})$ be a 0_ψ quiver space on $(P, \leq, *)$. Let ρ be a h_θ-*contraction* of $(E, \mathcal{F})$. Since h is invertible under the operation $\perp$ and the partial order $\leq$ is compatible with the operation $\perp$, then, for all p_1, p_2 in P,

$$p_1 \leq p_2 \Leftrightarrow h \perp p_1 \leq h \perp p_2 \Leftrightarrow \phi(p_1) \leq \phi(p_2)$$

Since $h < \theta(0_\mathbb{F})$, it follows that, for all $p \in P$,

$$\phi(p) = h \perp p \leq \theta(0_\mathbb{F}) \perp p \leq p.$$

Then, ϕ in $\mathcal{E}mb_i(P, \leq)$, such that,

$$\forall x, y \in E, \quad f_{\rho(x),\rho(y)} \leq h \perp \psi(f_{x,y}) = \phi(\psi(f_{x,y})) = (\phi \circ \psi)(f_{x,y}) \leq \psi(f_{x,y}).$$

It follows that, ρ is the right r_ϕ-function of $(E, \mathcal{F})$. □

We define convergence and Cauchy sequence in a 0_ψ quiver space on the positive cone $(P, \leq, *)$ of an ordered field $(\mathbb{F}, *, \perp, <)$ and we define the completeness of a 0_ψ quiver space on the positive cone $(P, \leq, *)$ of an ordered field $(\mathbb{F}, *, \perp, <)$ as follows:

Definition 4.10 *Let $(E, \mathcal{F})$ be a 0_ψ quiver space on the positive cone $(P, \leq, *)$ of a given ordered field $(\mathbb{F}, *, \perp, <)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E, then,*

- *The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be convergent to $x \in E$ under $*$, if for every $g > 0_\mathbb{F}$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow f_{x_n,x} \leq g$, and one writes* $\lim x_n(*) = x$.

- *The sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence under $*$, if for every $g > 0_\mathbb{F}$, there exists a positive integer n_0 such that,*

$$\forall m, n \in \mathbb{N} \quad m, n \geq n_0 \Rightarrow f_{x_m,x_n} \leq g.$$

- *$(E, \mathcal{F})$ is said to be Cauchy-complete under $*$, if, every cauchy sequence under $*$ has a limit under $*$.*

- *A selfmap ρ from E to itself, is said to be continous, if, for all $x \in E$ and for all sequence $(x_n)_{n\in\mathbb{N}}$ in E such that, $\lim x_n(*) = x$ implies that the sequence $(\rho(x_n))_{n\in\mathbb{N}}$ converges and $\lim \rho(x_n)(*) = \rho(x)$.*

Remark 4.6

- If, there is no ambiguity we will only say that a sequence converges without mentioning the operation. The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be converge to $x \in E$, if and only if the sequence $(f_{x_n,x})_{n\in\mathbb{N}}$ in P converge to $0_\mathbb{F}$.
- Let $(\mathbb{F}, *, \perp, <)$ be an ordered field equiped with the positive cone P and let $(E, \mathcal{F})$ be a 0_ψ quiver space on $(P, \leq, *)$ where $\psi \in \mathcal{E}mb_i(P, \leq)$. Let ρ be a h_θ-contraction of $(E, \mathcal{F})$, then ρ is continous. Indeed, let $x \in E$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in E that converges to x, then, we have,
$$f_{\rho(x_n),\rho(x)} \leq h \perp \psi(f_{x_n,x}) \leq h \perp f_{x_n,x}.$$
Since $(f_{x_n,x})_{n\in\mathbb{N}}$ converges to $0_\mathbb{F}$ then, $(f_{\rho(x_n),\rho(x)})_{n\in\mathbb{N}}$ converges also to $0_\mathbb{F}$. It follows that, the sequence $(\rho(x_n))_{n\in\mathbb{N}}$ converges to $\rho(x)$, then ρ is continous.

In his study of the *semi-metric* space (E, d) which only satisfies the properties of the separation and the symmetry [15], *Wilson* introduces the following axiom: for each pair of *distinct* points x, y in E, there exists a number $r_{x,y} > 0$, such that, for all $z \in E$, $r_{x,y} \leq d(x, z) + d(z, y)$. Let $r > 0$, and let $\mathcal{B}(x, r) = \{y \in E \mid d(x, y) < r\}$. The previous axiom is equivalent to the assertions that,

- Do not exist *distinct* points $x, y \in E$, and a sequence $(z_n)_{n\in\mathbb{N}}$ in E, such that, $d(x, z_n) + d(z, z_n)$ converges to 0.

- The limits are unique.

- E is *Hausdorff* in the sense that given any two distinct points x, y in E, there exists positive numbers r_x and r_y, such that, $\mathcal{B}(x, r_x) \cap \mathcal{B}(y, r_y) = \varnothing$. This suggests the presence of a topology.

Let ψ be an order embeding from P the positive cone of an ordered field $(\mathbb{F}, \leq)$ to itself. Since $0_\mathbb{F} \leq \psi(0_\mathbb{F})$ and for all x in P, $0_\mathbb{F} < x \Leftrightarrow \psi(0_\mathbb{F}) < \psi(x)$, then, the mixed transitivity implies $0_\mathbb{F} < \psi(x)$, $\forall x \in P^*$. Let $(E, \mathcal{F})$ a 0_ψ quiver space on $(P, \leq, *)$ the positive cone of an ordered field $\mathbb{F}$, then, for all *distinct* points x, y in E,

$$0_\mathbb{F} < \psi(f_{x,y}) \leq f_{x,z} * f_{z,y}, \text{ for all } z \in E.$$

Therefore, the *limits are unique.*

Theorem 4.5 *Let $(\mathbb{F}, *, \perp, <)$ be a cauchy-complete archimedean ordered field equiped with the positive cone P and let $(E, \mathcal{F})$ be a Cauchy-complete 0_ψ quiver space on $(P, \leq, *)$ where $\psi \in \mathcal{E}mb_i(P, \leq)$. Therefore, every ρ a h_θ-contraction of $(E, \mathcal{F})$ has one and only one fixed point.*

Proof: Let $(\mathbb{F}, *, \perp, <)$ be a cauchy-complete archimedean ordered field equiped with the positive cone P and let $(E, \mathcal{F})$ be a Cauchy-complete 0_ψ quiver space on $(P, \leq, *)$ where $\psi \in \mathcal{E}mb_i(P, \leq)$. Let ρ be a h_θ-contraction of $(E, \mathcal{F})$, then, there exists $h \in P$ where $h < \theta(0_\mathbb{F})$, such that,

$$\forall x, y \in E, \ f_{\rho(x),\rho(y)} \leq h \perp \psi(f_{x,y}).$$

Therefore, according to The remark 4.6, for any pair of *distinct* fixed points x and y of E, $f_{x,y} = f_{\rho(x),\rho(y)} \leq h \perp \psi(f_{x,y}) < \theta(0_\mathbb{F}) \perp \psi(f_{x,y}) = \psi(f_{x,y})$. Since $\psi \in \mathcal{E}mb_i(P, \leq)$ it follows that, $\psi(f_{x,y}) \leq f_{x,y}$

then, ρ have at most one fixed point in E. For all n in $\mathbb{N}^*$, $f_{\rho^n(x),\rho^n(y)} \leq \phi^n(f_{x,y})$, where $\phi(p) = h \perp \psi(p)$ for all $p \in P$. Since $\psi \in \mathcal{E}mb_i(P, \leq)$, then, for all $x, y \in E$ and for all $n \in \mathbb{N}$,

$$f_{\rho^n(x),\rho^n(y)} \leq h^n \perp \psi(f_{x,y}) \leq h^n \perp f_{x,y}.$$

Observe that, if $x = \rho(y)$, it follows that, for all $y \in E$, and for all $n \in \mathbb{N}$,

$$f_{\rho^{n+1}(y),\rho^n(y)} = f_{\rho^n(x),\rho^n(y)} \leq h^n \perp \psi(f_{\rho(y),y}) \leq h^n \perp f_{\rho(y),y}.$$

Let $n, p \in \mathbb{N}$, and $x \in E$ and define the iterative sequence by $x_{n+1} = \rho(x_n)$ for all $n \in \mathbb{N}$ with $x_0 = x$. According to the lemma 4.4 and since $h < 1_{\mathbb{F}}$ which means that $1_{\mathbb{F}} * h_{-1}$ is invertible under $\perp$, therefore,

$$\begin{aligned} f_{x_{n+p},x_n} &= f_{\rho^{n+p}(x),\rho^n(x)} \\ &\leq f_{\rho^{n+p}(x),\rho^{n+p-1}(x)} * f_{\rho^{n+p-1}(x),\rho^{n+p-2}(x)} * \ldots * f_{\rho^{n+1}(x),\rho^n(x)} \\ &\leq (h^{n+p} \perp f_{\rho(x),x}) * (h^{n+p-1} \perp f_{\rho(x),x}) * \ldots * (h^n \perp f_{\rho(x),x}) \\ &= (h^{n+p} * h^{n+p-1} * \ldots * h^n) \perp f_{\rho(x),x} \\ &= (1_{\mathbb{F}} * h * \ldots * h^p) \perp h^n \perp f_{\rho(x),x} \\ &= (1_{\mathbb{F}} * h_{-1})^{-1} \perp (1_{\mathbb{F}} * (h^{p+1})_{-1}) \perp h^n \perp f_{\rho(x),x} \\ &= (1_{\mathbb{F}} * h_{-1})^{-1} \perp (1_{\mathbb{F}} * (h^{p+1})_{-1}) \perp h^n \perp f_{x_1,x}. \end{aligned}$$

Since $(1_{\mathbb{F}} * (h^{p+1})_{-1} < 1_{\mathbb{F}})$, then, for all $n, p \in \mathbb{N}$ and for all $x_0 = x \in E$,

$$f_{x_{n+p},x_n} \leq (1_{\mathbb{F}} * h_{-1})^{-1} \perp f_{x_1,x} \perp h^n.$$

According to the Theorem 4.3, and since $\mathbb{F}$ is an archimedean ordered field, then, the sequence $(h^n)_{n\in\mathbb{N}}$ converges to $0_{\mathbb{F}}$. Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence in E. Hence, since $(E, \mathcal{F})$ is cauchy-complete 0_ψ quiver space on $(P, \leq, *)$, then, there exists γ in E such that, $\lim x_n = \gamma$ and since ρ is a continuous function, then,

$$\gamma = \lim x_n(*) = \lim x_{n+1}(*) = \lim \rho(x_n)(*) = \rho(\lim x_n(*)) = \rho(\gamma)$$

Therefore γ is a fixed point of ρ. □

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