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On the Nonlinear Fuzzy Hybrid ψ-Hilfer Fractional Differential Equations

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ABSTRACT: This manuscript aims to highlight the existence result for a class of nonlinear fuzzy hybrid ψ-Hilfer fractional differential equations. Our approach is based on the application of an extended ψ-Hilfer fractional derivative of order $q, \sigma \in (0,1)$ valid on fuzzy functions paired with Dhage fixed point theorem. As an example of application, we provide one at the end of this paper to show how the results can be used.

Key Words: Fuzzy hybrid fractional differential equation, Fuzzy ψ-Hilfer fractional derivative, Fuzzy ψ-Riemann-Liouville fractional integral, Fixed point theorem.

Contents

1. Introduction

Perturbation techniques play a vital role in the study of nonlinear dynamical systems that are not easily solvable or analyzed. The nonlinearity of such a dynamical system is not smooth for studying the existence or some other characterizations of the solutions, however, perturbing such a problem in some way allows the problem to be studied with available methods for different aspects of the solutions. The nonlinear dynamical systems perturbed in this way are called hybrid differential equations. They are results of perturbation techniques as explained in [1]. Almost all perturbed nonlinear systems are usually approached using hybrid fixed point theory. In several areas, mathematical, physics, chemical technology, and population dynamics, problems are modeled by quadratically perturbed systems of the form:

$$\begin{cases} \frac{d}{dt}\left(\frac{u(t)}{h(t,u(t))}\right) = w(t,u(t)), \quad t \in J = [0,b] \\ \\ u(t_0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $h \in C(J \times \mathbb{R}, \mathbb{R}^*)$ and $w \in C(J \times \mathbb{R}, \mathbb{R})$. It is well known that the inversion of these quadratically perturbed systems gives rise to the operator equation involving the product of the operator like $AuBu = u, \quad u \in S$, where S is a closed, convex, and bounded subset of a Banach algebra X, and A, B are two operators. A useful tool to deal with quadratically perturbed systems is the celebrated fixed point theorem due to Dhage [13]. Therefore, the hybrid fixed point theory developed on the lines of a Dhage fixed point theorem was used in the study of these quadratically perturbed systems. There is a lot of work that has been done on the theory of hybrid differential equations, among which we find [2], [3], [4],

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[5], [6], [7].

The main motivation for this work comes from the works mentioned above, with the purpose of studying the existence of solutions of the following nonlinear fuzzy hybrid fractional differential equation involving ψ-Hilfer fractional derivative.

$$\begin{cases} {}^{H}D_{0_+}^{q,\sigma,\psi}\left(u(t)\tilde{\div}f(t,u(t))\right) = g(t,u(t)), \quad t\in[0,b] \\ (\psi(t)-\psi(0))^{1-\zeta}u(t)|_{t=0} = u_0 \in \mathbb{R}_{\mathbb{F}}, \end{cases} \tag{1.2}$$

where $0<q\leq 1$, $0<\sigma\leq 1$, $\zeta = q+\sigma(1-q)$, $f\in C(J\times\mathbb{R}_{\mathbb{F}},\mathbb{R}_{\mathbb{F}}\setminus\{\widehat{0}\})$, $g\in C(J\times\mathbb{R}_{\mathbb{F}},\mathbb{R}_{\mathbb{F}})$, $\mathbb{R}_{\mathbb{F}}$ is the space of fuzzy sets in $\mathbb{R}$, ${}^{H}D_{0_+}^{q,\sigma,\psi}(.)$ is the ψ-Hilfer fractional derivative of order q and type σ and $(\tilde{\div})$ is an approximated fuzzy division given in the section 4. The division of fuzzy numbers does not always exist (exists under certain conditions) [19]. To solve this problem, the approximated fuzzy division between fuzzy numbers was proposed, a division that always exists. For more details on the fuzzy fractional differential equations. See [20], [19], and [11].

2. Preliminaires

Denote by $\mathbb{R}_{\mathbb{F}} = \{u:\mathbb{R}\longrightarrow[0,1]\}$ the space of fuzzy sets in $\mathbb{R}$ with the following properties:

i) u is normal, that is, there exists an $x_0\in\mathbb{R}$ such that $u(x_0)=1$.

ii) u is fuzzy convex, that is, for $x,y\in\mathbb{R}$ and $(0<\lambda\leq 1)$ such that,

$$u(\lambda x+(1-\lambda)y)\geq min(u(x),u(y)).$$

iii) u is upper semicontinuous.

iv) u has compact support i.e $cl\{x\in\mathbb{R}/u(x)>0\}$ is compact, such that we denote by $cl(A)$ the closed set of A.

Then, $\mathbb{R}_{\mathbb{F}}$ is called the space of fuzzy numbers. For $0<\alpha\leq 1$, denote $[u]^\alpha = \{x\in\mathbb{R}; u(x)\geq\alpha\}$. Then from $(i)-(iv)$, it follows that the α-level set $[u]^\alpha$ is a closed bounded interval, then a fuzzy number u is completely determined by the interval $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$ for all $0<\alpha\leq 1$.

Definition 2.1 *[8,9] If u is a fuzzy number and $[u]^\alpha=[u_1^\alpha,u_2^\alpha]$ the α-level of u for all $\alpha\in[0,1]$, we have*

i) u_1^α is abounded monotonic nondecreasing left-continuous function for all $\alpha\ \in]0,1]$, and right-continuous for $\alpha=0$.

ii) u_2^α is abounded monotonic nonincreasing left-continuous function for all $\alpha\ \in]0,1]$, and right-continuous for $\alpha=0$.

iii) $u_1^\alpha\leq u_2^\alpha$, for all $\alpha\in[0,1]$.

Theorem 2.1 *[8] Let $u\in\mathbb{R}_{\mathbb{F}}$ for $\alpha\in[0,1]$ we have*

i) $[u]^\alpha$ is a nonempty compact convex set in $\mathbb{R}$ for each $\alpha\in[0,1]$.

ii) $[u]^\beta\subseteq[u]^\alpha$ for $0<\alpha\leq\beta\leq 1$.

iii) $[u]^\alpha = \cap_{i=1}^\infty[u]^{\alpha_i}$, for any nondecreasing sequence $\alpha_i\longrightarrow\alpha$ on $[0,1]$.

Definition 2.2 *[10] Let $u,v\in\mathbb{R}_{\mathbb{F}}$ and $\lambda\in\mathbb{R}$. Then, we have*

i) $[u+v]^\alpha = \{x+y/x\in[u]^\alpha, y\in[v]^\alpha\} = [u_1^\alpha+v_1^\alpha, u_2^\alpha+v_2^\alpha]$ for each $\alpha\in[0,1]$.

ii) $[\lambda u]^\alpha = \{\lambda x/x\in[u]^\alpha\} = \lambda[u]^\alpha$, for each $\alpha\in[0,1]$.

iii) $[uv]^{\alpha} = [min(u_1^{\alpha}v_1^{\alpha}, u_1^{\alpha}v_2^{\alpha}, u_2^{\alpha}v_1^{\alpha}, u_2^{\alpha}v_2^{\alpha}, max(u_1^{\alpha}v_1^{\alpha}, u_1^{\alpha}v_2^{\alpha}, u_2^{\alpha}v_1^{\alpha}, u_2^{\alpha}v_2^{\alpha})]$, *for each* $\alpha \in [0,1]$.

We define $\widehat{0} \in \mathbb{R}_{\mathbb{F}}$ as

$$\widehat{0}(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x \neq 0. \end{cases}$$

We define the metric space of fuzzy numbers on $\mathbb{R}_{\mathbb{F}}$ by

$$\begin{aligned} D(u,v) &= sup_{\alpha\in[0,1]} max\{|u_1^{\alpha} - v_1^{\alpha}|, |u_2^{\alpha} - v_2^{\alpha}|\} \\ &= sup_{\alpha\in[0,1]}\{D_H([u]^{\alpha}, [v]^{\alpha})\}, \end{aligned}$$

where D_H is the Hausdorff metric defined as

$$D_H([u]^{\alpha}, [v]^{\alpha}) = max\{|u_1^{\alpha} - v_1^{\alpha}|, |u_2^{\alpha} - v_2^{\alpha}|\}.$$

We have the following results:

Theorem 2.2 *[10]*

1) $(\mathbb{R}_{\mathbb{F}}, D)$ *is a metric space.*
2) $D(u+w, v+w) = D(u,v)$, *for all* $u, v, w \in \mathbb{R}_{\mathbb{F}}$.
3) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$, *for all* $u, v \in \mathbb{R}_{\mathbb{F}}$ *and* $\lambda \in \mathbb{R}$.
4) $D(u+v, w+e) \leq D(u,w) + d(v,e)$, *for all* $u, v, w, e \in \mathbb{R}_{\mathbb{F}}$.

Theorem 2.3 *[10]* $(\mathbb{R}_{\mathbb{F}}, d)$ *is a complete metric space.*

Definition 2.3 *[1] The left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order q, $(n-1 < q < n)$ for an integrable function $\Phi : [0,b] \to \mathbb{R}$ with respect to another function $\psi : [0,b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi^{'}(t) \neq 0$, for all $t \in [0,b]$, $(b \leq +\infty)$, are respectively defined as follows:*

$$I_{0_+}^{q;\psi}\Phi(t) = \frac{1}{\Gamma(q)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{q-1}\Phi(s)ds,$$

and

$$D_{0_+}^{q;\psi}\Phi(t) = \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{n-q;\psi}\Phi(t),$$

where $\Gamma(.)$ is the Euler gamma function defined by:

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt, z > 0.$$

Definition 2.4 *[12] Let $(n-1 < q < n)$, $n \in \mathbb{N}$, with $\psi \in C^n([0,b], \mathbb{R})$ a function such that $\psi(t)$ is increasing and $\psi^{'}(t) \neq 0$ for all $t \in [0,b]$. The ψ-Hilfer fractional derivative (left-sided) of function $\Phi \in C^n([0,b], \mathbb{R})$ of order q and type $\sigma \in [0,1]$ is determined as*

$$^{H}D_{0_+}^{q,\sigma,\psi}\Phi(t) = I_{0_+}^{\sigma(n-q);\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-q);\psi}\Phi(t),$$

in other way

$$^{H}D_{0_+}^{q,\sigma,\psi}\Phi(t) = I_{0_+}^{\sigma(n-q);\psi} D_{0_+}^{\zeta,\psi}\Phi(t),$$

where

$$D_{0_+}^{\zeta,\psi}\Phi(t) = \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-q);\psi}\Phi(t),$$

with $\zeta = q + \sigma(n-q)$.

In particular, the ψ-Hilfer fractional derivative of order $q \in (0,1)$ and type $\sigma \in [0,1]$, can be written in the following form:

$$^{H}D_{0_+}^{q;\sigma;\psi}\Phi(t) = \frac{1}{\Gamma(\zeta-q)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\zeta-q-1}D_{0_+}^{\zeta,\psi}\Phi(s)ds,$$

where $\zeta = q+\sigma(1-q)$, and $D_{0_+}^{\zeta,\psi}\Phi(t) = \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{1-\zeta;\psi}\Phi(t)$.

Lemma 2.1 *[12] Let $\Phi \in C^n([0,b],\mathbb{R})$, $n-1<q<n$, $0\leq\sigma\leq 1$, and $\zeta = q+\sigma-\sigma q$. Then, for all $t\in(0,b]$*

$$^{H}D_{0_+}^{q;\sigma;\psi}I_{0_+}^{q;\psi}\Phi(t) = \Phi(t),$$

and

$$I_{0_+}^{q;\psi H}D_{0_+}^{q;\sigma;\psi}\Phi(t) = \Phi(t) - \sum_{k=1}^{n}\chi_{\psi}^{\zeta-k}(t,0)\Phi_{\psi}^{[n-k]}I_{0_+}^{(1-\sigma)(n-q);\psi}\Phi(0),$$

where $\chi_{\psi}^{\zeta-k}(t,0) = \frac{(\psi(t)-\psi(0))^{(\zeta-k)}}{\Gamma(\zeta-k+1)}$ and $\Phi_{\psi}^{[n-k]}\Phi(t) = \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^{n-k}\Phi(t)$.
In particular, if $0<q<1$, we have

$$I_{0_+}^{q;\psi H}D_{0_+}^{q;\sigma;\psi}\Phi(t) = \Phi(t) - \chi_{\psi}^{\zeta-1}(t,0)I_{0_+}^{(1-\sigma)(1-q);\psi}\Phi(0).$$

Let $J=[0,b]$ be a finite interval of the real line $\mathbb{R}$. We denote by $C(J,\mathbb{R}_{\mathbb{F}})$ the space of all continuous fuzzy functions on J.
Let $\psi\in C^1([0,b],\mathbb{R})$ be an increasing function such that $\psi^{'}(t)\neq 0$, $\forall t\in[0,b]$, we consider the weighted space:

$$C_{(1-\zeta,\psi)}(J,\mathbb{R}_{\mathbb{F}}) = \left\{u : J\to\mathbb{R}_{\mathbb{F}}, (\psi(t)-\psi(0))^{1-\zeta}u(t)\in C^1(J,\mathbb{R}_{\mathbb{F}})\right\}.$$

Clearly, $C_{1-\zeta,\psi}(J,\mathbb{R})$ is a complete metric space with the norm

$$N_{C_{(1-\zeta,\psi)}(J,\mathbb{R}_{\mathbb{F}})}(u,v) = max_{t\in J}|(\psi(t)-\psi(0))^{1-\zeta}|D(u(t),v(t)).$$

Also, the norme of a function $u : J\longrightarrow\mathbb{R}_{\mathbb{F}}$ will be denoted by

$$N_{C_{(1-\zeta,\psi)}(J,\mathbb{R}_{\mathbb{F}})}(u,\widehat{0}) = \|u\|_{C_{(1-\zeta,\psi)}(J,\mathbb{R}_{\mathbb{F}})}$$

Let $u : J\longrightarrow\mathbb{R}_{\mathbb{F}}$ be a fuzzy function. We denote

$$[u(t)]^{\alpha} = [u_1^{\alpha}(t), u_2^{\alpha}(t)], t\in J \quad and \quad \alpha\in[0,1].$$

We define the derivative $u^{'}(t)$ of a fuzzy function u by [17]:

$$[u^{'}(t)]^{\alpha} = [(u_1^{\alpha})^{'}(t), (u_2^{\alpha})^{'}(t)], t\in J \quad and \quad \alpha\in[0,1],$$

and we define the fuzzy integral $\int_0^b u(t)dt$ by [17]:

$$\left[\int_0^b u(t)dt\right]^{\alpha} = \left[\int_0^b u_1^{\alpha}(t)dt, \int_0^b u_2^{\alpha}(t)dt\right], \alpha\in[0,1],$$

provided that the Lebesgue integrals on the right exist.

3. Fuzzy ψ-Riemann-Liouville fractional integral and fuzzy ψ-Hilfer fractional derivative

Let $q > 0$ and $u : J \longrightarrow \mathbb{R}_{\mathbb{F}}$ be a fuzzy function such that $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$ and $u_1^\alpha, u_2^\alpha \in C([0,b],\mathbb{R}) \cap L^1([0,b],\mathbb{R})$ for all $t \in J$ and $\alpha \in [0,1]$.

Definition 3.1 *[1] Define the left-sided fuzzy ψ-Riemann-Liouville fractional integral of order q, $(n-1 < q < n)$ for an integrable function $u : [0,b] \to \mathbb{R}_{\mathbb{F}}$ with respect to another function $\psi : [0,b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi'(t) \neq 0$ and $\psi'(t) < +\infty$,*

$$I_{0_+}^{q;\psi} u(t) = \frac{1}{\Gamma(q)} \int_0^t \Omega_\psi^q(t,s) u(s) ds,$$

by

$$[I_{0_+}^{q;\psi} u(t)]^\alpha = \frac{1}{\Gamma(q)} \left[\int_0^t \Omega_\psi^q(t,s) u_1^\alpha(s) ds, \int_0^t \Omega_\psi^q(t,s) u_2^\alpha(s) ds \right],$$

where $\Omega_\psi^q(t,s) = \psi'(s)(\psi(t) - \psi(s))^{q-1}$.

Lemma 3.1 *The family,*

$$I_\alpha = \frac{1}{\Gamma(q)} \left[\int_0^t \Omega_\psi^q(t,s) u_1^\alpha(s) ds, \int_0^t \Omega_\psi^q(t,s) u_2^\alpha(s) ds \right],$$

$\alpha \in [0,1]$ defined a fuzzy number.

Proof: As we have $u_1^\alpha, u_2^\alpha \in C([0,b],\mathbb{R}) \cap L^1([0,b],\mathbb{R})$ for all $t \in J$ and $\alpha \in [0,1]$, then it is clear that I_α is a nonempty compact convex set in $\mathbb{R}$. For $\alpha < \beta$, using the fact that ψ is an increasing function and u is a fuzzy number we get

$$\begin{aligned} u_1^\alpha \leq u_1^\beta &\Rightarrow \Omega_\psi^q(t,s) u_1^\alpha(s) \leq \Omega_\psi^q(t,s) u_1^\beta(s) \\ &\Rightarrow I_{0_+}^{q;\psi} u(t)_1^\alpha \leq I_{0_+}^{q;\psi} u(t)_1^\beta, \end{aligned}$$

and

$$\begin{aligned} u_2^\alpha \geq u_2^\beta &\Rightarrow \Omega_\psi^q(t,s) u_2^\alpha(s) \geq \Omega_\psi^q(t,s) u_2^\beta(s) \\ &\Rightarrow I_{0_+}^{q;\psi} u_2^\alpha(t) \geq I_{0_+}^{q;\psi} u_2^\beta(t), \end{aligned}$$

this implies that $I_\beta \subset I_\alpha$.
Let α_n be a nondecreasing sequence in (0, 1], such that $\alpha_n \longrightarrow \alpha$ as $n \longrightarrow +\infty$, we prove that $I_{\alpha_n} = \cap_{i=1}^n I_{\alpha_i}$. Since $u_k^0(s) \leq u_k^{\alpha_n}(s) \leq u_k^1(s)$, $(k = 1,2)$, then we have

$$|\Omega_\psi^q(t,s) u_k^{\alpha_n}(s)| \leq max|\Omega_\psi^q(b,0) u_k^0(s), \Omega_\psi^q(b,0) u_k^1(s)|.$$

Clearly, $max|\Omega_\psi^q(b,0)u_k^0(s), \Omega_\psi^q(b,0)u_k^1(s)|, k = 1,2$ is Lebesgue integrable on $(0,b]$. Therefore, by the Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n \to +\infty} \int_0^t \Omega_\psi^q(t,s) u_k^{\alpha_n}(s) ds = \int_0^t \Omega_\psi^q(t,s) u_k^\alpha(s) ds, k = 1,2,$$

this implies that $I_{\alpha_n} = \cap_{i=1}^n I_{\alpha_i}$. From theorem (2.1), I_α is a fuzzy number. □

Definition 3.2 *[1] Define the left-sided fuzzy ψ-Riemann-Liouville fractional derivative of order $(0 < q < 1)$ and type $\sigma \in [0,1]$ for an integrable function $u : [0,b] \to \mathbb{R}_{\mathbb{F}}$ with respect to another function $\psi : [0,b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi'(t) \neq 0$,*

$${}^H D_{0_+}^{q;\sigma;\psi} u(t) = \frac{1}{\Gamma(\zeta - q)} \int_0^t \Omega_\psi^{\zeta-q}(t,s) D_{0_+}^{\zeta;\psi} u(s) ds,$$

by

$$[{}^H D_{0_+}^{q;\sigma;\psi} u(t)]^\alpha = \left[I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta,\psi} u_1^\alpha(t), I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta,\psi} u_2^\alpha(t) \right],$$

where $\zeta = q + \sigma(1-q)$, *and* $D_{0_+}^{\zeta,\psi}(.)$ *is the left-sided* ψ*-Riemann-Liouville fractional derivative. Provided that the above equation defines a fuzzy number* ${}^H D_{0_+}^{q;\sigma;\psi} u(t) \in \mathbb{R}_{\mathbb{F}}$, *such that*

$$\left[{}^H D_{0_+}^{q;\sigma;\psi} u(t) \right]^\alpha = \left[{}^H D_{0_+}^{q;\sigma;\psi} u_1^\alpha(t), {}^H D_{0_+}^{q;\sigma;\psi} u_2^\alpha(t) \right].$$

Let us recall some properties of Fuzzy ψ-Riemann-Liouville fractional integral and fuzzy ψ-Hilfer fractional derivative.

Lemma 3.2 *[16], [1] Let* $\alpha, \beta > 0$ *and* $\Phi \in C([0,b], \mathbb{R})$. *Then we have*
i) $I_{0_+}^{\alpha;\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(0))^{\alpha+\beta-1}$.
ii) ${}^H D_{0_+}^{\alpha;\sigma;\psi} (\psi(t) - \psi(0))^{\zeta-1} = 0$.
iii) $\lim_{t\to 0^+} I_{0_+}^{\alpha;\psi} \Phi(t) = I_{0_+}^{\alpha;\psi} \Phi(0) = 0$.

Lemma 3.3 *If* $u \in C([0,b], \mathbb{R}_{\mathbb{F}}) \bigcap L^1([0,b], \mathbb{R}_{\mathbb{F}})$ *and* $q, p > 0$. *Then, we have*

i) $I^{q,\psi}(u+v)(t) = I^{q,\psi} u(t) + I^{q,\psi} v(t)$.

ii) $I_{0_+}^{q;\psi} I_{0_+}^{p;\psi} u(t) = I_{0_+}^{q+p;\psi} u(t)$.

Proof: According to definition (2.2)(i) and by linearity of the integral we obtain

$$\left[I^{q,\psi}(u+v)(t) \right]^\alpha = \left[I^{q,\psi} u(t) \right]^\alpha + \left[I^{q,\psi} v(t) \right]^\alpha,$$

this implies that $I^{q,\psi}(u+v)(t) = I^{q,\psi} u(t) + I^{q,\psi} v(t)$.
Let $u, I_{0_+}^{p;\psi} u(t) \in L^1([0,b], \mathbb{R}_{\mathbb{F}})$, to prove ii) we take the simple case when $\psi(t) = t$, then we have

$$\begin{aligned} \left[I_{0_+}^{q;\psi} I_{0_+}^{p;\psi} u(t) \right]^\alpha &= \left[I_{0_+}^{q} I_{0_+}^{p} u(t) \right]^\alpha \\ &= \left[I_{0_+}^{q} I_{0_+}^{p} u_1^\alpha(t), I_{0_+}^{q} I_{0_+}^{p} u_2^\alpha(t) \right] \\ &= \left[I_{0_+}^{q} I_{\tau_+}^{p} u_1^\alpha(t), I_{0_+}^{q} I_{\tau_+}^{p} u_2^\alpha(t) \right], \end{aligned}$$

if we pose $s = \tau + \theta(t-\tau)$, we get

$$I_{0_+}^{q} I_{\tau_+}^{p} u_i^\alpha(t) = \int_0^t (t-\tau)^{p+q-1} u_i^\alpha(\tau) d\tau \int_0^1 \theta^{p-1}(1-\theta)^{q-1} d\theta, \quad i = 1, 2,$$

by using the fact that $\int_0^1 \theta^{p-1}(1-\theta)^{q-1} d\theta = \frac{\Gamma(q)\Gamma(p)}{\Gamma(p+q)}$ we obtain

$$\begin{aligned} \left[I_{0_+}^{q} I_{0_+}^{p} u(t) \right]^\alpha &= \left[I_{0_+}^{p+q} u_1^\alpha(t), I_{0_+}^{p+q} u_2^\alpha(t) \right] \\ &= \left[I_{0_+}^{p+q} u(t) \right]^\alpha, \end{aligned}$$

this implies that $I_{0_+}^{q;\psi} I_{0_+}^{p;\psi} u(t) = I_{0_+}^{q+p;\psi} u(t)$. □

Lemma 3.4 *Let* $u, v \in C([0,b], \mathbb{R}_{\mathbb{F}}) \cap L^1([0,b], \mathbb{R}_{\mathbb{F}})$, $0 \le \sigma \le 1$ *and* $q > 0$, *then we have*

$${}^H D_{0_+}^{q;\sigma;\psi} (u+v)(t) = {}^H D_{0_+}^{q;\sigma;\psi} u(t) + {}^H D_{0_+}^{q;\sigma;\psi} v(t).$$

Proof: Obviously,

$$
\begin{aligned}
{}^H D_{0_+}^{q;\sigma;\psi} u(t) &= \frac{1}{\Gamma(\zeta-q)} \int_0^t \Omega_\psi^{\zeta-q}(t,s) D_{0_+}^{\zeta,\psi} u(s)ds \\
&= I_{0_+}^{\zeta-q;\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{0+}^{n-\zeta;\psi} u(t),
\end{aligned}
$$

using lemma (3.3) i) we get

$$
{}^H D_{0_+}^{q;\sigma;\psi}(u+v)(t) = {}^H D_{0_+}^{q;\sigma;\psi} u(t) + {}^H D_{0_+}^{q;\sigma;\psi} v(t).
$$

□

Lemma 3.5 *Let $u \in C([0,b],\mathbb{R}_\mathbb{F}) \bigcap L^1([0,b],\mathbb{R}_\mathbb{F})$ and $0 \le q \le 1$. Then, we have*

$$
{}^H D_{0_+}^{q;\sigma;\psi} I_{0_+}^{q;\psi} u(t) = u(t).
$$

Proof: We have

$$
\begin{aligned}
&\left[{}^H D_{0_+}^{q;\sigma;\psi} I_{0_+}^{q;\psi} u(t)\right]^\alpha \\
&= \left[I_{0_+}^{\sigma(n-q);\psi}\left(\tfrac{1}{\psi'(t)}\tfrac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-q);\psi} I_{0_+}^{q;\psi} u_1^\alpha(t), I_{0_+}^{\sigma(n-q);\psi}\left(\tfrac{1}{\psi'(t)}\tfrac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-q);\psi} I_{0_+}^{q;\psi} u_2^\alpha(t)\right] \\
&= \left[I_{0_+}^{\sigma(n-q);\psi}\left(\tfrac{1}{\psi'(t)}\tfrac{d}{dt}\right)^n I_{0_+}^{n-\sigma n+\sigma q;\psi} u_1^\alpha(t), I_{0_+}^{\sigma(n-q);\psi}\left(\tfrac{1}{\psi'(t)}\tfrac{d}{dt}\right)^n I_{0_+}^{n-\sigma n+\sigma q;\psi} u_2^\alpha(t)\right] \\
&= \left[I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta-q;\psi} u_1^\alpha(t), I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta-q;\psi} u_2^\alpha(t)\right],
\end{aligned}
$$

using Lemmas (2.1) and (3.3), we get

$$
\begin{aligned}
&\left[{}^H D_{0_+}^{q;\sigma;\psi} I_{0_+}^{q;\psi} u(t)\right]^\alpha \\
&= \left[I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta-q;\psi} u_1^\alpha(t), I_{0_+}^{\zeta-q;\psi} D_{0_+}^{\zeta-q;\psi} u_2^\alpha(t)\right] \\
&= \left[u_1^\alpha(t) - \Lambda_{n,1}^{\zeta,\psi}(t,0), u_2^\alpha(t) - \Lambda_{n,2}^{\zeta,\psi}(t,0)\ \right] \\
&= [u_1^\alpha(t), u_2^\alpha(t)] = [u(t)]^\alpha,
\end{aligned}
$$

where $\Lambda_{n,i}^{\zeta,\psi}(t,0) = \sum_{k=1}^n \chi_\psi^{\zeta-k}(t,0) u_\psi^{[n-k]} I_{0_+}^{(1-\sigma)(n-q);\psi} u_i^\alpha(0)$ $i = 1,2$, then, this implies that

$$
{}^H D_{0_+}^{q;\sigma;\psi} I_{0_+}^{q;\psi} u(t) = u(t).
$$

□

Theorem 3.1 *[13] Let S be a closed, convex and bounded subset of the Banach algebra X. Suppose that $A : X \to X$ and $B : S \to X$ are two operators such that:*

a) A is Lipschitzian with a Lipschitz constant δ.

b) B is completely continuous.

c) $u = AuBv \Rightarrow u \in S$, for all $v \in S$.

d) $\delta M < 1$, Where $M = \|B(S)\|$ Then, the operator equation $u = AuBu$ has a solution in S.

4. Generalized division of fuzzy numbers

Definition 4.1 *[18] Let $u, v \in \mathbb{R}_{\mathbb{F}}$, such that $0 \notin [v]^\alpha$ for all $\alpha \in [0,1]$ The generalized division (g-division)is the fuzzy number $w = u \div_g v \in \mathbb{R}_{\mathbb{F}}$ ($[w]^\alpha = [w_1^\alpha, w_2^\alpha]$) defined by*

$$[u]^\alpha \div_g [v]^\alpha = [w]^\alpha \Leftrightarrow \begin{cases} (i)[u]^\alpha = [v]^\alpha [w]^\alpha \\ \\ Or(ii)[v]^\alpha = [u]^\alpha ([w]^\alpha)^{-1}, \end{cases}$$

where $([w]^\alpha)^{-1} = [\frac{1}{w_2^\alpha}, \frac{1}{w_1^\alpha}]$.
Provided that w is a proper fuzzy number, where the multiplications between intervals are performed in the standard interval arithmetic setting.

Generalized division of fuzzy numbers does not always exist, can be exist but the resulting intervals are not the α-cuts of fuzzy number. To solve this problem, in [18] an approximated fuzzy division between fuzzy numbers was proposed, a division that always exists.

Definition 4.2 *Let $u, v \in \mathbb{R}_{\mathbb{F}}$, such that $0 \notin [v]^\alpha$ for all $\alpha \in [0,1]$, the approximated fuzzy division between fuzzy numbers u and v is defined by*

$$[u \tilde{\div} v]^\alpha = cl\left(\cup_{\beta \geq \alpha}([u]^\beta \div_g [v]^\beta)\right). \tag{4.1}$$

As each g-division $[u]^\beta \div_g [v]^\beta$ exists for $\beta \in [0,1]$, $z = u \tilde{\div} v$ can be considered as a generalization of division of fuzzy numbers, existing for any $u, v \in \mathbb{R}_{\mathbb{F}}$ with $0 \notin [v]^\beta$ for all $\beta \in [0,1]$.

Proposition 4.1 *[14] The approximated fuzzy division (4.1) is given by the expression*

$$[u \tilde{\div} v]^\alpha = \left[inf_{\beta \geq \alpha} min\left\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\right\}, sup_{\beta \geq \alpha} max\left\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\right\}\right]. \tag{4.2}$$

Proposition 4.2 *[14] For any two fuzzy numbers $u, v \in \mathbb{R}_{\mathbb{F}}$ the two approximated fuzzy divisions $[u \tilde{\div} v]^\alpha$ and $[v \tilde{\div} u]^\alpha$ exist and, for any $\alpha \in [0,1]$, we have $[u \tilde{\div} v]^\alpha = ([v \tilde{\div} u]^\alpha)^{-1}$ with $0 \notin [v]^\beta$, $0 \notin [u]^\beta$ and*

$$[u \tilde{\div} v]^\alpha = [w_1^\alpha, w_2^\alpha] \quad and \quad [v \tilde{\div} u]^\alpha = \left[\frac{1}{w_2^\alpha}, \frac{1}{w_1^\alpha}\right],$$

where

$$w_1^\alpha = inf(W^\alpha), \quad w_2^\alpha = sup(W^\alpha),$$

and

$$W^\alpha = \left\{\frac{u_1^\beta}{v_1^\beta} | \beta \geq \alpha\right\} \cup \left\{\frac{u_1^\beta}{v_2^\beta} | \beta \geq \alpha\right\} \cup \left\{\frac{u_2^\beta}{v_1^\beta} | \beta \geq \alpha\right\} \cup \left\{\frac{u_2^\beta}{v_2^\beta} | \beta \geq \alpha\right\}.$$

Proposition 4.3 *[14] Let $u, v \in \mathbb{R}_{\mathbb{F}}$, $\alpha \in [0,1]$ (here 1 is the same as $\{1\}$), We have*

1) *$v \tilde{\div} u = u \div_g v$ if $0 \notin [v]^\alpha$ $\forall \alpha \in [0,1]$ whenever the expression on the right exists; in particular $u \tilde{\div} u = 1$ if $0 \notin [u]^\alpha$.*

2) *If $0 \notin [v]^\alpha$, then $(uv)\tilde{\div}u = u$.*

3) *If $0 \notin [v]^\alpha$, then $1 \tilde{\div} v = v^{-1}$ and $1\tilde{\div}v^{-1} = v$.*

4) *If $0 \notin [u]^\alpha$ and If $0 \notin [v]^\alpha$ $\forall \alpha \in [0,1]$, then $1\tilde{\div}(v\tilde{\div}u) = u \tilde{\div} v$.*

5) *$v \tilde{\div} u = u \tilde{\div} v = w$ if and only if $w = w^{-1}$, furthermore, $w = 1$ if and only if $u = v$.*

5. Existence Results

In this partition, we prove the existence of the solution to the given problem (1.2). We first present the following important result through which we can prove our major results.
In the fuzzy case, the fuzzy hybrid fractional differential equation (1.2) is not always equivalent to the following fuzzy fractional integral equation (see [15] in the non-fuzzy case):

$$u(t) = f(t,u(t))(u_0 \tilde{\div} f(0,u(0))(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g(t,u(t)). \tag{5.1}$$

Lemma 5.1 *If $f \in C(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}} \setminus \{\widehat{0}\}) \bigcap L^1(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}} \setminus \{\widehat{0}\})$, $g \in C(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}}) \bigcap L^1(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}})$ and $u(t) \tilde{\div} f(t,u(t))$ existe (in sense (i)), then an integral solution of (5.1) is also a solution of the fuzzy probleme (1.2).*

Proof: Indeed, using the integral equation (5.1) and applying the ψ-Hilfer fractional derivative operator ${}^H D_{0_+}^{\alpha,\sigma,\psi}$ on both sides, we get

$${}^H D_{0_+}^{q,\sigma,\psi}(u(t) \tilde{\div} f(t,u(t))) = {}^H D_{0_+}^{q,\sigma,\psi}(u_0 \tilde{\div} f(0,u(0)))(\psi(t)-\psi(0))^{\zeta-1} + {}^H D_{0_+}^{q,\sigma,\psi} I_{0^+}^{q;\psi} g(t,u(t)), \tag{5.2}$$

if we pose $w = (u_0 \tilde{\div} f(0,u(0))) \in \mathbb{R}_{\mathbb{F}}$, such that $[w]^\alpha = [w_1^\alpha, w_2^\alpha]$, then by using lemma (3.2)(ii) we have

$$\begin{aligned}
&\left[{}^H D_{0_+}^{q,\sigma,\psi} w(\psi(t)-\psi(0))^{\zeta-1}\right]^\alpha \\
&= \left[w_1^\alpha({}^H D_{0_+}^{q,\sigma,\psi}(\psi(t)-\psi(0))^{\zeta-1}), w_2^\alpha({}^H D_{0_+}^{q,\sigma,\psi}(\psi(t)-\psi(0))^{\zeta-1})\right] \\
&= 0[w]^\alpha \\
&= \{0\}.
\end{aligned}$$

Using the above result and lemma (3.5) we get

$${}^H D_{0_+}^{q,\sigma,\psi}(u(t) \tilde{\div} f(t,u(t))) = g(t,u(t)), \quad \forall t \in [0,b].$$

To end the proof it is sufficient to show that : $\lim_{t \mapsto 0^+}(\psi(t)-\psi(0))^{\zeta-1}u(t) = u_0$, From (5.1) we have

$$(\psi(t)-\psi(0))^{1-\zeta})u(t) = f(t,u(t))(u_0 \tilde{\div} f(0,u(0))) + (\psi(t)-\psi(0))^{1-\zeta} f(t,u(t)) I_{0^+}^{q;\psi} g(t,u(t)),$$

passing to the limit $t \mapsto 0^+$ we get the result

$$(\psi(t)-\psi(0))^{1-\zeta})u(t)|_{t=0} = u_0.$$

□

Next, we introduce the following hypotheses:
(H_1) The function $f \in C(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}} \setminus \{\widehat{0}\})$ is bounded and there exists constant $\delta > 0$ such that for all $u, v \in \mathbb{R}_{\mathbb{F}}$, and $t \in J$ we have:

$$D(f(t,u), f(t,v)) \leq \delta D(u,v), \quad \textit{and} \quad D(f(t,u), \widehat{0}) \leq L, \quad L > 0.$$

(H_1) The function $g \in C(J \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}})$ and there exists a function $K \in C_{1-\zeta}(J, \mathbb{R}_{\mathbb{F}})$ such that:

$$D(g(t,u), \widehat{0}) \leq (\psi(t)-\psi(0))^{1-\zeta} D(K(t), \widehat{0}) \quad t \in J, u \in \mathbb{R}_{\mathbb{F}}.$$

Let $X := (C_{1-\zeta,\psi}(J, \mathbb{R}_{\mathbb{F}}), N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})})$. Then X is a complete metric space. Define,

$$S = \left\{u \in X, N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(u, \widehat{0}) \leq R\right\},$$

where

$$R = L\left(D(u_0 \tilde{\div} f(0,v(0)), \widehat{0}) + \frac{(\psi(b)-\psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}}(K, \widehat{0})\right).$$

Clearly, S is a closed, convex, and bounded subset of X. Define the operators $A : X \to X$ and $B : S \to X$ by

$$Au(t) = f(t, u(t)),$$

$$Bu(t) = \left\{ u_0 \tilde{\div} f(0, u(0))(\psi(t) - \psi(0))^{\zeta-1} + I_{0_+}^{\alpha;\psi} g(t, u(t)) \right\}.$$

We consider the mapping $T : S \to X$ defined by $Tu(t) = Au(t)Bu(t)$.

Theorem 5.1 *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then, the Hybrid fuzzy fractional differential equation* (1.2) *has a solution u provided:*

$$\delta \left\{ D(u_0 \tilde{\div} f(0, u(0)), \widehat{0}) + \frac{(\psi(b) - \psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}}(K, \widehat{0}) \right\} < 1. \tag{5.3}$$

Proof: To prove that $u \in C_{1-\zeta,\psi}(J, \mathbb{R}_{\mathbb{F}})$ is a solution of problem (1.2) is equivalent to proving that the mapping T has a fixed point, we show that the operators A and B satisfy the conditions of the theorem (3.1).
The proof is given in the several steps:
Step 1: $A : X \to X$ is Lipschitz operator:
Using the hypothesis (H_1), we obtain:

$$\begin{aligned}
&N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(Au(t) - Av(t)) \\
&= max_{t\in J}(\psi(t) - \psi(0))^{1-\zeta} D(f(t, u(t)), f(t, v(t))) \\
&\leq \delta max_{t\in J}(\psi(t) - \psi(0))^{1-\zeta} D(u(t), v(t)) \\
&\leq \delta N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(u, v),
\end{aligned}$$

therefore, A is Lipschitz operator with Lipschitz constant δ.
Step 2: $B : S \to X$ is completely continuous:
i) $B : S \to X$ is continuous:
Let $(u_n)_{n\in\mathbb{N}}$ be any sequence in S such that $u_n \to u$ as $n \to \infty$ in S. We prove that $Bu_n \to Bu$ as $n \to \infty$ in S. We have

$$\begin{aligned}
&(\psi(t) - \psi(0))^{1-\zeta} D(Bu_n, Bu) \\
&= (\psi(t) - \psi(0))^{1-\zeta} D(I_{0_+}^{q;\psi} g(t, u_n(t)), I_{0_+}^{q;\psi} g(t, u(t)))) \\
&\leq (\psi(t) - \psi(0))^{1-\zeta} \frac{1}{\Gamma(q)} \int_0^t \Omega_\psi^q(t, s) D(g(s, u_n(s)), g(s, u(s)))ds.
\end{aligned}$$

By continuity of g and Lebesgue dominated convergence theorem, from the above inequality, we obtain: $N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(Bu_n, Bu) \to 0$ as $n \to +\infty$.
This proves that $B : S \to X$ is continuous.
ii) $B(S) = \{Bu : u \in S\}$ is uniformly bounded.

Using hypothesis (H_2) for any $u \in S$ and $t \in J$, we have

$$\begin{aligned}
&(\psi(t)-\psi(0))^{1-\zeta} D(Bu(t),\widehat{0})\\
&= (\psi(t)-\psi(0))^{1-\zeta} D\left(u_0 \tilde{\div} f(0,u(0))(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g(t,u(t))),\widehat{0}\right)\\
&\le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + (\psi(t)-\psi(0))^{1-\zeta} D\left(I_{0+}^{q;\psi} g(t,u(t))),\widehat{0}\right)\\
&\le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + \frac{(\psi(t)-\psi(0))^{1-\zeta}}{\Gamma(q)} \int_0^t \Omega_\psi^q(t,s) D\left(g(s,u(s)),\widehat{0}\right) ds\\
&\le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + (\psi(t)-\psi(0))^{1-\zeta} I_{0+}^{q;\psi} (\psi(t)-\psi(0))^{1-\zeta} D\left(K(t),\widehat{0}\right)\\
&\le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + \frac{(\psi(t)-\psi(0))^{1-\zeta}}{\Gamma(q)} N_{C_{1-\zeta,\psi}}(K,\widehat{0}) \int_0^t \Omega_\psi^q(t,s) ds\\
&\le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + \frac{(\psi(b)-\psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}}(K,\widehat{0}),
\end{aligned}$$

therefore,

$$N_{C_{1-\zeta,\psi}}(Bu,\widehat{0}) \le D\left(u_0 \tilde{\div} f(0,u(0)),\widehat{0}\right) + \frac{(\psi(b)-\psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}}(K,\widehat{0}). \tag{5.4}$$

iii) $B(S)$ is equicontinuous:
Let any $u \in S$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then using hypothesis (H_2), we have:

$$\begin{aligned}
&D\left((\psi(t_2)-\psi(0))^{1-\zeta} Bu(t_2), (\psi(t_1)-\psi(0))^{1-\zeta} Bu(t_1)\right)\\
&= D\left((\psi(t_2)-\psi(0))^{1-\zeta} I_{0+}^{q;\psi} g(t_2,u(t_2)), (\psi(t_1)-\psi(0))^{1-\zeta} I_{0+}^{q;\psi} g(t_1,u(t_1))\right)\\
&\le \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} D\left(\int_0^{t_2} \Omega_\psi^q(t,s) g(s,u(s)) ds, \int_0^{t_1} \Omega_\psi^q(t,s) g(s,u(s) ds\right)\\
&\le \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} D\left(\int_0^{t_1} \Omega_\psi^q(t_2,s) g(s,u(s)) ds + \int_{t_1}^{t_2} \Omega_\psi^q(t_2,s) g(s,u(s)) ds,\right.\\
&\left.\int_0^{t_1} \Omega_\psi^q(t_1,s)(g(s,u(s)) ds\right)\\
&\le \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} D\left(\int_0^{t_1} \Omega_\psi^q(t_2,s) g(s,u(s)) ds, \int_0^{t_1} \Omega_\psi^q(t_1,s) g(s,u(s)) ds\right)\\
&+ \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} D\left(\int_{t_1}^{t_2} \Omega_\psi^q(t_2,s) g(s,u(s)) ds, \widehat{0}\right)\\
&\le \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} \int_0^{t_1} \Omega_\psi^q(t_1,s) D(g(s,u(s)), g(s,u(s)) ds\\
&+ \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} N_{C_{1-\zeta,\psi}}(K,\widehat{0}) \int_{t_1}^{t_2} \Omega_\psi^q(t_2,s) ds\\
&\le \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q)} \int_0^{t_1} \Omega_\psi^q(t_1,s) D(g(s,u(s)), g(s,u(s)) ds\\
&+ \frac{(\psi(b)-\psi(0))^{1-\zeta}}{\Gamma(q+1)} N_{C_(1-\zeta,\psi)}(K,\widehat{0}) (\psi(t_2)-\psi(t_1))^q.
\end{aligned}$$

By the continuity of ψ and the fact that $D(g(s,u(s)), g(s,u(s)) = 0$, from the above inequality it follows that
if $|t_1 - t_2| \to 0$ then $D\left((\psi(t_2)-\psi(t_1))^{1-\zeta} Bu(t_2), (\psi(t_1)-\psi(0))^{1-\zeta} Bu(t_1)\right) \to 0$.

From the parts (ii) and (iii), it follows that B(S) is uniformly bounded and equicontinuous set in X. Then by Arzela-Ascoli theorem, B(S) is relatively compact. Since $B : S \to X$ is the continuous and compact operator, it is completely continuous.

Step 3:

Let any $u \in C_{1-\zeta,\psi}(J, \mathbb{R}_{\mathbb{F}})$ and $v \in S$ such that $u(t) = Au(t)Bv(t)$ then, for all $t \in [0, b]$ we have:

$$
\begin{aligned}
&(\psi(t) - \psi(0))^{1-\zeta} D\left(u(t), \widehat{0}\right) \\
&= (\psi(t) - \psi(0))^{1-\zeta} D\left(f(t, u(t))(u_0 \dot{\div} f(0, v(0))(\psi(t) - \psi(0))^{\zeta-1} + I_{0_+}^{q;\psi} g(t, v(t))), \widehat{0}\right) \\
&\leq D\left(f(t, u(t))(u_0 \dot{\div} f(0, v(0))), \widehat{0}\right) + (\psi(b) - \psi(0))^{1-\zeta} D\left(f(t, u(t)) I_{0_+}^{q;\psi} g(t, v(t)), \widehat{0}\right) \\
&\leq D\left(f(t, u(t)), \widehat{0}\right) D\left(u_0 \dot{\div} f(0, v(0)), \widehat{0}\right) + (\psi(b) - \psi(0))^{1-\zeta} D\left(f(t, u(t)), \widehat{0}) D(I_{0_+}^{q;\psi} g(t, v(t)), \widehat{0}\right) \\
&\leq L\left\{D\left(u_0 \dot{\div} f(0, v(0)), \widehat{0}\right) + \frac{(\psi(b) - \psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(K, \widehat{0})\right\},
\end{aligned}
$$

this gives, $N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(u, \widehat{0}) \leq R$, then $u \in S$.

Step 4:

Let $M = N_{C_{1-\zeta,\psi}}(B(S), \widehat{0}) = sup\{N_{C_{1-\zeta,\psi}(J,\mathbb{R}_{\mathbb{F}})}(u, \widehat{0}) : u \in S\}$.

From inequality (5.3) and (5.4), we have:

$$
\begin{aligned}
\delta M &\leq \delta\left\{D\left(u_0 \dot{\div} f(0, u(0)), \widehat{0}\right) + \frac{(\psi(b) - \psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{(1-\zeta,\psi)}}(K, \widehat{0})\right\} \\
&< 1.
\end{aligned}
$$

From steps 1 to 4, it follows that all the conditions of the theorem (3.1) are fulfilled. Hence operator T has a solution in S.

This implies that the nonlinear fuzzy hybrid fractional differential equation (1.2) has a solution in $C_{1-\zeta,\psi}(J, \mathbb{R}_{\mathbb{F}})$. □

Note that the solution of the problem (1.2), although it exists is not always defined as a fuzzy number (see [14] in the non-fractional case). As an explication, we have the following results:

We consider the problem (1.2), then according to the extension principle of Zadeh, we have the following definition of $f(t, u(t))$ and $g(t, u(t))$ when are a fuzzy numbers

$$f(t, u)y = sup\{u(x) : y = f(t, x), x \in \mathbb{R}\},$$

$$g(t, u)y = sup\{u(x) : y = g(t, x), x \in \mathbb{R}\}.$$

It follows that

$[f(t, u)]^\alpha = [min\{f(t, x) : x \in [u_1^\alpha, u_2^\alpha]\}, max\{f(t, x) : x \in [u_1^\alpha, u_2^\alpha]\}] = [f_1^\alpha, f_2^\alpha]$,

and

$[g(t, u)]^\alpha = [min\{g(t, x) : x \in [u_1^\alpha, u_2^\alpha]\}, max\{g(t, x) : x \in [u_1^\alpha, u_2^\alpha]\}] = [g_1^\alpha, g_2^\alpha]$.

Then the problem (1.2) is equivalent to the following problem

$$
\begin{cases}
\left[{}^H D_{0_+}^{\alpha,\sigma,\psi}\left(u(t) \dot{\div} f(t, u(t))\right)\right]^\alpha = \left[g(t, u(t))\right]^\alpha, \quad t \in J \\
\left[(\psi(t) - \psi(0))^{1-\zeta} u(t)|_{t=0}\right]^\alpha = [u_0]^\alpha,
\end{cases}
\tag{5.5}
$$

where $\alpha \in [0, 1]$ and $[u_0]^\alpha = [u_{01}^\alpha, u_{02}^\alpha]$.

Let $r(t) = f(t, u(t)) \dot{\div} (u(t) \dot{\div} f(t, u(t))(\psi(t) - \psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g(t, u(t)))$, such that $0 \notin [u(t) \dot{\div} f(t, u(t))(\psi(t) - \psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g(t, u(t))]^\alpha$.

Then we will show that the intervals $[u_1^\alpha, u_2^\alpha]$ define a fuzzy number $u \in \mathbb{R}_{\mathbb{F}}$ when $r(t)$ exists (in the sense i)) and an inequality that we will see later is satisfied. For simplicity assume $[u(0)]^\alpha \leq 0$, $[f(t,u(t))]^\alpha > 0$, $[g(t,u)]^\alpha < 0$ and $\tilde{u}(t) = (u_1^\alpha(t)), u_2^\alpha(t))$for all $\alpha \in [0,1]$, then we have two cases:

Case 1:

According to the problem 5.5, we have the two following ψ-Hilfer HDE

$$\begin{cases} {}^H D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u_1^\alpha(t)}{f_2^\alpha(t,\tilde{u}(t))}\right) = g_1^\alpha(t,\tilde{u}(t)), \quad t \in J \\ (\psi(t)-\psi(0))^{1-\zeta} u_1^\alpha(t)|_{t=0} = u_{01}, \end{cases} \tag{5.6}$$

and

$$\begin{cases} {}^H D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u_2^\alpha(t)}{f_1^\alpha(t,\tilde{u}(t))}\right) = g_2^\alpha(t,\tilde{u}(t)), \quad t \in J \\ (\psi(t)-\psi(0))^{1-\zeta} u_2^\alpha(t)|_{t=0} = u_{02}. \end{cases} \tag{5.7}$$

The solution to problems (5.6) $-$ (5.7) are respectively

$$u_1^\alpha(t) = f_2^\alpha(t,\tilde{u}(t))(\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_1^\alpha(t,\tilde{u}(t)))$$

and

$$u_2^\alpha(t) = f_1^\alpha(t,\tilde{u}(t))(\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_2^\alpha(t,\tilde{u}(t))).$$

Then, we will show that $[u_1^\alpha(t), u_2^\alpha(t)]$ is a fuzzy number for all $\alpha \in [0,1]$ and $t \in J$, by verifying the three conditions of theorem (2.1)

(1): We prove that $u_1^\alpha(t) \leq u_2^\alpha(t)$, for all $\alpha \in [0,1]$ and $t \in J$.

Indeed, for all $\alpha \in [0,1]$ and $t \in J$ we have: $f_1^\alpha(t,\tilde{u}(t)) \leq f_2^\alpha(t,\tilde{u}(t))$, $g_1^\alpha(t,\tilde{u}(t)) \leq g_2^\alpha(t,\tilde{u}(t))$ and $u_1^\alpha(0) \leq u_2^\alpha(0)$, this implies that

$$\begin{aligned} &\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + \frac{1}{\Gamma(q)}\int_0^t \Omega_\psi^q(t,s) g_1^\alpha(s,\tilde{u}(s))ds \\ &\leq \frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + \frac{1}{\Gamma(q)}\int_0^t \Omega_\psi^q(t,s) g_2^\alpha(s,\tilde{u}(s))ds, \end{aligned}$$

then by classical arithmetic we have

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,\tilde{u}(t))(\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_1^\alpha(t,\tilde{u}(t))) \\ &\leq u_2^\alpha(t) = f_1^\alpha(t,\tilde{u}(t))(\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_2^\alpha(t,\tilde{u}(t))). \end{aligned}$$

(2): We prove that $[u_1^\beta(t), u_2^\beta(t)] \subseteq [u_1^\alpha(t), u_2^\alpha(t)]$, for all $0 \leq \alpha \leq \beta \leq 1$.

Let $0 \leq \alpha \leq \beta \leq 1$. Since $f_2^\beta(t,\tilde{u}(t)) \leq f_2^\alpha(t,\tilde{u}(t))$, $g_1^\alpha(t,\tilde{u}(t)) \leq g_1^\beta(t,\tilde{u}(t))$ and $u_1^\alpha(0) \leq u_2^\beta(0)$, this implies that

$$\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_1^\alpha(t,\tilde{u}(t)) \leq \frac{u_{01}^\beta}{f_2^\beta(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_1^\beta(t,\tilde{u}(t)),$$

then,

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,\tilde{u}(t))\left\{\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_1^\alpha(t,\tilde{u}(t))\right\} \\ &\leq u_1^\beta(t) = f_2^\beta(t,\tilde{u}(t))\left\{\frac{u_{01}^\beta}{f_2^\beta(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q;\psi} g_1^\beta(t,\tilde{u}(t))\right\}. \end{aligned}$$

Similarly, we have $f_1^\alpha(t,\tilde{u}(t)) \leq f_1^\beta(t,\tilde{u}(t))$, $g_2^\beta(t,\tilde{u}(t)) \leq g_2^\alpha(t,\tilde{u}(t))$ and $u_2^\beta(0) \leq u_2^\alpha(0)$, this implies that

$$\frac{u_{02}^\beta}{f_1^\beta(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^\beta(t,\tilde{u}(t)) \leq \frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^\alpha(t,\tilde{u}(t)),$$

then,

$$\begin{aligned} u_2^\beta(t) &= f_1^\alpha(t,\tilde{u}(t))\left\{\frac{u_{02}^\beta}{f_1^\beta(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^\beta(t,\tilde{u}(t))\right\} \\ &\leq u_2^\alpha(t) = f_1^\alpha(t,\tilde{u}(t))\left\{\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^\alpha(t,\tilde{u}(t))\right\}, \end{aligned}$$

this proves that $[u_1^\beta(t), u_2^\beta(t)] \subseteq [u_1^\alpha(t), u_2^\alpha(t)]$, for all $0 \leq \alpha \leq \beta \leq 1$.

(3): Let α_i be a nondecreasing sequence in $[0,1]$, such that $\alpha_i \to \alpha \in [0,1]$, prove that $[u_1^\alpha(t), u_2^\alpha(t)] = \cap_{i=1}^\infty [u_1^{\alpha_i}(t), u_2^{\alpha_i}(t)]$.

Using Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \lim_{\alpha_i\to\alpha} u_1^{\alpha_i}(t) &= \lim_{\alpha_i\to\alpha}(f_2^{\alpha_i}(t,\tilde{u}(t))(\frac{u_{01}^{\alpha_i}}{f_2^{\alpha_i}(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_1^{\alpha_i}(t,\tilde{u}(t))) \\ &= f_2^\alpha(t,\tilde{u}(t))(\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_1^\alpha(t,\tilde{u}(t))) = u_1^\alpha(t), \end{aligned}$$

and

$$\begin{aligned} \lim_{\alpha_i\to\alpha} u_2^{\alpha_i}(t) &= \lim_{\alpha_i\to\alpha}(f_1^{\alpha_i}(t,\tilde{u}(t))(\frac{u_{02}^{\alpha_i}}{f_1^{\alpha_i}(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^{\alpha_i}(t,\tilde{u}(t))) \\ &= f_1^\alpha(t,\tilde{u}(t))(\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q;\psi} g_2^\alpha(t,\tilde{u}(t))) = u_2^\alpha(t). \end{aligned}$$

This implies that $[u_1^\alpha(t), u_2^\alpha(t)] = \cap_{i=1}^\infty [u_1^{\alpha_i}(t), u_2^{\alpha_i}(t)]$.

It follows that all the conditions of theorem (2.1) are fulfilled. Hence $[u_1^\alpha(t), u_2^\alpha(t)]$ define a fuzzy number $u(t) \in \mathbb{R}_\mathbb{F}$ for all $\alpha \in [0,1]$ and $t \in J$.

Case 2: By problem (5.5), we have the two following ψ-Hilfer HFDE:

$$\begin{cases} {}^H D_{0^+}^{\alpha,\sigma,\psi}\left(\frac{u_1^\alpha(t)}{f_2^\alpha(t,\tilde{u}(t))}\right) = g_2^\alpha(t,\tilde{u}(t)), \quad t \in J \\ (\psi(t)-\psi(0))^{1-\zeta} u_1^\alpha(t)|_{t=0} = u_{01}, \end{cases} \tag{5.8}$$

and

$$\begin{cases} {}^H D_{0^+}^{\alpha,\sigma,\psi}\left(\frac{u_2^\alpha(t)}{f_1^\alpha(t,\tilde{u}(t))}\right) = g_1^\alpha(t,\tilde{u}(t)), \quad t \in J \\ (\psi(t)-\psi(0))^{1-\zeta} u_2^\alpha(t)|_{t=0} = u_{02}. \end{cases} \tag{5.9}$$

The solution to problems (5.8) − (5.9) are respectively

$$u_1^\alpha(t) = f_2^\alpha(t,\tilde{u}(t))(\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g_2^\alpha(t,\tilde{u}(t))),$$

and

$$u_2^\alpha(t) = f_1^\alpha(t,\tilde{u}(t))(\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g_1^\alpha(t,\tilde{u}(t))).$$

Let $0 \notin \left[\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g_1^\alpha(t,\tilde{u}(t))\right]^\alpha$
and $0 \notin \left[\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0^+}^{q,\psi} g_2^\alpha(t,\tilde{u}(t))\right]^\alpha$.

Then, in this case we have:

$$\begin{array}{rl} & u_1^\alpha(t) \leq u_2^\alpha(t) \\ \Rightarrow & f_2^\alpha(t,\tilde{u}(t))(\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_2^\alpha(t,\tilde{u}(t))) \\ \leq & f_1^\alpha(t,\tilde{u}(t))(\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_1^\alpha(t,\tilde{u}(t))) \\ \Rightarrow & \frac{f_2^\alpha(t,\tilde{u}(t))}{\frac{u_{02}^\alpha}{f_1^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_1^\alpha(t,\tilde{u}(t))} \\ \leq & \frac{f_1^\alpha(t,\tilde{u}(t))}{\frac{u_{01}^\alpha}{f_2^\alpha(t,\tilde{u}(0))}(\psi(t)-\psi(0))^{\zeta-1} + I_{0+}^{q,\psi} g_2^\alpha(t,\tilde{u}(t))}. \end{array} \tag{5.10}$$

Then, for $[u_1^\alpha(t), u_2^\alpha(t)]$ to be a fuzzy number (all conditions of Theorem (2.1) satisfied) it suffices that inequality (5.10) holds. If the inequality (5.10) does not hold (i.e, $u_2^\alpha(t) \leq u_1^\alpha(t)$), according to theorem (2.1) u(t) is not a fuzzy number.

6. An illustrative example

In this section, we give an illustrative example to show the efficiency of our obtained results. We consider the particular case when $\psi(t) = t$ and $\sigma = 1$.
Let $f(t, u(t) = 1 + \frac{cos(t)}{9} u(t)$ and $g(t, u(t)) = u(t) \tilde{\div} (1 + u(t))$ be two fuzzy functions, (here 1 is the same as $\{1\}$). Then, we consider the following nonlinear fuzzy hybrid fractional differential equation:

$$\begin{cases} {}^H D_{0_+}^{\frac{1}{2},1,t} \left(u(t) \tilde{\div} (1 + \frac{cos(t)}{9} u(t)) \right) = u(t) \tilde{\div} (1 + u(t)), \quad t \in [0,1] \\ \\ u(0) = \widehat{0}. \end{cases} \tag{6.1}$$

Comparing the problem (6.1) with the system of FHFDE (1.2). Then, $q = \frac{1}{2}$, $\sigma = 1$, $\zeta = 1$, and $J = [0, 1]$. We now move on to verity hypotheses H_1 and H_2. Then, for all $t \in [0, 1]$ and $u, v \in \mathbb{R}_F$ we have:

$$\begin{aligned} D\left(f(t,u(t)), f(t,v(t))\right) &= D\left(1 + \frac{cos(t)}{9} u(t), 1 + \frac{cos(t)}{9} v(t)\right) \\ &\leq \frac{1}{9} D(u, v), \end{aligned}$$

and

$$\begin{aligned} D\left(g(t,u(t)), \widehat{0}\right) &= D\left(u(t) \tilde{\div} (1 + u(t)), \widehat{0}\right) \\ &\leq 1, \end{aligned}$$

therefore, the hypotheses H_1 and H_2 are satisfied with $\delta = \frac{1}{9}$ and $K(t) = 1$.
Next, we check for condition (5.3). Then, we have:

$$\begin{aligned} &\delta \left\{ D(u_0 \tilde{\div} f(0, u(0)), \widehat{0}) + \frac{(\psi(b) - \psi(0))^{q+1-\zeta}}{\Gamma(q+1)} N_{C_{1-\zeta,\psi}}(K, \widehat{0}) \right\} \\ &= \frac{1}{9} \left\{ D\left((\widehat{0} \tilde{\div} f(0, u(0))), \widehat{0} \right) + \frac{1}{\Gamma(\frac{1}{2} + 1)} \right\} \\ &\leq \frac{1}{9\Gamma(\frac{3}{2})} < 1. \end{aligned}$$

We note that all the conditions of Theorem 5.1 are satisfied. Then, the nonlinear fuzzy hybrid fractional differential equation (6.1) has a solution in $C_{1-\zeta,\psi}(J, \mathbb{R}_{\mathbb{F}})$.

7. Conclusion

In this paper, we have based on the concept of the fuzzy hybrid differential equations involving ψ-Hilfer fractional derivative. An existence theorem for the fuzzy hybrid fractional differential equations is proved under the classical technique of Dhage fixed point theorem. Finally, we provided an illustrative example to illustrate our main results.

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