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Polynomial stabilizability of a plate equation in a waveguide with dissipation at infinity *

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ABSTRACT: We consider the dissipative plate equation in a waveguide, in the case where the usual Geometric Control Condition of Bardos, Lebeau and Rauch [5] is not necessarily satisfied. More precisely, assuming that the damping is concentrated close to infinity. We prove polynomial energy decay with respect to a stronger norm of the initial data.

Key Words: Energy decay, dissipative plate equation, waveguides, semiclassical analysis, resolvent estimates .

Contents

1. Introduction and statement of the main results

This article concerns the energy decay rate for the damped plate equation in waveguide. This is an important and interesting question in the general understanding of PDE without the usual geometric control condition (GCC) of Bardos, , Lebeau and Rauch [5] and a relevant situation for application. This paper is a natural continuation of ones [17], we are motivate to study this topic due its importance and its application in the field of mechanical engineering. It appears especially in various problems of linear elasticity, for example when looking at displacement of a plate. Our investigation is motivated by analyzing this displacement when the damping is effective. The result seems new and interesting especially when the domain is not compact.

Let $d, n \in \mathbb{N}^*$, and let ω be a bounded, open, smooth and connected subset of $\mathbb{R}^n$. We denote by Ω the straight wave guide $\mathbb{R}^d \times \omega \subset \mathbb{R}^{d+n}$. Everywhere in the paper we denote by (x, y) a general point in Ω, with $x \in \mathbb{R}^d$ and $y \in \omega$. Given $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$, we consider on Ω the dissipative plate equation

$$\partial_t^2 u + \Delta^2 u + a\partial_t u = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \tag{1.1}$$

subject to the initial conditions

$$u(0) = u_0, \quad \partial_t u(0) = u_1 \tag{1.2}$$

with the so called "hinged" boundary conditions

$$u = \triangle u = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega. \tag{1.3}$$

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In (1.1), u stands for a mechanical variable denoting the vertical displacement of plate while the function a stands for a damping variable. Assuming that a is bounded, smooth and there exists a compact subset K of $\overline{\Omega}$ and $c_0 > 0$ such that

$$\forall (x,y) \in \Omega \setminus K, \quad a(x,y) \geqslant c_0. \tag{1.4}$$

The energy $E(t)$ of solution of Eqs (1.1)-(1.3) is given by the following expression

$$E(t) := \int_{\Omega} |\triangle u(t)|^2 \, dx + \int_{\Omega} |\partial_t u(t)|^2 \, dx. \tag{1.5}$$

We can easily check that every sufficiently smooth solution of Eqs (1.1)-(1.3) satisfies the energy identity

$$E(t_2) - E(t_1) = -2 \int_{t_1}^{t_2} \int_{\Omega} a \left|\partial_t u(t)\right|^2 \, dxdt,$$

for all $t_2 \geqslant t_1$, and therefore, the energy is a non-increasing function of time variable t. Among the existing results available in the literature for plate equation, many of them concern the plate equation with boundary conditions $u|_{\partial\Omega} = 0$ and $\triangle u|_{\partial\Omega} = 0$. This set of boundary is known as "hinged" boundary conditions. We first mention the sharp result obtained in [13]. This result implies in particular that the plate equation can be stabilized by means of a feedback acting in arbitrary sub-domain of the rectangle. The same geometry and the same technique can be found in [20]. For a special geometric condition with boundary damping term, an exponential stability is established for square plate by moment feedbacks only in [3]. In [18], a polynomial decay is proved if the last geometric condition in [3] is not fulfilled.
For the "clamped" boundary conditions :$u = \partial_n u = 0$ on $\mathbb{R}_+ \times \partial\Omega$, where ∂_n is the outward normal to $\partial\Omega$, the literature for plate type equations give us a few available results. We first mention the important result obtained [19], on compact Remanian manifold with boundary, a log-type estimate decay for energy is proved. In the whole domain with a damping term of Kelvin voigt-type, which corresponds to the form term $\partial_t \triangle$, an exponential decay have been proved in [10].
The geometry of trajectories decide whether a uniform energy exists or not. We have uniform decay for the energy of the damped equation under the so-called geometric control condition (GCC) [5]. Roughly speaking, the assumption is that any (generalized) bicaracteristic (or classical trajectory, or ray of geometric optics) meets the damping region (in the interior of the domain or at the boundary) or escape at infinity. If only the bounded trajectories go through the damping region, we are limited to investigate the local energy, which measure the fact that the energy escape to infinity. Closely related problems have obtained in ([2], [16]) for the dissipative wave equation outside a compact obstacle of the Euclidean space (with dissipation at the boundary or in interior domain). If all the classical trajectories go through the damping region and not only the bounded ones, it is possible to study the decay of the total energy. This means that all the energy is dissipated by the medium. We mention for instance [1] for the wave equation in a exterior domain with the damping at infinity and [22] for the damped Schrödinger equation in wave guide. Here, we consider the domain neither compact nor close to the Euclidean space at infinity in any usual sense, so that properties of both compact and Euclidean domains will appear in our analysis. In our setting, when the damping coefficients vanish inside a compact K, the geometric control condition fails. For instance, if there exists $x_0 \in \mathbb{R}^d$ such that $a(x_0, y) = 0$ for all $y \in \omega$ then any trajectory staying in $\{x_0\} \times \omega$ will neither see the damping nor escape to infinity (see the same geometry in [17]). In this case, most of the classical trajectories meet the damping region. Indeed, except the bounded trajectories, all the rays which have a non zero velocity in the x direction and escape to infinity. Moreover, outside some bounded subset all rays meet the damping region. Closely related problems have been considered in [7] (about the dissipative wave equation on partially rectangular domains on both ends) and [1] (about the wave equation with dissipation a damping at infinity). Therefore, no uniform decay estimate for the energy with respect to energy norm of initial data is possible. However, if we allow some loss of regularity, we may have some energy decay. The main purpose of the present article is to estimate the resolvent operator. In spectral point of view, since 0 belongs to the resolvent, there is no low frequencies effect. Thus, the energy decay is only limited by the contribution of high frequencies. Notice that, when 0 is a singularity and the damping is effective at infinity, the contribution of low frequencies is more appreciated and the general rate of decay is governed by the contribution of low frequencies. Furthermore, for certain

works one recover the phenomenon of diffusion comes from the contribution of low frequencies we refer to ([17], [1]), but here there is no "low frequencies effect" at all. The semi classical defect measure and the now usual contradiction argument will be central in the analysis here.
To formulate the problem (1.1)-(1.3) in the semigroup setting, introducing the unknown function $v = \partial_t u$, we rewrite it in matrix form:

$$\begin{cases} \partial_t U(t) = -AU(t) \quad \text{in } \mathbb{R}_+ \times \Omega, \\ U = \triangle U = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega \\ U(0) = U_0 = (u_0, v_0), \end{cases} \tag{1.6}$$

where we have set $U = (u, v)$ and

$$A = \begin{pmatrix} 0 & -I \\ \triangle^2 & a \end{pmatrix}. \tag{1.7}$$

A is an unbounded operator defined on the Hilbert space $H = H_0^2(\Omega) \times L^2(\Omega)$, where $H_0^2(\Omega) = \{u \in H^2(\Omega), u = \triangle u = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega\}$ with respect to the norm

$$\|u\|^2 = \int_\Omega (|\triangle u|)^2 dx \tag{1.8}$$

with the domain

$$D(A) = (H^4(\Omega) \cap H_0^2(\Omega))) \times H_0^2(\Omega). \tag{1.9}$$

Now let $U_0 = (u_0, u_1) \in \mathcal{D}(A)$. It is standard that u is a solution of Eqs (1.1)-(1.3) if and only if $U : t \mapsto (u(t), \partial_t u(t))$ is solution of (1.6). Since the operator A is maximal accretive (see Proposition 2.2 below), we know from the Hille-Yosida theorem that $-A$ generates a contractions semigroup, so that the problem (1.6) has a unique solution $U : t \mapsto e^{-tA}U_0 \in C^0(\mathbb{R}_+, \mathcal{D}(A)) \cap C^1(\mathbb{R}_+, H)$.
In order to characterize the rate of energy decay, we give the growth estimate of the resolvent which is the main result of this paper.

Theorem 1.1 (Resolvent-estimate) *Let $\sigma \in \mathbb{R}$. Then, the operator $A - i\sigma$ is invertible on H and there exist $C > 0$ such that*

$$\left\|(A - i\sigma)^{-1}\right\|_{\mathcal{L}(H(\Omega))} \leqslant C(1 + |\sigma|^2)$$

From the resolvent estimate of Theorem1.1, we obtain the following energy decay for the damped plate equation

Theorem 1.2 (Energy-decay) *Let $k \in \mathbb{N}^*$. Then, there exists $C \geqslant 0$ such that for $t \geqslant 1$ and $U_0 \in \mathcal{D}(A^k)$, we have*

$$\left\|e^{-tA}U_0\right\|_H \leqslant Ct^{-k/2}\left\|A^k U_0\right\|_H.$$

To prove Theorem 1.1, first we show that the resolvent $(A - i\sigma)^{-1}$ is defined on the imaginary axis (there is no singularity for $\sigma = 0$). Then, by contradiction arguments based on semiclassical measures introduced by Gerard and Leichtnam [12](see also [15,8,14,23]) ,we shall prove that the semiclassical measure is identically null (see section 3). In Theorem 1.2, we obtain a polynomial energy decay. Here, there is no singularity at 0, so we can convert resolvent estimate into estimate for propagator. The proof of Theorem 1.2 is a slight adaptation of [6, Theorem 2.4] and because of this, we shall omit its proof. The same kind of results can be found in [15,7,17].
The structure of the article is as follows:
In section 2, we give general properties of the resolvent. We recall some basic properties of the hinged bi-Laplacian operator in waveguide which we can use to write the resolvent as a series of resolvents on $\mathbb{R}^d$ shifted by eigenvalues of bi-Laplacian operator on the section ω. Section 3 is devoted to estimate the resolvent and the proof of Theorem 1.1. Finally, Section 4 is devoted to discuss some problems close to (1.1)-(1.3): The damped plate equation on the waveguide with "hinged" boundary condition with dissipation everywhere on the domain and the damped plate equation on the waveguide with "clamped" boundary condition.

2. Properties of the Resolvent

We recall that an operator A on some Hilbert space $\mathcal{K}$ with domain $\mathcal{D}(A)$ is said to be accretive if for all $\varphi \in \mathcal{D}(A)$ we have

$$\operatorname{Re} \langle A\varphi, \varphi \rangle_{\mathcal{K}} \geqslant 0.$$

Moreover A is said to be maximal if it has no other accretive extension on $\mathcal{K}$ than itself. We know that the accretive operator A is maximal accretive if and only if $(A - \zeta)$ is boundedly invertible for certain (therefore any) $\zeta \in \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta) < 0\}$. Finally, the operator A is said to be (maximal) dissipative if iA is (maximal) accretive. If A is both dissipative and accretive, then it is maximal dissipative if and only if it is maximal accretive.

2.1. Some basic properties of the resolvent

As usual for the damped plate equation, the resolvent $(A - z)^{-1}$ on H will be expressed in terms of the resolvent of $\triangle^2 - za$, where $\triangle^2$ is the bi-Laplacian operator on Ω. We denote

$$\mathbb{C}_- := \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta) < 0\} \quad \text{ and } \mathbb{C}^+ := \{\zeta \in \mathbb{C} : \operatorname{Im}(\zeta) > 0\}.$$

Proposition 2.1 *Let $z \in \mathbb{C}_-$. Then $-z^2$ belongs to the resolvent set of the operator $\triangle^2 - az$.*

Proof: The bi-Laplacian is a model of fourth-order operator $\triangle^2 := \triangle \circ \triangle$ with the domain $D(\triangle^2) = H^4(\Omega) \cap H_0^2(\Omega))$ is a self-adjoint and non-negative operator on $L^2(\Omega)$. If $\operatorname{Im}(z) = 0$ then $-az$ is a bounded and non-negative operator, so $\triangle^2 - az$ is selfadjoint and non-negative. Since $-z^2$ is has negative real part, it belongs to its resolvent set. Now assume that $\operatorname{Im}(z) > 0$. Then $-az$ is bounded and dissipative, so $\triangle^2 - az$ is maximal dissipative. Thus its resolvent set contains $\mathbb{C}^+$ and in particular $-z^2$. Finally, if $\operatorname{Im}(z) < 0$ then $-(\triangle^2 - az)$ is maximal dissipative and z^2 belongs to $\mathbb{C}^+$, so we can conclude similarly. □

For $z \in \mathbb{C}_-$, we set

$$\mathcal{R} = \left(\triangle^2 - az + z^2\right)^{-1}.$$

Proposition 2.2 *The operator A is maximal accretive on H. Moreover for $z \in \mathbb{C}_-$ and $F \in H$ we have*

$$(A - z)^{-1} F = \begin{pmatrix} \mathcal{R}(a - z) & \mathcal{R} \\ -I - \mathcal{R}(az - z^2) & -z\mathcal{R} \end{pmatrix} F. \tag{2.1}$$

Proof: For $U = (u, v) \in \mathcal{D}(A)$ we have

$$\operatorname{Re} \langle AU, U \rangle_H = \langle av, v \rangle_{L^2(\Omega)} \geqslant 0,$$

so A is accretive on H. Then

$$\|(A + 1)U\|_H^2 = \|AU\|_H^2 + \|U\|_H^2 + 2 \operatorname{Re} \langle AU, U \rangle_H \geqslant \|U\|_H^2,$$

so $(A + 1)$ is injective. It remains to prove that $A + 1$ is surjective in H. Let $F = (f, g) \in H$. For $U = (u, v) \in \mathcal{D}(A)$ we have

$$(A + 1)U = F \Longleftrightarrow \begin{cases} u = \mathcal{R}(-1)(g + af + f) \\ v = u - f. \end{cases} \tag{2.2}$$

Since $-1 \in \mathbb{C}_-$, so $\mathcal{R}(-1)$ is a bounded operator from $L^2(\Omega)$ to $H_0^2(\Omega))$. Defined this way, $U = (u, v)$ indeed belongs to $\mathcal{D}(A)$ so $F \in \operatorname{Ran}(A + 1)$. This proves that $(A + 1)$ has a bounded inverse in $\mathcal{L}(H)$, so A is maximal accretive. In particular any $z \in \mathbb{C}_-$ belongs to the resolvent set of A. Then if we denote by $\mathcal{R}_A(z)F$ the right-hand side of (2.1), we can check by straightforward computation that $\mathcal{R}_A(z)F \in \mathcal{D}(A)$ and

$$(A - z)\mathcal{R}_A(z)F = F.$$

This proves that $(A - z)^{-1} = \mathcal{R}_A(z)$ on H. □

2.2. Specific properties of the resolvent on a flat waveguide

On a flat waveguide $\Omega \simeq \mathbb{R}^d \times \omega$, The bi-Laplacian operator can be split as the sum

$$\triangle^2 = \triangle_x^2 + \triangle_y^2 + 2\triangle_x\triangle_y, \tag{2.3}$$

where $\triangle_x^2$ is the usual bi-Laplacian on $\mathbb{R}^d$, $\triangle_y^2$ is the hinged bi-Laplacian operator, $-\triangle_x$ is the usual Laplacian on $\mathbb{R}^d$ and $-\triangle_y$ is the Dirichlet Laplacian operator on ω. In the hinged boundary conditions, we can deduce eigenvalues of bi-Laplacian Δ_y^2 from eigenvalues of Laplacian $-\Delta_y$. Indeed, μ is a eigenvalue of Δ_y^2 if only if $\sqrt{\mu}$ is a eigenvalue of $-\Delta_y$ (see introduction in [19]). Furthermore, $\triangle_y^2$ and $-\triangle_y$ are non-negative self-adjoint operators on $L^2(\omega)$ with compact resolvent. Therefore, there exist an orthonormal basis for $L^2(\omega)$ made of $(\varphi_k)_{k\in\mathbb{N}}$ of eigenfunctions: $\forall k \in \mathbb{N}^*$, we have $\|\varphi_k\|_{L^2(\omega)} = 1$, $\varphi_k \in \mathcal{D}(\triangle_y^2)$, $\triangle_y^2\varphi_k = \mu_k\varphi_k$ and $-\triangle_y\varphi_k = \sqrt{\mu_k}\varphi_k$, with

$$0 < \mu_1 \leqslant \ldots \leqslant \mu_k \leqslant \ldots : \lim_{j\longrightarrow\infty} \mu_j = +\infty.$$

Let $u \in L^2(\Omega)$. For almost all $x \in \mathbb{R}^d$ we have $u(x,\cdot) \in L^2(\omega)$, so there exists a sequence $(u_k(x))_{k\in\mathbb{N}^*}$ such that in $L^2(\omega)$ we have

$$u(x,\cdot) = \sum_{k\in\mathbb{N}^*} u_k(x)\varphi_k \tag{2.4}$$

and

$$\|u(x,\cdot)\|_{L^2(\omega)}^2 = \sum_{k\in\mathbb{N}^*} |u_k(x)|^2 .$$

After integration over $x \in \mathbb{R}^d$ we obtain that $u_k \in L^2(\mathbb{R}^d)$ for all $k \in \mathbb{N}^*$ and

$$\|u\|_{L^2(\Omega)}^2 = \sum_{k\in\mathbb{N}^*} \|u_k\|_{L^2(\mathbb{R}^d)}^2 .$$

For $m \in \mathbb{N}^*$ and $z \in \mathbb{C}_-$, we set

$$g_m = \sum_{k=1}^m f_k\varphi_k \quad \text{and} \quad v_m = \sum_{k=1}^m \left(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x - a(x)z + (z^2+\mu_k)\right)^{-1} f_k\varphi_k,$$

where $(f_k)_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R}^d)$. Then for all $m \in \mathbb{N}^*$ we have $v_m \in \mathcal{D}(\triangle^2)$ and by direct computation $(\triangle^2 - a(x)z + z^2)v_m = g_m$. Therefore $v_m = (\triangle^2 - a(x)z + z^2)^{-1}g_m$. At the limit $m \to \infty$ we obtain

$$(\triangle^2 - a(x)z + z^2)^{-1}f = \sum_{k\in\mathbb{N}^*} \left(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x - a(x)z + (z^2+\mu_k))\right)^{-1} f_k\varphi_k. \tag{2.5}$$

2.3. Extension of the resolvent on the imaginary axis

Proposition 2.3 *Let $\sigma \in \mathbb{R}$. Then the resolvent $\mathcal{R}(i\sigma)$ is well defined and extends to an operator in $\mathcal{L}(L^2(\Omega), H_0^2(\Omega))$.*

Proof: • First case: $\sigma > 0$
Let $\tilde{a} = a + c_0 1_K$. Then $\tilde{a} \geqslant c_0$ everywhere on Ω. This implies that the operator $\triangle^2 - i\sigma(\tilde{a} - c_0)$ is maximal dissipative, so its spectrum is contained in the lower half-plane. Therefore the spectrum of $\triangle^2 - i\sigma\tilde{a}(x)$ (and in particular its essential spectrum) is a subset of $\{\mathrm{Im}(\zeta) \leqslant -\sigma c_0\}$. Since $\triangle^2 - i\sigma a(x)$ is a bounded and relatively compact perturbation of $\triangle^2 - i\sigma\tilde{a}(x)$, we deduce by the Weyl Theorem (see [17, Theorem B.1 in appendix], applied with a connected component containing $\{\mathrm{Im}(z) > -\tau c_0\}$ that its essential spectrum is included in $\{\mathrm{Im}(\zeta) \leqslant -\sigma c_0\}$. Then σ^2 belongs to the spectrum of $\triangle^2 - i\sigma a$ if and only it is an eigenvalue. Now assume that $u \in \mathcal{D}(\triangle^2)$ is such that $(\triangle^2 - i\sigma a - \sigma^2)u = 0$. Then

$$\int_\Omega a\,|u|^2 = -\frac{1}{\sigma}\,\mathrm{Im}\left\langle(\triangle^2 - i\sigma a - \sigma^2)u, u\right\rangle = 0. \tag{2.6}$$

Since $a \geqslant 0$, this implies that $u \equiv 0$ on *suppa*. Hence $(\triangle^2 - \sigma^2)u = 0$. But this impossible owing to the unique continuation result [11].

• Second case: $\sigma < 0$

The operator $-(\triangle^2 - i\sigma(\tilde{a} - c_0))$ is maximal dissipative, and we conclude similarly as the first case exluding the possibility that $-\sigma^2$ is an eigenvalue of the operator of $(\triangle^2 - i\sigma(\tilde{a} - c_0))$.

• Third case: $\sigma = 0$

For $u = \sum_{k\in\mathbb{N}^*} u_k(x)\tilde{\varphi}_k(y) \in \mathcal{D}(\triangle^2)$ we can write

$$\begin{aligned}\left\langle \triangle^2 u, u\right\rangle_{L^2(\Omega)} &= \sum_{k\in\mathbb{N}^*} \left\langle (\triangle_x^2 + \mu_k - 2\sqrt{\mu_k}\triangle_x)u_k, u_k\right\rangle_{L^2(\mathbb{R}^d)} \\ &\geqslant \mu_1 \sum_{k\in\mathbb{N}^*} \|u_k\|^2_{L^2(\mathbb{R}^d)} \\ &\geqslant \mu_1 \|u\|^2_{L^2(\Omega}\,. \end{aligned} \tag{2.7}$$

So $\triangle^2$ is injective which implies that $\ker \triangle^2 = \ker(\triangle^2)^* = \{0\}$. Then

$$[\mathrm{Ran}(\triangle^2)]^\perp = \ker(\triangle^2)^* = \{0\}, \tag{2.8}$$

which proves that $Ran(\triangle^2)$ has dense range. It remains to prove that $Ran(\triangle^2)$ is closed in $L^2(\Omega)$ by showing that $Ran(\triangle^2)$ is a complete space. Let (g_n) be a Cauchy sequence in $\mathrm{Ran}(\triangle^2)$, so there exists a sequence $(f_n) \in \mathcal{D}(\triangle^2)$ such that $g_n := \triangle^2 f_n$. From (2.7), we have

$$\|f_n - f_m\|_{L^2(\Omega)} \leqslant (\sqrt{\mu_1})^{-1}(\|g_n - g_m\|_{L^2(\Omega)}\,. \tag{2.9}$$

This implies that (f_n) is a Cauchy sequence in $L^2(\Omega)$ which is complete, there exists $f \in L^2(\Omega)$ such that $f_n \longrightarrow f \in L^2(\Omega)$. Since $\triangle^2$ is self-adjoint, so it is closed. As a consequence, $g_n \longrightarrow \triangle^2 f$. This proves that the resolvent $\mathcal{R}(0)$ is well defined and $\mathcal{R}(i\sigma)$ extends to a bounded operator from $(L^2(\Omega)$ to $H_0^2(\Omega))$. □

3. Resolvent estimate and proof of Theorem 1.1

3.1. High frequency estimates

We are now in the position to capture the high frequency behavior estimates.

Proposition 3.1 *Let $k \in \mathbb{N}^*$ and α be a non-negative function on $\mathbb{R}^d$ such that $\alpha \geqslant c_0$ for some $c_0 > 0$ outside some bounded subset of $\mathbb{R}^d$. Then there exist $\sigma_0 \geqslant 0$ and $c \geqslant 0$ such that for $|\sigma| \geqslant \sigma_0$ we have*

$$\left\|\left(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x - i\sigma\alpha - \sigma^2\right)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant \frac{c}{\sigma}.$$

We refer to [23] for general results about semiclassical analysis.

Proof: For the rescaled variable $h = \frac{1}{\sigma}$ the estimate reads

$$\left\|\left[h^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - ih\alpha - 1\right]^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim \frac{1}{h}, \qquad 0 \leqslant h \ll 1. \tag{3.1}$$

We define $P(\alpha, h) := h^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - ih\alpha - 1$ and its enough to show there exists a positive constant C such that

$$\|u\|_{L^2(\mathbb{R}^d)} \leqslant \frac{C}{h}\|P(\alpha, h)u\|_{L^2(\mathbb{R}^d)}\,. \tag{3.2}$$

We argue by contradiction. If the assertion were false, then for $m = 1, 2, ...$ there would exist $0 < h_m \leqslant \frac{1}{m}$ and $(u_m)_{m\in\mathbb{N}} \in H^2(\mathbb{R}^d)^{\mathbb{N}}$ such that $h_m \to 0$ and

$$\|P(\alpha, h_m)u_m\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim \frac{h_m}{m}\|u_m\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\,. \tag{3.3}$$

By dividing $\|u_m\|_{L^2(\mathbb{R}^d)}$ by suitable constants, we suppose that $\|u_m\|_{L^2(\mathbb{R}^d)} = 1$ then

$$\|P(\alpha, h_m)u_m\|_{L^2(\mathbb{R}^d)} = o(h_m). \tag{3.4}$$

The sequence (u_m) is bounded in $L^2(\mathbb{R}^d)$ so, after extraction of a subsequence if necessary, there exists a Radon measure μ on $\mathbb{R}^{2d}$ such that for all $q \in C_0^\infty(\mathbb{R}^{2d})$ we have

$$\left\langle \mathrm{Op}^w_{h_m}(q)u_m, u_m \right\rangle \xrightarrow[m\to\infty]{} \int_{\mathbb{R}^{2d}} q\, d\mu, \tag{3.5}$$

where the Weyl quantization operator Op^w_h is given by the formula

$$\mathrm{Op}^w_{h_m}(q)u_m(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i}{h_m}<x-y,\xi>} q(\frac{x+y}{2}, \xi)u_m(y)dyd\xi.$$

We first observe that

$$\begin{aligned} \|\alpha u_m\|^2_{L^2(\mathbb{R}^d)} \quad & \lesssim \left\|\sqrt{\alpha}u_m\right\|^2_{L^2(\mathbb{R}^d)} \\ & = \frac{1}{h_m}\operatorname{Im}\left\langle P(\alpha, h_m)u_m, u_m\right\rangle \\ & \xrightarrow[m\to\infty]{} 0. \end{aligned} \tag{3.6}$$

• Let $\mathrm{Car}(P)$ be is the characteristic variety of p:

$$\mathrm{Car}(P) := \{(x, \xi) \in T^*\mathbb{R}^d \times \Omega : p(x, \xi) = 0\} \tag{3.7}$$

where $p(x, \xi) = |\xi|^4 - 1$ is the principal symbol of the operator $h^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - 1$. In the first, we show that μ is supported on $\mathrm{Car}(P)$. Let $q \in C_0^\infty(\mathbb{R}^{2d})$ be equal to 0 near $\mathrm{Car}(P)$ which is meaning that $\operatorname{supp} a \cap p^{-1}(0) = \emptyset$. We must show that

$$\int_{\mathbb{R}^{2d}} q d\mu = 0. \tag{3.8}$$

We can define

$$b(x, \xi) = \frac{q(x, \xi)}{|\xi|^4 - 1} \tag{3.9}$$

is well-defined from the assumption on q. From symbolic calculus, we have

$$\mathrm{Op}^w_{h_m}(q) = \mathrm{Op}^w_{h_m}(b)\left[h_m^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - 1\right] + O_{L^2\to L^2}(h_m). \tag{3.10}$$

Therefore,

$$\begin{aligned} \lim_{m\to\infty}\left\langle \mathrm{Op}^w_{h_m}(q)u_m, u_m\right\rangle \quad & = \lim_{m\to\infty}\left\langle \mathrm{Op}^w_{h_m}(b)\left[h_m^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - 1\right]u_m, u_m\right\rangle \\ & = \lim_{m\to\infty}\left\langle \mathrm{Op}^w_{h_m}(b)[h_m^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - ih_m\alpha - 1]u_m, u_m\right\rangle \\ & = \lim_{m\to\infty}\left\langle \mathrm{Op}^w_{h_m}(b)(P(\alpha, h_m))u_m, u_m\right\rangle \\ & = o(1), \text{ as } m \to \infty. \end{aligned} \tag{3.11}$$

Let $\chi \in C_0^\infty(\mathbb{R}^d, [0, 1])$ be such that $\alpha(x) \geqslant c_0$ on a neighborhood of $\operatorname{supp}(1 - \chi)$. (3.11) and (3.6) imply that μ is supported in $\operatorname{supp}(\chi) \times \{\xi \in T^*\mathbb{R}^d : p(x, \xi) = 0\}$.

• Moreover, for $\tilde{\chi} \in C_0^\infty(\mathbb{R}^d, [0, 1])$ such that $\tilde{\chi}(\xi) = 1$ if $\left| |\xi|^4 - 1\right| = 0$ we have

$$\int_{\mathbb{R}^{2d}} \chi(x)\tilde{\chi}(\xi)\, d\mu(x, \xi) = \lim_{m\to\infty}\left\langle \mathrm{Op}^w_{h_m}(\chi(x)\tilde{\chi}(\xi))u_m, u_m\right\rangle = \lim_{m\to\infty}\langle u_m, u_m\rangle = 1, \tag{3.12}$$

so $\mu \neq 0$. We will derive a contradiction to this.

• Hereafter write $P_m := P(\alpha, h_m)$. Then

$$P_m := h_m^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - ih_m\alpha - 1, \tag{3.13}$$

and

$$P_m^* := h_m^2(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) + ih_m\alpha - 1 \tag{3.14}$$

and therefore

$$P_m - P_m^* = -2i\alpha h_m. \tag{3.15}$$

Now, select $c \in C_0^\infty(\mathbb{R}^{2d})$ takes real-valued and setting $C_m = \mathrm{Op}_{h_m}^w(c)(x, h_m D)$. Then $C_m = C_m^*$. Using (3.4) and (3.15), we calculate that

$$\begin{aligned} o(h_m) \qquad &= 2iIm\left\langle C_m P_m u_m, u_m\right\rangle \\ &= \left\langle C_m P_m u_m, u_m\right\rangle - \left\langle u_m, C_m P_m u_m\right\rangle \\ &= \left\langle (C_m P_m - C_m P_m^*) u_m, u_m\right\rangle \\ &= \left\langle [C_m, P_m] u_m, u_m\right\rangle + \left\langle (P_m - P_m^*) C_m u_m, u_m\right\rangle \\ &= \frac{h_m}{i}\left\langle \mathrm{Op}_{h_m}^w(\{c,p\}) u_m, u_m\right\rangle - 2ih_m\left\langle \mathrm{Op}_{h_m}^w(\alpha c) u_m, u_m\right\rangle \end{aligned} \tag{3.16}$$

Dividing by $\frac{h_m}{i}$ and let $m \longrightarrow \infty$, we obtain

$$\langle \mu, \{c,p\} + 2\alpha c\rangle = 0, \tag{3.17}$$

Now, Let be b a function such that $\{c,p\} + 2\alpha c > 0$ on $\operatorname{supp}\alpha$ (for the buillding of this function, we refer to [23, Theorem 5.9, P113]). This imply that $\mu \equiv 0$, wchih is a contradiction to (3.12) and achieve the proof of the proposition 3.1. □

3.2. Proof of Theorem 1.1

The main point will be to estimate the resolvent $(A - i\sigma)^{-1} : H \longrightarrow H$ for $\sigma \in \mathbb{R}$ and $|\sigma| \geqslant 1$. Theorem 1.1 is a direct consequence of the following result

Proposition 3.2 *Let Ω be the straight waveguide $\mathbb{R}^d \times \omega \subset \mathbb{R}^{d+n}$, and we consider the stationary problem:*

$$\begin{cases} (\triangle^2 - ia\sigma - \sigma^2)u = f \\ u|_{\partial\Omega} = 0 \\ \triangle u|_{\partial\Omega} = 0 \end{cases} \tag{3.18}$$

Here $\sigma \in \mathbb{R}$ and $|\sigma| \geqslant 1$. Assuming that a is bounded and there exists a compact subset K of $\overline{\Omega}$ and $c_0 > 0$ such that $\forall (x,y) \in \Omega \setminus K, \quad a(x,y) \geqslant c_0$. Then, we have

$$\|u\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}^2 + \sigma^2 \int_\Omega a(x,y)\,|u|^2\,. \tag{3.19}$$

Proof: When establishing (3.19) it is sufficient to do so when $a = a(x)$ is a function of x only. Assume that (3.19) has already been established in the special case. Indeed, if $a = a(x,y)$, there exist a_1 such $0 \leqslant a_1 \leqslant a$ and $a_1 \geqslant c_0$ outside some bounded subset of compact and a_1 depend only on $x \in \mathbb{R}^d$.
Now if $u \in H_0^2(\Omega)$ satisfies

$$(\triangle^2 - ia\sigma - \sigma^2)u = f, \tag{3.20}$$

then

$$(\triangle^2 - ia_1\sigma - \sigma^2)u = f - i\sigma(a_1 - a)u. \tag{3.21}$$

Applying (3.19) and (3.21) we get,

$$\|u\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}^2 + \sigma^2\,\|(a - a_1)u\|_{L^2(\Omega)}^2 + \sigma^2 \int_\Omega a_1\,|u|^2\,. \tag{3.22}$$

Since $a - a_1$ is bounded and positive, this yields

$$\|u\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}^2 + \sigma^2 \int_\Omega (a - a_1)\,|u|^2 + \tau^2 \int_\Omega a_1\,|u|^2\,.$$

The bound (3.19) follows in general case. In what follows, when proving (3.19), we shall therefore assume that $a = a(x)$ is a function of x only. In this case, from section .2, we can apply the seperation of variables, we observe that the imaginary axis is included in the resolvent set of the operator $\triangle^2 - i\alpha$. In particular (2.5) applied for $z = i\sigma$, so for all $k \in \mathbb{N}^*$ we have in $L^2(\mathbb{R}^d)$

$$u_k = \left[(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - i\sigma a - (\sigma^2 - \mu_k)\right]^{-1} f_k. \tag{3.23}$$

Let $k \in \mathbb{N}^*$. The operator $(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - i\sigma a$ is maximal accretive (if $\sigma > 0$ it is accretive and maximal dissipative, and if $\sigma < 0$ we consider its adjoint) so if $\sigma^2 - \mu_k \leqslant -1$ we have

$$\|u_k\|_{L^2(\mathbb{R}^d)} \lesssim \|f_k\|_{L^2(\mathbb{R}^d)}. \tag{3.24}$$

Let now σ_0 be given by Proposition 3.1. By continuity of the resolvent $\zeta \mapsto ((\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - i\sigma a - \zeta)^{-1}$ there exists $C \geqslant 0$ such that if $\sigma^2 - \mu_k \in [-1, \sigma_0^2]$ we have

$$\|u_k\|_{L^2(\mathbb{R}^d)} \leqslant C\,\|f_k\|_{L^2(\mathbb{R}^d)}. \tag{3.25}$$

It remains to consider the case $\sigma^2 - \mu_k \geqslant \sigma_0^2$. Let $\tau = \sqrt{\sigma^2 - \mu_k} \in [\sigma_0, \sigma]$. We have

$$\left[(\triangle_x^2 - 2\sqrt{\mu_k}\triangle_x) - i\tau a - \tau^2\right]u_k = f_k + i(\sigma - \tau)a u_k \tag{3.26}$$

By Proposition 3.1 we obtain

$$\|u_k\|_{L^2(\mathbb{R}^d)} \lesssim \|f_k\|_{L^2(\mathbb{R}^d)} + |\sigma|\,\|a u_k\|_{L^2(\mathbb{R}^d)}. \tag{3.27}$$

Hence, summing the square of the norms with respect to k, we achieve the proof of the Proposition 3.2. □

Proposition 3.3 *Let Ω be the straight wave guide $\mathbb{R}^d \times \omega \subset \mathbb{R}^{d+n}$, Assuming that a is bounded and there exist a compact subset K of $\overline{\Omega}$ and $c_0 > 0$ such that $\forall (x, y) \in \Omega \setminus K, \quad a(x, y) \geqslant c_0$. Then, we have*

$$\mathcal{R}(i\sigma) = O(|\sigma|) : L^2(\Omega) \mapsto L^2(\Omega) \quad |\sigma| \geqslant 1. \tag{3.28}$$

Proof: Multiplying the equation (3.18)by $\overline{u}$, integreating by part and taking the imaginary part. Then, we have

$$\sigma^2 \int_\Omega a\,|u|^2 \leqslant \sigma\,\|f\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}. \tag{3.29}$$

Combining (3.19), (3.29) and using the Young inequality, we deduce that

$$\|u\|_{L^2(\Omega)} \leqslant O(|\sigma|)\,\|f\|_{L^2(\Omega)}. \tag{3.30}$$

□

We are now in the position to estimate the global resolvent $(A - i\sigma)^{-1}$ on the imaginary axis. In doing so, we shall derive bounds of the following operators:

$$\begin{aligned}
&\mathcal{R}(i\sigma) : L^2(\Omega) \longrightarrow L^2(\Omega),\\
&\mathcal{R}(i\sigma) : L^2(\Omega) \longrightarrow H_0^2(\Omega),\\
&\mathcal{R}(i\sigma)(ia\sigma + \sigma^2) + I : H_0^2(\Omega) \longrightarrow L^2(\Omega)\\
&\quad\text{and}\\
&\mathcal{R}(i\sigma)(a - i\sigma) : H_0^2(\Omega) \longrightarrow H_0^2(\Omega).
\end{aligned}$$

With the proposition 3.3 available , we have

$$\mathcal{R}(i\sigma) = (O(|\sigma|)) : L^2(\Omega) \longrightarrow L^2(\Omega). \tag{3.31}$$

Multiplying the equation (3.18) by $u = \mathcal{R}(i\sigma)f$ and integreating by parts. Then, for $f \in L^2(\Omega)$ we have

$$\|\triangle\mathcal{R}(i\sigma)f\|^2_{L^2(\Omega)} = \langle f, \mathcal{R}(i\sigma)f\rangle + i\sigma\,\langle a\mathcal{R}(i\sigma)f, \mathcal{R}(i\sigma)f\rangle + \sigma^2\,\|\mathcal{R}(i\sigma)f\|^2_{L^2(\Omega)}\,. \tag{3.32}$$

Proposition 3.3 together with 3.32 shows that

$$\mathcal{R}(i\sigma) = (O(\sigma^2)) : L^2(\Omega) \longrightarrow H^2_0(\Omega). \tag{3.33}$$

So by duality,

$$\mathcal{R}(i\sigma) = (O(\sigma^2)) : (H^2_0)'(\Omega) \longrightarrow L^2(\Omega). \tag{3.34}$$

We see that $\mathcal{R}(i\sigma)(\sigma^2 + ia\sigma) + I = \mathcal{R}(i\sigma)\triangle^2$ with the fact that the operator $\triangle^2 : (H^2_0)(\Omega) \longrightarrow (H^2_0)'(\Omega)$ is continuous, we deduce that

$$\mathcal{R}(i\sigma)(\sigma^2 + ia\sigma) + I = O(\sigma^2) : H^2_0(\Omega) \longrightarrow L^2(\Omega). \tag{3.35}$$

It remains to estimate the norm of $\mathcal{R}(i\sigma)(a - \sigma)$, we write

$$\mathcal{R}(i\sigma)(a - i\sigma) = -\frac{i}{\sigma}[\mathcal{R}(i\sigma)\triangle^2 - I]. \tag{3.36}$$

If $f \in H^2_0(\Omega)$ and $u = \mathcal{R}(i\sigma)\triangle^2 f \in H^2_0(\Omega)$, then

$$(\triangle^2 + a(x)i\sigma + \sigma^2)u = \triangle^2 f \in (H^2_0)'(\Omega). \tag{3.37}$$

Multiplying (3.37) by $\overline{u}$, integrating by parts and taking the real part, we get

$$\|u\|^2_{H^2_0(\Omega)} - \sigma^2\|u\|^2_{L^2(\Omega)} \leqslant \|\triangle^2 f\|_{(H^2_0)'(\Omega)}\|u\|_{H^2_0(\Omega)} \lesssim \|f\|_{H^2_0(\Omega)}\|u\|_{H^2_0(\Omega)}, \tag{3.38}$$

therefore

$$\|u\|^2_{H^2_0(\Omega)} \leqslant O(1)\big(\sigma^2\|u\|^2_{L^2(\Omega)} + \|f\|^2_{H^2_0(\Omega)}\big). \tag{3.39}$$

Combining the estimate (3.39) with (3.35) , we get

$$\|u\|_{H^2_0(\Omega)} \leqslant O(|\sigma|^3)\|f\|_{H^2_0(\Omega)}. \tag{3.40}$$

Therefore

$$\mathcal{R}(i\sigma)(a - i\sigma) = O(|\sigma|^2) : H^2_0(\Omega) \longrightarrow H^2_0(\Omega). \tag{3.41}$$

Combining together (3.31), (3.33), (3.35) and (3.41) with the fact that A has no imaginary spectrum, we get the basic bound:

$$\left\|(A - i\sigma)^{-1}\right\|_{\mathcal{L}(H(\Omega))} \leqslant C(1 + |\sigma|^2), \quad \sigma \in \mathbb{R}.$$

The proof of Theorem 1.1 is complete.

4. Connection problems

If $a \equiv 1$ then in particular the Geometric Control Condition (GCC) holds and we easily get better high frequency resolvent estimates. If $\sigma \in \mathbb{R}\setminus\{0\}$, $u \in H^4(\Omega) \cap H^2_0(\Omega)$ and $f \in L^2(\Omega)$ are such that

$$(\triangle^2 - i\sigma - \sigma^2)u = f,$$

then

$$\sigma\,\|u\|^2_{L^2(\Omega)} = -\operatorname{Im}\langle f, u\rangle \leqslant \|f\|\,\|u\|\,,$$

from which we deduce that

$$\|\mathcal{R}(i\sigma)\|_{\mathcal{L}(L^2(\Omega)} \lesssim \frac{1}{\sigma}$$

and consequently that $(A - i\sigma)^{-1}$ is uniformly bounded for $|\sigma| \geqslant 1$. Furthermore, there is no "low frequencies effects", so the energy decay is limited by the contribution of high energy. In this case we know that the contribution of high frequencies decays exponentially without any loss of derivative.

Theorem 4.1 (Energy-decay) *Let $U_0 \in H$, then there exists $C \geqslant 0$ and $\alpha \geqslant 0$ such that for $t \geqslant 1$ we have*

$$\left\|e^{-tA}U_0\right\|_H \leqslant Ce^{-\alpha t}\,\|U_0\|_H\,.$$

4.1. Improved decay estimate.

It is well understood that plate equation with "hinged" boundary condition is linked to Schroïdinger equation since $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$. However, even without Geometric Control Condition an observability estimate or equivalently an exponential decay rate for the damped Schroïdinger equation can hold (see [9]). The energy decay for the damped plate equation in the waveguide may be an exponential one. To improve the decay rate of this paper, it may be an open problem.

4.2. The damped plate equation on the wave guide with "clamped" boundary condition.

Now we discuss the damped plate equation in the wave guide Ω with "clamped" boundary condition. Consider the following equation:

$$\partial_t^2 u + \Delta^2 u + a\partial_t u = 0 \quad \text{on } \mathbb{R}_+ \times \Omega, \tag{4.1}$$

subject to the initial conditions

$$u(0,x) = u_0 \quad \partial_t u(0,x) = u_1 \tag{4.2}$$

and" clamped" boundary conditions

$$u = \partial_n u = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega. \tag{4.3}$$

In a spectral point view, the main difficulty that occurs in the problem (4.1)-(4.3) that we can not deduce the eigenvalues of bi-Laplacian Δ_y^2 from eigenvalues of Laplace $-\Delta_y$. Indeed, let μ be a eigenvalue of Δ_y^2, it can not $\sqrt{\mu}$ is an eigenvalue of $-D_y$. To do this, we consider a function $\phi \in H^2$ such that

$$\Delta_y^2 \phi = \mu\phi \quad \textit{and} \quad \phi = \partial_\nu \phi = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \tag{4.4}$$

from the unique continuation, ϕ vanishes identically (see some comment in introduction [19]). Let discuss now about resolvent estimates, for the high frequencies part, we use the same ideas as in Proposition 3.1. The main difference with the hinged boundary conditions is that in this case the transverse operators bi-Laplacian Δ_y^2 and Laplacian $-\Delta_y$ on ω, whose first eigenvalues are nulls, Therefore, there is a singularity at 0. Thus, there is a low frequency effect which need to study the contribution of low frequencies. Extending the results of this article to this case is an interesting problem.

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