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On Representations of Real Quadratic Integers as Sums of Unitary Fractions*

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ABSTRACT: We verify that each natural number $t > 1$ can be expressed as a sum of unitary fractions of different natural numbers such that the least common multiple among them is coprime with a fixed value v. Using this fact, we characterize the real quadratic integers which appear in the representation of 1 as a unitary fraction sum over the real quadratic integer ring.

Key Words: Representation of unity, unitary fractions, real quadratic integers.

Contents

1. Introduction

The representation of rational numbers as sums of unit fractions dates back to the time of ancient Egypt. An s-uple $\{x_i\}_{i\leq s} \in \mathbb{Z}^s$ is called a *representation of* $t \in \mathbb{N}$ as unit fractions of length s if it solves the Diophantine equation $t = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_s}$. Sylvester [7] considered the sequence given by $A_1 = 2$ and $A_{i+1} = 1 + \prod_{j=1}^{i} A_j$, where $\{A_1, A_2, \ldots, A_{i-1}, A_i - 1\}$ is a representation of 1 such that $A_i - 1 = \operatorname{lcm}(A_1, A_2, \ldots, A_{i-1}, A_i - 1)$. Furthermore, note that such sequence is a solution to

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{s-1}} + \frac{1}{\prod x_i}.$$

For a fixed s, Curtiss [4] proved that Sylvester sequence maximizes the product $\prod x_i$ among all solutions of the above equation. Note that, in Sylvester's solution the smallest and the greatest term are 2 and $\operatorname{lcm}(x_1, x_2, \ldots, x_{s-1})$, respectively. However, there are representations of 1 whose smallest term is greater than 2. For instance, given a t-perfect number n, there exists a representation of 1 with smallest term t, which means that the set of all divisors of n is a representation of t. It is easy to check that there is no solution of length an even number s where every x_i is an odd number. Burshtein proved in [3] that the aforementioned Diophantine equation does not have solutions for $s \leq 7$ whenever x_i are odd numbers. He showed, moreover, that there are only five solutions for $s = 9$. Arce-Nazario et al. [2], found all possible representations of 1 of length $s = 11$ where each x_i is an odd number.

Note that all representations of 1 found in [2,3] satisfy that each x_i is coprime with 2. It is reasonable to ask whether there are representations of $t > 1$ such that each x_i is coprime with a fixed number v. Therefore, we will restrict our discussion to representations $\{x_1 < x_2 < \cdots < x_s\}$ of 1 over $\mathbb{N}$ of the form

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{s-1}} + \frac{1}{\operatorname{lcm}(x_1, x_2, \ldots, x_{s-1})}, \tag{1.1}$$

such that x_s is coprime with v.

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Recent studies have explored the representation of integers as sums of unit fractions in relation to various mathematical topics. Harris and Louwsma [6] examined arithmetical structures on complete graphs, while Archer and collaborators [1] investigated the critical groups of arithmetical structures on both star and complete graphs. They observed that arithmetical structures on the star graph S_n, consisting of n leaves connected to a central vertex v_0, are in bijection with the number of distinct representations of the positive integer d_0 (associated to v_0) as sums of n unit fractions. In a different approach, Elsholtz and Planitzer [5] focused on integer representations motivated by a generalized Erdős-Straus conjecture. This conjecture addresses the number of distinct representations of the rational number $\frac{m}{n}$, with fixed integers m and n, as a sum of k unit fractions.

Our purpose is to study such kind of representations. This will help us to characterize how do the unitary fractions of real quadratic integers appear in the representations of 1. In Section 2, we will fix notation and the tools which are required in the proof of our auxiliary theorem, whose proof is given in Section 3. In the final section, we will describe how do the real quadratic integers appear in the representations of 1 over the real quadratic integer ring using its standard norm N and trace T.

Summarizing, the main theorem of this paper is:

Theorem 1.1 *Let $x = a + b\sqrt{d}$ be a real quadratic integer such that $N(x) > 1$ and $a > 0$. There exists a representation $\{x_1, x_2, \ldots, x_s\}$ of 1 such that $x_1 = x$, $x_i = a_i + b_i\sqrt{d}$, $N(x_i) > 1$ and $a_i > 0$, if and only if, $T(x) \leq N(x)$.*

2. Basic Facts

In this section we introduce some basic notions and terminology useful to show our main result. Furthermore, we give an algorithm to determine the numbers in Lemma 2.3.

Lemma 2.1 *Let $\{x_1, x_2, \ldots, x_s\}$ be a representation of some integer t such that x_s is divisible by x_i for each $i \leq s$. Then $|x_s| = \operatorname{lcm}(x_1, x_2, \ldots, x_{s-1})$.*

Proof: Suppose contrary to our claim that this is not true. Then there exists a prime number p that divides x_s and the highest power of p in x_s does not divide every x_i for $i < s$. Since

$$t = \sum_{i=1}^{s} \frac{1}{x_i},$$

we have

$$tx_s = \sum_{i=1}^{s-1} \frac{x_s}{x_i} + 1 \equiv 1 \mod p.$$

This contradicts the fact $tx_s \equiv 0 \mod p$. Our claim is proved. □

From now on, we fix the natural number $v \geq 2$. It follows an adaptation of a construction from [8]. For any $i \geq 0$ define $a_i := 1 + iv$ and $P_s := \prod_{i=1}^{s} a_i$. A divisor d^* of P_s is called a *convenient divisor* of P_s, or a *stardivisor*, if d^* is a product of different numbers a_i.

Lemma 2.2 *[8, Lemma 3.2] Every integer m with $\frac{v^3}{2} \leq m \leq v^4$ can be written as the sum of $2v$ or less distinct terms a_i with $i \leq v^2$.*

Next lemma gives us more information about the terms a_i that appear in the sum of m. In view of describing these terms we take $1 \leq r \leq v$ with $r \equiv m \mod v$, and consider $R = v^3 r - \frac{vr^2}{2} + \frac{vr}{2} + r$. Note that

$$\begin{aligned}
a_{v^2-r+1} + a_{v^2-r+2} + \cdots + a_{v^2} &= \sum_{i=1}^{r} a_{v^2-(r-i)} \\
&= \sum_{i=1}^{r} (1 + v(v^2 - (r-i))) \\
&= r + v^3 r - vr^2 = v\frac{r(r+1)}{2} \\
&= v^3 r - \frac{vr^2}{2} + \frac{vr}{2} + r \\
&= R,
\end{aligned}$$

and $v^3 \leq R \leq v^4$. Furthermore,

$$\begin{aligned}
a_{v^2-v-r+1} + a_{v^2-v-r+2} + \cdots + a_{v^2} &= \sum_{i=1}^{v+r} a_{v^2-(v+r-i)} \\
&= v^4 + v^3 r - \frac{v^3}{2} - v^2 r - \frac{vr^2}{2} + \frac{v^2}{2} + vr + v + r \\
&\geq v^4 + v^3 r - 2v^3 + \frac{v^2}{2} + vr + v + r \\
&\geq v^4
\end{aligned}$$

Lemma 2.3 *Let $v \geq 4$ and $m \in [v^3, v^4]$ be integers. If $m \leq R$, then there exist r distinct terms a_{i_k} such that $m = \sum_{k=1}^{r} a_{i_k}$. Otherwise, there exist $v + r$ distinct terms in the sum. In both cases, $1 \leq i_k \leq v^2$.*

Proof: First, we will assume that $m \leq R$. Set $h_0 = a_1 + a_2 + \cdots + a_{r-1} + a_r \leq v^3 \leq m$, and note that $h_0 \leq v^3 \leq m$. So, it is enough to consider $h_0 < m$. Follow from $r \equiv m$ and $1 \equiv a_i \mod v$ that $m - h_0 \equiv 0 \mod v$. Hence, $h_0 + v \leq m$. It allows us to take a maximal index $i_1 \leq v^2$ such that $h_1 = h_0 - a_r + a_{i_1} \leq m$. If $h_1 = m$, we obtain our result. We now proceed by recurrence, and consider $i_1 > i_2 > \cdots > i_k$ such that $h_k = a_1 + a_2 + \cdots + a_{r-k} + a_{i_k} + a_{i_{k-1}} + \cdots + a_{i_1} \leq m$. If $h_k = m$ the result is obtained. Otherwise, we can replace $r - k$ by i_{k+1} in a similar way to the case of h_0 where $i_{k+1} < i_k$. It is clear that there exists a step in which $h_k = m$ as a consequence of $R = a_{v^2-r+1} + a_{v^2-r+2} + \cdots + a_{v^2}$.

If $m > R$, take $h_0 = a_1 + a_2 + \cdots + a_{v+r-1} + a_{v+r}$. For any $v \geq 5$, it follows from $0 \leq v^2 - 5v \leq v^2 - 3v - 2r$ and $0 \leq rv^2 - 3rv - 2r^2 < (2r-1)v^2 - (2r+1)v - 2r^2$ that $0 < (2r-1)v^2 - (2r+1)v - 2r^2$. Thus,

$$v^2 + 2vr + v + r^2 < 2v^2 r - r^2.$$

Therefore,

$$(v + r)(v + r + 1) < 2v^2 r - r^2 + r.$$

Multiplying by v, the last inequality means that $h_0 - v < R$. Since $h_0 - R \equiv 0 \mod v$, then $h_0 \leq R$. Now, in a way analogous to our discussion in the former case, we can choose $i_1 \leq v^2$ with the same properties and so we can apply the above recurrence in a similar manner. Again, there exists a step in which $h_k = m$ as a consequence of $v^4 < a_{v^2-v-r+1} + a_{v^2-v-r+2} + \cdots + a_{v^2}$. Case $v = 4$ is easily computable. □

In order to obtain the numbers a_i, the following algorithm depends on the inputs v and m.
Input: v and m as in Lemma 2.3.
Output: $i_1, i_2, \ldots, i_r$ such that $\sum_{k=1}^{r} a_{i_k} = m$.

```
1. r=m mod v;
2. R=v^3r - vr^2/2 + vr/2 + r;
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If m=R, then Output v^2-r+1,v^2-r+2, ... ,v^2 and halt.
If m<R
    H(r+1)=v^2,
    For i=1,2, ...,r, H(i)=i,
    h_0= sum of a_{H(i)} for i less than r+1, where a_k=1+kv,
    i=0,
    While h_0 is less  than m
        j=H(r-i),
        While h_0+v <= m and j < H(r-i+1)
             h_0+=v,
             j++,
        H(r-i)=j,
        i++,
    Output H(1),H(2), ... ,H(r).
Else
    H(v+r+1)=v^2,
    For i=1,2, ..., v+r, H(i)=i,
    h_0= sum of a_{H(i)} for i less than v+r+1, where a_k=1+kv,
    i=0,
    While h_0 is less  than m
        j=H(r-i),
        While h_0+v <= m and j < H(r-i+1)
             h_0+=v,
             j++,
        H(r-i)=j,
        i++,
    Output H(1),H(2), ... ,H(v+r).
```

In order to clarify in a better way the result of the latter lemma, we give two examples to use the above code:

Example 2.1 *The code has as input the values v and m. Consider $v = 10$ and $m = 2025$. Thus, we obtain $r = m \mod v$ as $r = 5$. These values give $R = 4905$. The conditional $m < R$ is true, then we calculate the vector $H = [1, 2, 3, 4, 5, 100]$. Hence, $h_0 = 155$. In this part, the while cicle gives $j = 100$. Thus, the new vector $H = [1, 2, 3, 4, 100, 100]$ and $h_1 = 1105$. In the next step, we modify $H(4)$ to $j = 96$. So, $H = [1, 2, 3, 96, 100, 100]$ and $2025 = a_1 + a_2 + a_3 + a_{96} + a_{100}$. The output of the algorithm is $H = [1, 2, 3, 96, 100, 100]$.*

Example 2.2 *Now, consider $v = 7$ and $m = 2025$. Thus, we obtain $r = 2$. These values give $R = 681$ and the conditional $R < m$ is true. We calculate the vector $H = [1, 2, 3, 4, 5, 6, 7, 8, 9, 49]$. Hence, $h_0 = 324$. In this part, the while gives $j = 49$. Thus, the new vector $H = [1, 2, 3, 4, 5, 6, 7, 8, 49, 49]$ and $h_1 = 604$. In the next step, we modify $H(8)$ to $j = 48$. So, $H = [1, 2, 3, 4, 5, 6, 7, 48, 49, 49]$ and $h_2 = 884$. Following the algorithm, we obtain*

$$
\begin{aligned}
&i_9 = 49, && h_1 = 604, && H = [1, 2, 3, 4, 5, 6, 7, 8, 49, 49],\\
&i_8 = 48, && h_2 = 884, && H = [1, 2, 3, 4, 5, 6, 7, 48, 49, 49],\\
&i_7 = 47, && h_3 = 1164, && H = [1, 2, 3, 4, 5, 6, 47, 48, 49, 49],\\
&i_6 = 46, && h_4 = 1444, && H = [1, 2, 3, 4, 5, 46, 47, 48, 49, 49],\\
&i_5 = 45, && h_5 = 1724, && H = [1, 2, 3, 4, 45, 46, 47, 48, 49, 49],\\
&i_4 = 44, && h_6 = 2004, && H = [1, 2, 3, 44, 45, 46, 47, 48, 49, 49],\\
&i_3 = 6, && h_6 = 2025, && H = [1, 2, 6, 44, 45, 46, 47, 48, 49, 49].
\end{aligned}
$$

Then, the output of the algorithm is $H = [1, 2, 6, 44, 45, 46, 47, 48, 49, 49]$.

From the last lemma, we can consider that for any $v^3 \le m \le v^4$, the term $a_0 = 1$ appears in the sum of m. To increase the interval we shall make a slight change of hypothesis in Lemma 3.3 from [8] by imposing the conditions $s \ge v^2$ and

$$\sum_{i=1}^{s-1} \frac{1}{a_i} > 1, \tag{2.1}$$

namely, we prove the following lemma.

Lemma 2.4 *Let v and s satisfy the latter conditions. If $v^3 \le m \le P_s$, then m is the sum of distinct convenient divisors of P_s including $a_0 = 1$.*

Proof: Note that for any $i < v^2$, the number a_i is a convenient divisor of P_s. So, it is enough to prove our lemma for $m > v^4$. To do this, we subtract from m some convenient distinct divisors greater than a_{v^2} in order to obtain a number in the interval $[v^3, v^4]$. Let $d_1 < d_2 < \cdots < d_q$ be all proper convenient divisors of P_s greater than a_{v^2}. Note that for every $i < s$ and $i \ne v^2$, the integer $P_s/a_i > a_{v^2}$. If $i = v^2$,

$$\begin{aligned} P_s/a_{v^2} &> a_{v^2-2}a_{v^2-1} \\ &= (v^3 - 2v + 1)(v^3 - v + 1) \\ &= (v^3 + 1) + (v^3 + 1)(v^3 - 3v) + 2v^2 \\ &> (v^3 + 1) = a_{v^2}. \end{aligned}$$

Thus, for every $i < s$, the integer P_s/a_i is some d_k in the list. Furthermore, $d_1 = v^3 + v + 1 < m - v^3 + 1$ and from Equation (2.1)

$$\begin{aligned} d_1 + d_2 + \cdots + d_q &\ge \sum_{i=1}^{s-1} \frac{P_s}{a_i} \\ &= P_s \sum_{i=1}^{s-1} \frac{1}{a_i} > P_s \\ &> m - v^3 + 1. \end{aligned}$$

For any natural number $k < 2^q$, define $S(k) = \sum_{\alpha_i = 1} d_{i+1}$ where $k = \alpha_0 + \alpha_1 2 + \alpha_2 2^2 + \cdots + \alpha_q 2^q$. Choose the greatest integer $M < 2^{q+1}$ such that $S(M) < m - v^3 + 1$. It holds that $m - S(M) < v^4 + 1$. In fact, assume that $m - S(M) > v^4$. Due to $m - S(M+1) < v^3$, then

$$S(M+1) - S(M) > v^4 - v^3,$$

and there exists some $j > 0$ such that

$$d_j - (d_1 + d_2 + \cdots + d_{j-1}) > v^4 - v^3. \tag{2.2}$$

Since $d_2 < v^4 - v^3$, then $j > 2$ is the smallest value such that $\alpha_j = 0$ in the binary expansion of M. Now, taking J as the smallest number for which inequality (2.2) holds, from minimality it follows that $d_J > 2d_{J-1}$. Since for any $h > 1$ the inequality $2a_{h-1} > a_h$ holds, we conclude that d_J is the product of at least 2 terms. If $a_h > 1$ is a term in the factorization of d_J, then

$$2d_{J-1} \ge 2d_J \frac{a_{h-1}}{a_h} > d_J > 2d_{J-1}.$$

Hence, $v^3 \le m - S(M) \le v^4$. Using the last lemma, $m - S(M)$ is the sum of some a_i in which a_0 appears, concluding that m is the sum of convenient divisors. □

3. Representation of positive integers

In this section we employ the above lemmas to prove a theorem which allows us to obtain a representation of 1 with the first term being an arbitrary positive integer number, namely.

Theorem 3.1 *Let $t > 1$ and v be natural numbers. Then there exists $N = N(t, v) \in \mathbb{N}$ such that for every $s \geq N$ with $s \equiv N \mod 2$, there is a representation of t of length s satisfying that $x_1 = 1$ and v is coprime with x_s. Furthermore, if v is an even number, then $s \equiv t \mod 2$.*

Note that parity condition cannot be omitted. In fact,

$$t = \sum_{i=1}^{s} \frac{1}{x_i},$$

where $x_1, \ldots, x_s$ are odd numbers, and putting $P = x_1 x_2 \cdots x_s$ we have

$$Pt = \sum_{i=1}^{s} \frac{P}{x_i}.$$

Since $Pt \equiv t \mod 2$ and every number $\dfrac{P}{x_i}$ is an odd number, it holds that $s \equiv t \mod 2$.

Proof: Let $t > 1$ be an integer, $v \geq 4$ and a_i as before. There exists $m > 0$ such that $\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{m-1}} < t \leq \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_m}$. Thus $t = \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{m-1}} + \frac{k}{P_m}$ for some $0 < k < P_m$. Let $n \geq m$ satisfies Equation (2.1) and be such that $\frac{S}{P_n} = \frac{k}{P_m}$ for some $v^3 < S < P_n$. Last lemma allows us to write S as a sum of distinct convenient divisors of P_n, including 1. Therefore, $\frac{S}{P_n}$ is the sum of unitary fractions where P_n is the greatest denominator. We denote such denominators by $x_1 < x_2 < \cdots < x_d = P_n$. Since $\frac{S}{P_n} \leq \frac{1}{a_m}$, it is clear that $a_m \leq x_1$, hence $a_1 < a_2 < \cdots < a_{m-1} < x_1 < x_2 < \cdots < x_d = P_n$. Thus, t is expressed as a sum of unitary fractions where every denominator is coprime with v. Furthermore, from Lemma 2.1, it follows that P_n is the least common multiple of all denominators. This means, $\{1, a_1, a_2, \ldots, a_{m-1}, x_1, x_2, \ldots, x_{d-1}, x_d\}$ is a representation of t of length $N = N(t, v) = m + d$. Set $P_n = AB$ with

$$A = a_1 a_2 \cdots a_{v-3}(v-1) \quad \text{and} \quad B = (v-1)a_{v-1} \cdots a_n.$$

Clearly, $A \equiv B \equiv -1 \mod v$ and

$$\frac{1}{AB} = \frac{1}{A(A+B+1)} + \frac{1}{B(A+B+1)} + \frac{1}{AB(A+B+1)},$$

where $A(A+B+1) \equiv B(A+B+1) \equiv 1 \mod v$ and $AB(A+B+1) \equiv -1 \mod v$. Note that $AB(A+B+1)$ is the least common multiple of all denominators. Thus,

$$\{1, a_1, \ldots, a_{m-1}, x_1, \ldots, x_{d-1}, A(A+B+1), B(A+B+1), AB(A+B+1)\}$$

is a representation of t of length $N+2$. Now, we have $B' = B(A+B+1) \equiv 1 \mod v$, hence, $A+B'+1 \equiv 1 \mod v$ and we use recursion to obtain representations of t for arbitrary length $s \geq N$ where $s \equiv N \mod 2$. □

4. Unit fractions in the real quadratic integers

In this section we analyze the real quadratic integer ring and we give conditions to obtain a representation of 1 in terms of quadratic integers with positive norm. Summarizing, we know that the ring of real quadratic integers $\mathcal{O}_{\sqrt{d}}$ has a basis over $\mathbb{Z}$ given by $\{1, w\}$ where

$$w = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \mod 4, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4. \end{cases}$$

So, for any free square natural number $d > 0$, we have $a + b\sqrt{d} \in \mathcal{O}_{\sqrt{d}}$ whether a and b are either both integers, or both halves of odd integers (only if $d \equiv 1 \mod 4$). We will use the notion of norm and trace of the real field $\mathbb{Q}(\sqrt{d})$ given by $N(a + b\sqrt{d}) = a^2 - b^2 d$ and $T(a + b\sqrt{d}) = 2a$.

We showed in the last section that any natural number greater than one is the smallest denominator in a representation of 1 as a sum of distinct (positive) unit fractions. In this section, we want to characterize the real quadratic integers with positive norm that appear in the representation of 1. This result is Theorem 1.1 and now we give its proof:

Proof: [Proof of Theorem 1.1]

If $T(x) \leq N(x)$, then there exists some integer $0 \leq t < 4N(x)$ such that

$$\frac{1}{a + b\sqrt{d}} + \frac{1}{a - b\sqrt{d}} + \frac{t}{4N(x)} = \frac{T(x)}{N(x)} + \frac{t}{4N(x)} = 1.$$

For $t = 0$ and $t = 1$ we have a representation of 1 of length 2 and 3, respectively. For any $1 < t < 4N(x)$, it follows from Theorem 3.1 with $v = 4N(x)$ that there exists a representation $\{y_1 = 1, y_2, y_3, \cdots, y_{N(t,v)}\}$ of t of length $N(t, v)$. So,

$$1 = \frac{1}{a + b\sqrt{d}} + \frac{1}{a - b\sqrt{d}} + \sum_{i=1}^{N(t,v)} \frac{1}{4y_i N(x)}.$$

Conversely, if $\{x_1 = x, x_2, x_3, \cdots, x_s\}$ is such a representation of 1, then

$$1 = \frac{a - b\sqrt{d}}{N(x)} + \sum_{i=2}^{s} \frac{a_i - b_i\sqrt{d}}{N(x_i)}$$

Thus, we get the following two equalities:

$$1 = \frac{a}{N(x)} + \sum_{i=2}^{s} \frac{a_i}{N(x_i)} \qquad \text{and} \qquad 0 = \frac{b}{N(x)} + \sum_{i=2}^{s} \frac{b_i}{N(x_i)}.$$

Since $N(a_i + b_i\sqrt{d}) > 0$, we have $\delta b_i\sqrt{d} < a_i$ where $\delta = \pm 1$. Hence,

$$0 = \frac{\delta b\sqrt{d}}{N(x)} + \sum_{i=2}^{s} \frac{\delta b_i\sqrt{d}}{N(x_i)} < \frac{\delta b\sqrt{d}}{N(x)} + \sum_{i=2}^{s} \frac{a_i}{N(x_i)} = \frac{\delta b\sqrt{d}}{N(x)} + 1 - \frac{a}{N(x)}.$$

This means that $a \pm b\sqrt{d} < N(x)$, which implies $1 < a \pm b\sqrt{d}$ because $a > 0$. Consequently, $0 < N(a - 1 + b\sqrt{d})$, hence $T(x) = 2a \leq a^2 - b^2 d = N(x)$. □

Note that in the case $T(x) < N(x)$, Theorem 3.1 guarantees the existence of representations of 1 with arbitrary length.

Given $x = a + b\sqrt{d}$ a real quadratic integer of positive norm such that $a > 0$ and $T(x) < N(x)$, from the last theorem remain open the following questions:

Question 1: Are there representations of 1 in which $a - b\sqrt{d}$ do not belong to the representation?

Question 2: Is it possible to find similar conditions like those given in Theorem 3.1? Here we want to impose $N(x_i) < N(x_{i+1})$. Is it possible to impose (in terms of ideals) the least common multiple as Equation 1.1?

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