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Global stability of a multi-dimensional system of rational difference equations of higher-order with Pell-coefficients

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ABSTRACT: This article considers a new multi-dimensional system of rational difference equations of higher-order with Pell-coefficients. In this system, the Pell-coefficients are allowed to Pell-sequence, while it is considered constant for this system. This system generalizes the same as the first-order system introduced in this article. We show that the solutions of this system are also associated with Pell-numbers. The global stability of positive solutions of this system is also established.

Key Words: Stability, Pell numbers, Pell-Lucas numbers, Binet formula, system of difference equations.

Contents

1. Introduction

The uni-dimensional bilinear difference equation has received the interest of several workers including Adamović (1971), Boole (1880), Brand (1955), Jordan (1956), Krechmar (1974), Mitrinović and Adamović (1980) and Mitrinović and Keckić (1984). As also observed by some workers (e.g. Stević et al. 2019 and references therein), this difference equation can be solved in several ways. More lately, Stević et al. [32] presented the definition of a two-dimensional bilinear system of difference equations which, according to their definition, is two-difference equations satisfying the equations

$$u_{n+1}^{(1)} = \frac{\alpha u_n^{(2)} + \beta}{\gamma u_n^{(2)} + \delta}, u_{n+1}^{(2)} = \frac{\alpha u_n^{(1)} + \beta}{\gamma u_n^{(1)} + \delta}, n \geq 0, \tag{1.1}$$

where α, β, γ and δ are satisfy some regularity conditions, such that the general solutions via the generalized Fibonacci sequence. In this regard, many papers have recently given formulas for solutions of difference equations and systems in terms of the Fibonacci sequence (see., [12], [31], [34], [36]). Furthermore, among the methods that proved the solvability of System (1.1) is to transform it into the most important system of homogeneous linear difference equation of the 2nd-order, which has the following form:

$$t_{n+1} = \alpha t_n + \beta t_{n-1}, r_{n+1} = \alpha r_n + \beta r_{n-1}, n \geq 1,$$

where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$ such that $\beta \neq 0$, in particular, we give information about the Pell sequence that establishes a significant part of our study, defined as follows

$$P_{n+1} = 2P_n + P_{n-1}, n \geq 1, \tag{1.2}$$

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with initial conditions $P_0 = 0$ and $P_1 = 1$. The following Binet formula of the Pell numbers gives the solution to equation (1.2),

$$P_n = \frac{a^n - b^n}{a - b},$$

where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$. Moreover, the Pell-Lucas sequence has the same homogeneous linear difference equation with constant coefficients as the Pell sequence,

$$Q_{n+1} = 2Q_n + Q_{n-1}, n \geq 1,$$

with distinct initial conditions $Q_0 = Q_1 = 2$ and the closed-form expressions for the Pell-Lucas numbers are $Q_n = a^n + b^n$. Now, the search for closed-form solutions of systems of difference equations is a classic problem (see., [4]-[17], [19]-[20], [21]-[22], [26]-[31], [35], [37]), so, this article gives formulas for solutions in terms of Pell sequence, we consider the following $m-$dimensional system of difference equations with Pell-coefficients,

$$u_{n+1}^{(i)} = \frac{P_{k+2} + P_{k+1} u_{n-l}^{(i+1) \bmod m}}{P_{k+3} + P_{k+2} u_{n-l}^{(i+1) \bmod m}}, n, k, l, m \in \mathbb{N}_0, i \in \{1, ..., m\}, \tag{1.3}$$

and the initial values $u_{-j}^{(i)}$, $i \in \{1, ..., m\}$, $j \in \{0, 1, ..., l\}$.

2. Main results

To solve system (1.3) we require to utilize the following lemmas.

Lemma 2.1 *Let $(P_n, n \geq 0)$ the Pell sequence and $(Q_n, n \geq 0)$ the Pell-Lucas sequence, we have some important relations, for $n, m \in \mathbb{N}$,*

$$\begin{aligned} P_m P_{n+1} + P_{m-1} P_n &= P_{m+n}, \\ P_{n-1} P_{n+1} - P_n^2 &= (-1)^n, \\ P_{n-1} + P_{n+1} &= Q_n, \\ P_{m(n+1)} P_{m+1} + (-1)^{m+1} P_{mn} &= P_m P_{m(n+1)+1}, \\ P_{m(n+1)} - P_{m+1} P_{mn} &= P_{mn-1} P_m \\ P_{m(n+1)+1} - P_{m+1} P_{mn+1} &= P_{mn} P_m \end{aligned}$$

Moreover, the sequence $(P_n P_{n+1}^{-1}, n \geq 1)$ converges to $a^{-1} = -b$.

Lemma 2.2 *Consider the homogeneous linear difference equation with constant coefficients*

$$w_{n+1} - Q_{m_k} w_n + (-1)^{m_k} w_{n-1} = 0, n \geq 0, \tag{2.1}$$

with initial conditions w_0, $w_{-1} \in \mathbb{R}^*$. Then,

$$\forall n \geq 0, \; w_n = \frac{P_{m_k(n+1)}}{P_{m_k}} w_0 + (-1)^{m_k+1} \frac{P_{m_k n}}{P_{m_k}} w_{-1},$$

where $(P_n, n \geq 0)$ is the Pell sequence and $(Q_n, n \geq 0)$ is the Pell-Lucas sequence.

Proof: Difference equation (2.1) is ordinarily solved by using the following characteristic polynomial, $\lambda^2 - Q_{m_k} \lambda + (-1)^{m_k} = (\lambda - a^{m_k})(\lambda - b^{m_k}) = 0$, roots of this equation are $\lambda_1 = a^{m_k}, \lambda_2 = b^{m_k}$. These roots are linked to the roots of the Pell number sequence. Then the closed form of general solution of the equation (2.1) is

$$\forall n \geq -1, \; w_n = c_1 a^{m_k n} + c_2 b^{m_k n},$$

where w_0, w_{-1} are initial values such that

$$\begin{cases} w_0 = c_1 + c_2 \\ w_{-1} = \dfrac{c_1}{a^{m_k}} + \dfrac{c_2}{b^{m_k}} \end{cases},$$

and we have

$$c_1 = \frac{a^{m_k} w_0 - (-1)^{m_k} w_{-1}}{a^{m_k} - b^{m_k}}, c_2 = \frac{(-1)^{m_k} w_{-1} - b^{m_k} w_0}{a^{m_k} - b^{m_k}},$$

after some calculations, we get

$$\begin{aligned} w_n &= \frac{a^{m_k} w_0 - (-1)^{m_k} w_{-1}}{a^{m_k} - b^{m_k}} a^{m_k n} + \frac{(-1)^{m_k} w_{-1} - b^{m_k} w_0}{a^{m_k} - b^{m_k}} b^{m_k n} \\ &= \frac{\left(a^{m_k(n+1)} - b^{m_k(n+1)}\right) w_0 + (-1)^{m_k} \left(b^{m_k n} - a^{m_k n}\right) w_{-1}}{a^{m_k} - b^{m_k}} \\ &= \frac{P_{m_k(n+1)} w_0 + (-1)^{m_k+1} P_{m_k n} w_{-1}}{P_{m_k}}. \end{aligned}$$

The lemma is proved.□ □

Lemma 2.3 *Consider the following rational difference equation,*

$$u_{n+1} = \frac{P_{m_k} + P_{m_k-1} u_n}{P_{m_k+1} + P_{m_k} u_n}, n \geq 0. \tag{2.2}$$

Then,

$$\forall n \geq 1, \; u_n = \frac{P_{m_k n-1} u_0 + P_{m_k n}}{P_{m_k n} u_0 + P_{m_k n+1}}.$$

Proof: Using the change of variables $v_n = P_{m_k+1} + P_{m_k} u_n$, we can write (2.2) as

$$v_{n+1} = \frac{\left(P_{m_k}^2 - P_{m_k-1} P_{m_k+1}\right) + \left(P_{m_k-1} + P_{m_k+1}\right) v_n}{v_n} = \frac{(-1)^{m_k+1} + Q_{m_k} v_n}{v_n}, n \geq 0, \tag{2.3}$$

and $v_n = \dfrac{w_n}{w_{n-1}}$, we get $w_{n+1} - Q_{m_k} w_n + (-1)^{m_k} w_{n-1} = 0, n \geq 0$, by Lemma 2.2, the closed form of general solution of the equation (2.3) is

$$v_n = \frac{P_{m_k(n+1)} w_0 + (-1)^{m_k+1} P_{m_k n} w_{-1}}{P_{m_k n} w_0 + (-1)^{m_k+1} P_{m_k(n-1)} w_{-1}} = \frac{P_{m_k(n+1)} v_0 + (-1)^{m_k+1} P_{m_k n}}{P_{m_k n} v_0 + (-1)^{m_k+1} P_{m_k(n-1)}}, n \geq 1,$$

then,

$$\begin{aligned} u_n &= P_{m_k}^{-1} \left(v_n - P_{m_k+1}\right) \\ &= P_{m_k}^{-1} \left(\frac{P_{m_k(n+1)} v_0 + (-1)^{m_k+1} P_{m_k n}}{P_{m_k n} v_0 + (-1)^{m_k+1} P_{m_k(n-1)}} - P_{m_k+1} \right) \\ &= P_{m_k}^{-1} \left(\frac{P_{m_k(n+1)} P_{m_k} u_0 + \left(P_{m_k(n+1)} P_{m_k+1} + (-1)^{m_k+1} P_{m_k n}\right)}{P_{m_k n} P_{m_k} u_0 + \left(P_{m_k n} P_{m_k+1} + (-1)^{m_k+1} P_{m_k(n-1)}\right)} - P_{m_k+1} \right) \\ &= P_{m_k}^{-1} \left(\frac{P_{m_k(n+1)} u_0 + P_{m_k(n+1)+1}}{P_{m_k n} u_0 + P_{m_k n+1}} - P_{m_k+1} \right) \\ &= \frac{P_{m_k n-1} u_0 + P_{m_k n}}{P_{m_k n} u_0 + P_{m_k n+1}}, n \geq 1. \end{aligned}$$

The lemma is proved.□ □

2.1. On the system (2.4)

In this subsection, we consider the following system of difference equations of 1st-order,

$$u_{n+1}^{(i)}=\frac{P_{k+2}+P_{k+1}u_n^{(i+1)\bmod m}}{P_{k+3}+P_{k+2}u_n^{(i+1)\bmod m}},n,k,m\in\mathbb{N}_0,i\in\{1,...,m\}.\tag{2.4}$$

Now, using the last difference equation in (2.4), we get

$$u_{n+1}^{(m-1)}=\frac{P_{k+2}+P_{k+1}u_n^{(m)}}{P_{k+3}+P_{k+2}u_n^{(m)}}=\frac{P_{2k+4}+P_{2k+3}u_{n-1}^{(1)}}{P_{2k+5}+P_{2k+4}u_{n-1}^{(1)}},n\geq 1,$$

similarly, we get

$$u_{n+1}^{(m-2)}=\frac{P_{k+2}+P_{k+1}u_n^{(m-1)}}{P_{k+3}+P_{k+2}u_n^{(m-1)}}=\frac{P_{3k+6}+P_{3k+5}u_{n-1}^{(1)}}{P_{3k+7}+P_{3k+6}u_{n-1}^{(1)}},n\geq 2,$$

and recursively for the above, we can get

$$u_{n+1}^{(1)}=\frac{P_{m(k+2)}+P_{m(k+2)-1}u_{n-(m-1)}^{(1)}}{P_{m(k+2)+1}+P_{m(k+2)}u_{n-(m-1)}^{(1)}},n\geq m-1.$$

System (2.4) can be written as the following rational difference equation of 3rd-order

$$u_{n+1}=\frac{P_{m_k}+P_{m_k-1}u_{n-(m-1)}}{P_{m_k+1}+P_{m_k}u_{n-(m-1)}},n\geq m-1.\tag{2.5}$$

where $m_k=m\,(k+2)$. Let $u_{n,s}=u_{mn+s}$, $s\in\{0,1,...,m-1\}$. For this, we have

$$u_{n+1,s}=\frac{P_{m_k}+P_{m_k-1}u_{n,s}}{P_{m_k+1}+P_{m_k}u_{n,s}},n\geq 0,s\in\{0,1,...,m-1\}.$$

By Lemma 2.3, the closed form of general solution of the equation (2.5) is easily obtained, in the following corollary

Corollary 2.1 *Let* $\{u_n,n\geq 0\}$ *be a solution of equation* (2.5)*. Then*

$$\forall n\geq m-1,\ u_{mn+s}=\frac{P_{m_kn-1}u_s+P_{m_kn}}{P_{m_kn}u_s+P_{m_kn+1}},s\in\{0,1,...,m-1\},$$

where $(P_n,n\geq 0)$ *is the Pell sequence.*

Through the above discussion, we can introduce the following Theorem

Theorem 2.1 *Let* $\left\{u_n^{(1)},u_n^{(2)},...,u_n^{(m)},n\geq 0\right\}$ *be a solution of system* (2.4)*. Then,*

$$u_{mn+s}^{(i)}=\frac{P_{m_kn+s_k}+P_{m_kn+s_k-1}u_0^{(s+i)\bmod(m)}}{P_{m_kn+s_k+1}+P_{m_kn+s_k}u_0^{(s+i)\bmod(m)}},s\in\{0,1,...,m-1\},i\in\{1,...,m\},$$

where $(P_n,n\geq 0)$ *is the Pell sequence.*

Proof: From Corollary 2.1, we have

$$\forall n\geq m-1,\ u_{mn+s}^{(1)}=\frac{P_{m_kn-1}u_s^{(1)}+P_{m_kn}}{P_{m_kn}u_s^{(1)}+P_{m_kn+1}},s\in\{0,1,...,m-1\},$$

and by system (2.4), we get

$$u_s^{(1)} = \frac{P_{s_k} + P_{s_k-1} u_0^{(s+1)\bmod(m)}}{P_{s_k+1} + P_{s_k} u_0^{(s+1)\bmod(m)}}, s \in \{0, 1, ..., m-1\}.$$

Now, using Lemma 2.2, we obtain

$$\begin{aligned} u_{mn+s}^{(1)} &= \frac{(P_{m_k n-1} P_{s_k} + P_{m_k n} P_{s_k+1}) + (P_{m_k n-1} P_{s_k-1} + P_{m_k n} P_{s_k}) u_0^{(s+1)\bmod(m)}}{(P_{m_k n} P_{s_k} + P_{m_k n+1} P_{s_k+1}) + (P_{m_k n} P_{s_k-1} + P_{m_k n+1} P_{s_k}) u_0^{(s+1)\bmod(m)}} \\ &= \frac{P_{m_k n+s_k} + P_{m_k n+s_k-1} u_0^{(s+1)\bmod(m)}}{P_{m_k n+s_k+1} + P_{m_k n+s_k} u_0^{(s+1)\bmod(m)}}, s \in \{0, 1, ..., m-1\}. \end{aligned}$$

The theorem is proved.□ □

2.2. On the system (1.3)

In this article, we study the System (1.3), which is an extension of System (2.4). Therefore, the System (1.3) can be written as follows

$$u_{(l+1)(n+1)-j}^{(i)} = \frac{P_{k+2} + P_{k+1} u_{(l+1)n-j}^{(i+1)\bmod m}}{P_{k+3} + P_{k+2} u_{(l+1)n-j}^{(i+1)\bmod m}}, \tag{2.6}$$

for $j \in \{0, 1, ..., l\}$, $i \in \{1, ..., m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$u_{n,j}^{(i)} = u_{(l+1)n-j}^{(i)}, j \in \{0, 1, ..., l\}, i \in \{1, ..., m\},$$

we can get $(l+1)$ −systems similar to System (2.4),

$$u_{n+1,j}^{(i)} = \frac{P_{k+2} + P_{k+1} u_{n,j}^{(i+1)\bmod m}}{P_{k+3} + P_{k+2} u_{n,j}^{(i+1)\bmod m}}, i \in \{1, ..., m\}, n \in \mathbb{N}_0, \tag{2.7}$$

for $j \in \{0, 1, ..., l\}$. Through the above discussion, we can introduce the following Theorem

Theorem 2.2 *Let $\left\{u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)}, n \geq 0\right\}$ be a solution of system (1.3). Then, for $j \in \{0, 1, ..., l\}$,*

$$u_{(l+1)(mn+s)-j}^{(i)} = \frac{P_{m_k n+s_k} + P_{m_k n+s_k-1} u_{-j}^{(s+i)\bmod(m)}}{P_{m_k n+s_k+1} + P_{m_k n+s_k} u_{-j}^{(s+i)\bmod(m)}}, s \in \{0, 1, ..., m-1\}, i \in \{1, ..., m\}$$

where $(P_n, n \geq 0)$ is the Pell sequence .

Proof: Let $\left\{u_{n,j}^{(1)}, u_{n;j}^{(2)}, ..., u_{n;j}^{(m)}, n \geq 0, j \in \{0, 1, ..., l\}\right\}$ be a solution of systems (2.7) with initial values $u_{0,j}^{(i)}$, $i \in \{1, ..., m\}$. Using Theirem 2.1, we obtain, for $j \in \{0, 1, ..., l\}$,

$$u_{mn+s,j}^{(i)} = \frac{P_{m_k n+s_k} + P_{m_k n+s_k-1} u_{0,j}^{(s+i)\bmod(m)}}{P_{m_k n+s_k+1} + P_{m_k n+s_k} u_{0;j}^{(s+i)\bmod(m)}}, s \in \{0, 1, ..., m-1\}, i \in \{1, ..., m\}.$$

Returning to the original notation, we obtain

$$u_{(l+1)(mn+s)-j}^{(i)} = \frac{P_{m_k n+s_k} + P_{m_k n+s_k-1} u_{-j}^{(s+i)\bmod(m)}}{P_{m_k n+s_k+1} + P_{m_k n+s_k} u_{-j}^{(s+i)\bmod(m)}},$$

for $s \in \{0, 1, ..., m-1\}$, $j \in \{0, 1, ..., l\}$, $i \in \{1, ..., m\}$.□ □

3. Global stability of positive solutions of (1.3)

In the following, we will study the global stability character of the solutions of system (1.3). Obviously, the positive equilibriums of system (1.3) are

$$E = \left(\overline{u^{(1)}}, \overline{u^{(2)}}, ..., \overline{u^{(m)}}\right) = -b\underline{1}'_{(m)},\ \Xi = -a\underline{1}'_{(m)},$$

where $\underline{1}_{(m)}$ denotes the vector of order $m\times 1$ whose entries are ones. Let the functions $h_i : (0, +\infty)^{m(l+1)} \to (0, +\infty)$, $i \in \{1, ..., m\}$ defined by

$$f_i\left(\left(\underline{x}^{(1)}_{0:l}\right)', \left(\underline{x}^{(2)}_{0:l}\right)', ..., \left(\underline{x}^{(m)}_{0:l}\right)'\right) = \frac{P_{k+2} + P_{k+1}x_l^{(i+1) \bmod m}}{P_{k+3} + P_{k+2}x_l^{(i+1) \bmod m}}, i \in \{1, ..., m\},$$

where $\underline{x}_{0:m} = (x_0, x_1, ..., x_m)'$. Now, it is usually useful to linearize the system (1.3) around the equilibrium point E in order to facilitate its study. For this purpose, introducing the vectors $\underline{X}'_n := \left(\left(\underline{X}^{(1)}_n\right)', \left(\underline{X}^{(2)}_n\right)', ..., \left(\underline{X}^{(m)}_n\right)'\right)$ where $\underline{X}^{(i)}_n = \left(x^{(i)}_n, x^{(i)}_{n-1}, ..., x^{(i)}_{n-l}\right)$, $i \in \{1, ..., m\}$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = \Lambda_l \underline{X}_n, \tag{3.1}$$

where

$$\Lambda_l = \begin{pmatrix} \underline{O}'_{(l-1)} & 0 & \underline{O}'_{(l-1)} & \frac{(-1)^k}{(P_{k+3}-P_{k+2}b)^2} & \cdots & O_{(l-1)} & 0 & O_{(l-1)} & 0 \\ I_{(l-1)} & \underline{O}_{(l-1)} & O_{(l-1)} & \underline{O}_{(l-1)} & \cdots & O_{(l-1)} & \underline{O}_{(l-1)} & O_{(l-1)} & \underline{O}_{(l-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ O_{(l-1)} & 0 & \underline{O}'_{(l-1)} & 0 & \cdots & \underline{O}'_{(l-1)} & 0 & \underline{O}'_{(l-1)} & \frac{(-1)^k}{(P_{k+3}-P_{k+2}b)^2} \\ O_{(l-1)} & \underline{O}_{(l-1)} & I_{(l-1)} & \underline{O}_{(l-1)} & \cdots & I_{(l-1)} & \underline{O}_{(l-1)} & O_{(l-1)} & \underline{O}_{(l-1)} \\ \underline{O}'_{(l-1)} & \frac{(-1)^k}{(P_{k+3}-P_{k+2}b)^2} & O_{(l-1)} & 0 & \cdots & O_{(l-1)} & 0 & \underline{O}'_{(l-1)} & 0 \\ O_{(l-1)} & \underline{O}_{(l-1)} & O_{(l-1)} & \underline{O}_{(l-1)} & \cdots & O_{(l-1)} & \underline{O}_{(l-1)} & I_{(l-1)} & \underline{O}_{(l-1)} \end{pmatrix},$$

with $O_{(k,l)}$ denotes the matrix of order $k\times l$ whose entries are zeros, for simplicity, we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 3.1 *The positive equilibrium point E is locally asymptotically stable.*

Proof: After some preliminary calculations, the characteristic polynomial of Λ_l is

$$P_{\Lambda_l}(\lambda) = \det\left(\Lambda_l - \lambda I_{(m(l+1))}\right) = (-1)^{m(l+1)} A_1(\lambda) + (-1)^l A_{2,k}(\lambda),$$

where $A_1(\lambda) = \lambda^{m(l+1)}$ and $A_{2,k}(\lambda) = \frac{(-1)^{mk}}{(P_{k+3}-P_{k+2}b)^{2m}}$, then $|A_{2,k}(\lambda)| < |A_1(\lambda)|, \forall\lambda : |\lambda| = 1$. So, according to Rouche's Theorem, all zeros of $A_1(\lambda) - A_{2,k}(\lambda) = 0$ lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point E is locally asymptotically stable.□ □

Corollary 3.1 *For every well defined solution of system (1.3), we have*

$$\lim u^{(i)}_n = -b,\ i \in \{1, ..., m\}.$$

Proof: From Theorem 2.2, we have

$$\begin{aligned} \lim u^{(i)}_{(l+1)(mn+s)-j} &= \lim \frac{P_{m_kn+s_k} + P_{m_kn+s_k-1}u^{(s+i) \bmod(m)}_{-j}}{P_{m_kn+s_k+1} + P_{m_kn+s_k}u^{(s+i) \bmod(m)}_{-j}}, s \in \{0, 1, ..., m-1\}, i \in \{1, ..., m\} \\ &= \lim \frac{1 + \frac{P_{m_kn+s_k-1}}{P_{m_kn+s_k}}u^{(s+i) \bmod(m)}_{-j}}{\frac{P_{m_kn+s_k+1}}{P_{m_kn+s_k}} + u^{(s+i) \bmod(m)}_{-j}} \\ &= \frac{1 - bu^{(s+i) \bmod(m)}_{-j}}{a + u^{(s+i) \bmod(m)}_{-j}}, \end{aligned}$$

for $i \in \{1, ..., m\}$, hence

$$\begin{aligned}
\lim u^{(i)}_{(l+1)(mn+s)-j} &= \frac{1-\left(1-\sqrt{2}\right)u^{(s+i)\bmod(m)}_{-j}}{\left(1+\sqrt{2}\right)+u^{(s+i)\bmod(m)}_{-j}} \\
&= \frac{\left(\left(1-u^{(s+i)\bmod(m)}_{-j}\right)+u^{(s+i)\bmod(m)}_{-j}\sqrt{2}\right)\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)-\sqrt{2}\right)}{\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)+\sqrt{2}\right)\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)-\sqrt{2}\right)} \\
&= \frac{-\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)^2-2\right)+\sqrt{2}\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)^2-2\right)}{\left(\left(1+u^{(s+i)\bmod(m)}_{-j}\right)^2-2\right)} \\
&= -b,\ i \in \{1, ..., m\}.\square
\end{aligned}$$

□

The following Corollary is an immediate consequence of Theorem 3.1 and Corollary 3.1.

Corollary 3.2 *The positive equilibrium point E is globally asymptotically stable.*

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Conflicts of Interest

The corresponding author declares no conflict of interest.

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