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Some Inequalities on Slant Submanifolds of Golden Riemannian Manifolds of Constant Golden Sectional Curvature

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ABSTRACT: The present paper is devoted to extend the Wintgen and Chen inequality for slant submanifolds of Golden Riemannian manifolds with constant G-sectional curvature.

Key Words: Golden Riemannian Manifolds, Wintgen Inequality, Chen Invariants, Scalar Curvature, Slant Submanifolds.

Contents

1. Introduction

Soon after the introduction of Riemannian manifolds, Schläfli conjectured that each Riemannian manifold could be locally considered as a submanifold of Euclidean space with sufficient higher co-dimensions. Janet, Cartan and Burstein later proved this conjecture in different steps. The main aim was to learn about the manifold's intrinsic properties using the knowledge of extrinsic world. Based on these embedding theorms some inequalities between the main intrinsic and extrinsic invariants where obtained. For example, Wintgen [30] obtained a sharp inequality that involves the extrinsic and intrinsic invariants for surface S^2 in E^4 by the following equation

$$\|H\|^2 \geq K + |K^{\perp}|, \tag{1.1}$$

where K, $K^{\perp}$ and H are used to denote Gauss curvature, normal curvature and mean curvature respectively. The above inequality was extended for surfaces of arbitrary codimension m in real space forms with constant sectional curvature $\bar{c}$ as (see [11,23])

$$\|H\|^2 + \bar{c} \geq K + |K^{\perp}|. \tag{1.2}$$

De Smet et al. [10] conjectured the above inequality as natural generalisation of Wintgen inequality and they brought into light that on an $(n+m)$-dimensional real space form $(\bar{M}, \bar{c})$, the following inequality is satisfied for every point of n-dimensional submanifod of $(\bar{M}, \bar{c})$

$$\rho^{\perp} \leq c - \rho + \|H\|^2 \tag{1.3}$$

where ρ and $\rho^{\perp}$ denote the normalised scalar curvature and normal curvature of submanifold respectively. This conjecture was named as DDVV conjecture. A vast investigation have been done on this conjecture for different submanifolds of different space forms (for instance see [17]- [20], [26]).

The another most important famous relation between the intrinsic and extrinsic invariants was brought into light by Chen [3]. The author established an optimum inequality involving the novel intrinsic

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invariants and key extrinsic invariants also known as first Chen invariant and is given by the following equation;

$$\delta_N(x) = \tau(x) - inf\{k(\pi); \pi \subset T_x\bar{M}, dim(\pi) = 2\}, \tag{1.4}$$

where τ and $k(\pi)$ being well known scalar curvature and sectional curvature of Riemannian manifold $\bar{M}$ respectively. The Chen invariants attained interest from most of the geometers and a lot of research has been done on chen invariants since their appearance for instance, see ([1]- [7], [9], [15], [28]- [29]) and references mentioned therein. The golden ratio emphasises the sense of harmony and good proportion, In 2008 the golden type structure on manifolds was introduced for the first time [8] and since then a plenty of work has been done on such manifolds with different ideas (for instance see [12,21,22,25,27]). The notion of Golden Riemannian manifolds with constant Golden sectional curvature was introduced recently by Sahin [24]. Motivated by Chen invariants and Sahin's work we obtain Wintgen inequality and Chen inequality on Golden Riemannian manifolds having constant G-sectional curvature.

2. Preliminaries

Let N be an m-dimensional immersed submanifold of an n-dimensional Riemannian manifold $(\bar{M}, G)$. Let $\bar{\nabla}$ and $\tilde{\nabla}$ be the Levi-Civita connection on $\bar{M}$ and the induced connection on N. Denote by B the second fundamental form of N and by A_N the shape operator. Then the Gauss and Weingarten formulas are written as;

$$\bar{\nabla}_{X_1}X_2 = \tilde{\nabla}_{X_1}X_2 + B(X_1, X_2),$$

and

$$\bar{\nabla}_{X_1}V = -A_V X_1 + \tilde{\nabla}^{\perp}_{X_1}V \tag{2.1}$$

for $X_1,\ X_2 \in \Gamma(TN)$, where $\tilde{\nabla}^{\perp}$ is connection on the normal bundle. The shape operator and second fundamental form satisfy

$$G(A_V X_1, X_2) = G(B(X_1, X_2), V)$$

for $X_1, X_2 \in \Gamma(TN)$. Let $\bar{R}$ and $\tilde{R}$ denote the Riemannian curvature of manifold and submanifold respectively, then the Gauss equation is given by

$$\begin{aligned}\bar{R}(X_1, X_2, X_3, X_4) = \tilde{R}(X_1, X_2, X_3, X_4) &- G(B(X_1, X_4), B(X_2, X_3)) \\ &+ G(B(X_1, X_3), B(X_2, X_4)).\end{aligned}$$

Consider an orthonormal tangent frame $\{e_i\}_{i=1}^m$ of TN of N and orthonormal normal frame $\{e_j\}_{j=m+1}^n$ of $T^{\perp}N$ of N in $\bar{M}$, then the mean curvature vector denoted by H is given by

$$\|H\| = \frac{1}{m}\sum_{i=1}^{m} B(e_i, e_i).$$

Furthermore the scalar curvature, the normalised scalar curvature and normalised scalar normal curvature at $x \in TN$ are defined as

$$2\tau(x) = \sum_{1\le i<j\le m} R(e_i, e_j, e_j, e_i), \tag{2.2}$$

$$\rho = \frac{2\tau}{m(m-1)}, \tag{2.3}$$

$$\rho^{\perp} = \frac{2}{m(m-1)}(\kappa^{\perp})^{\frac{1}{2}}, \tag{2.4}$$

respectively, where $\kappa^{\perp}$ denote the scalar normal curvature and is given by [16]

$$\kappa^{\perp} = \sqrt{\sum_{1\le a<b\le n-m}\ \sum_{1\le i<j\le m}[\sum_{\alpha=1}^{m}(B^a_{j\alpha}B^b_{i\alpha} - B^a_{i\alpha}B^b_{j\alpha}]^2}. \tag{2.5}$$

3. Golden Riemannian space form

Crassmareanu and Hretcanu brought into light the Golden Riemannian manifolds in [8] as follows. A (1,1) tensor filed $\hat{\Psi}$ on manifold $\bar{M}$ satisfying

$$\hat{\Psi}^2 = \hat{\Psi} + I. \tag{3.1}$$

known as the Golden structure on $\bar{M}$ and $(\bar{M}, \hat{\Psi})$ is called a Golden manifold. If the Riemannian manifold with Golden structure satisfies

$$G(\hat{\Psi}X_1, X_2) = G(X_1, \hat{\Psi}X_2), \tag{3.2}$$

then $(\bar{M}, G, \hat{\Psi})$ is called Golden Riemannian manifold. Furthermore if $\bar{\nabla}\hat{\Psi} = 0$, then $\bar{M}$ is called locally decomposable Golden Riemannian manifold. Recall that if P and Q denote the the tangential and normal component of $\hat{\Psi}$ then

$$\hat{\Psi}X_1 = PX_1 + QX_1, \tag{3.3}$$

for any $X_1 \in \Gamma(T\bar{M})$.

Furthermore from (3.1)and (3.2) it is easy to see that

$$G(\hat{\Psi}X_1, \hat{\Psi}X_2) = G(\hat{\Psi}X_1, X_2) + G(X_1, X_2). \tag{3.4}$$

let X_1 be nonzero vector tangent to N at $x \in N$. We denote by $\theta(X_1)$ the angle between $\Psi\hat{X}_1$ and T_xN, then submanifold N is said to be slant if $\theta(X_1)$ does not depend on the choice of $x \in N$ and $X_1 \in T_xN$.

Case.1 If $\theta = 0$ then N is invariant submanifold.

Case.2 If $\theta = \frac{\pi}{2}$ then N is anti-invariant submanifold.

Lemma 3.1 *[13] If N is slant submanifold of a Golden Riemannian manifold $(\bar{M}, G, \hat{\Psi})$ with slant angle θ, then*

$$\begin{aligned} G(PX_1, PX_2) &= cos^2\theta[G(X_1, PX_2) + G(X_1, X_2)], \\ G(QX_1, QX_2) &= sin^2\theta[G(X_1, PX_2) + G(X_1, X_2)], \end{aligned}$$

for any $X_1, X_2 \in \Gamma(TN)$.

Example 3.1 [12,13] Consider an Euclidean 4-space with standard coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Define a $(1,1)$ tensor $\hat{\Psi} : E^4 \to E^4$ such that

$$\hat{\Psi}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = ((1-\lambda)\alpha_1, \lambda\alpha_2, (1-\lambda)\alpha_3, \lambda\alpha_4),$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ and $1-\lambda = \frac{1-\sqrt{5}}{2}$ are the roots of equation $\alpha^2 = \alpha+1$. It is easy to see that $\hat{\Psi}^2 = \hat{\Psi}+I$. Moreover we get

$$G(\hat{\Psi}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) = G((\alpha_1, \alpha_2, \alpha_3, \alpha_4), \hat{\Psi}(\beta_1, \beta_2, \beta_3, \beta_4)),$$

for each $(\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4) \in E^4$, where G is standard metric on E^4. Thus $(E^4, G, \hat{\Psi})$ is Golden Riemannian manifold. Now consider a submanifold N of E^4 given by

$$\alpha(t_1, t_2) = ((1-\lambda)t_1, q\lambda t_2, (1-\lambda)t_1, q\lambda t_2,$$

for any $q \neq 0, 1$ then we have

$$\begin{aligned} e_1 = (1-\lambda, q\lambda, 0, 0), \quad e_2 = (0, 0, 1-\lambda, q\lambda), \quad and \\ \hat{\Psi}e_1 = (-1, -q, 0, 0), \quad \hat{\Psi}e_2 = (0, 0, -1, -q). \end{aligned}$$

So we get

$$G(\hat{\Psi}e_1, e_1) = G(\hat{\Psi}e_2, e_2) = (-q^2+1)\lambda - 1 \quad and \quad G(\hat{\Psi}e_1, e_2) = 0.$$

If the slant angle of N is represented by θ , then N becomes a slant submaniold with slant angle $\theta = cos^{-1}\left(\frac{(\lambda-1-q^2\lambda)}{\sqrt{q^2+1}}\right)$.

Let $(\bar{M}^n, G)$ be a locally decomposable Golden Riemannian manifold of constant G-sectional curvature $\bar{c}$ then we have [24]

$$\begin{aligned}\bar{R}(X_1,X_2)X_3 &= \frac{\bar{c}}{3}\{G(X_2,X_3)X_1 - G(X_1,X_3)X_2\\ &- G(X_2,\bar{\phi}X_3)X_1 - G(X_2,X_3)\bar{\phi}X_1\\ &+ 2G(X_2,\bar{\phi}X_3)\bar{\phi}X_1 + G(X_1,\bar{\phi}X_3)X_2\\ &+ G(X_1,X_3)\phi X_2 + 2G(X_1,\phi X_3)\phi X_2\}. \end{aligned} \tag{3.5}$$

4. Main results

In this section we obtain Wintgen and Chen inequality for locally decomposable Golden Riemannian manifold of constant Golden sectional curvture.

Theorem 4.1 *Let N^m represent a slant-submanifold of dimension m in locally decomposable golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\rho^{\perp} \leq \|H\|^2 - 2\rho + \frac{\bar{c}}{3}\left(2 + \frac{4(trP-m)(trP-cos^2\theta)}{m(m-1)}\right), \tag{4.1}$$

Furthermore the equality in the above relation holds if and only if for some real functions f_1, f_2, f_3 and f^* on N^m, the shape operator A takes the following form

$$A_{m+1} = \begin{pmatrix} f_1 & f^* & 0 & \dots & 0\\ f^* & f_1 & 0 & \dots & 0\\ 0 & 0 & f_1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & & f_1 \end{pmatrix}, \tag{4.2}$$

$$A_{m+2} = \begin{pmatrix} f_2+f^* & 0 & 0 & \dots & 0\\ 0 & f_2-f^* & 0 & \dots & 0\\ 0 & 0 & f_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & & f_2 \end{pmatrix}, \tag{4.3}$$

$$A_{m+3} = \begin{pmatrix} f_3 & 0 & 0 & \dots & 0\\ 0 & f_3 & 0 & \dots & 0\\ 0 & 0 & f_3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & & f_3 \end{pmatrix}, \quad A_{n+4} = \dots = A_m = 0 \tag{4.4}$$

Proof: Let us assume that $\{e_i\}_{i=1}^m$ and $\{e_r\}_{r=m+1}^n$ correspond to local orthonormal tangent and local orthonormal normal frame on N respectively, then from (3.5) and Gauss equation we get

$$\begin{aligned}\tilde{R}(X_1,X_2,X_3,X_4) &= \frac{\bar{c}}{3}\{G(X_2,X_3)G(X_1,X_4) - G(X_1,X_3)G(X_2,X_4)\\ &- G(X_2,\bar{\phi}X_3)G(X_1,X_3) - G(X_2,X_3)G(\bar{\phi}X_1,X_4)\\ &+ 2G(X_2,\bar{\phi}X_3)G(\bar{\phi}X_1,X_4)\\ &+ G(X_1,\bar{\phi}X_3)G(X_2,X_4) + G(X_1,X_3)G(\bar{\phi}X_2,X_4)\\ &+ 2G(X_1,\phi X_3)G(\bar{\phi}X_2,X_4)\}\\ &+ G(B(X_1,X_4),B(X_2,X_3))\\ &- G(B(X_1,X_3),B(X_2,X_4))\end{aligned}$$

In view of lemma (3.1) and (2.2), the foregoing equation reduces to

$$\sum_{1\leq i<j\leq m}\tilde{R}(e_i,e_j,e_j,e_i) = \frac{\bar{c}}{3}\{m(m-1)+2(trP-m)(trP-cos^2\theta)\} + \sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}[B_{ii}B_{jj}-(B_{ij})^2].$$

Again using (2.2) we have

$$2\tau = \frac{\bar{c}}{3}\{m(m-1)+2(trP-m)(trP-cos^2\theta)\} + \sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}[B_{ii}B_{jj}-(B_{ij})^2]. \tag{4.5}$$

On the other hand we have

$$m^2\|H\|^2 = \sum_{r=m+1}^{n}(\sum_{i=1}^{m}B_{ii}) = \frac{1}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(B^r_{ii}-B^r_{jj})^2 + \frac{2m}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}B^r_{ii}B^r_{jj}. \tag{4.6}$$

Thanks to [25] we have

$$\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(B_{ii}-B_{jj})^2 + 2m\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}(B_{ij})^2 \geq 2m\sqrt{\sum_{m+1\leq a<b\leq n-m}\sum_{1\leq i<j\leq m}[\sum_{\alpha=1}^{m}(B^a_{j\alpha}B^b_{i\alpha}-B^a_{i\alpha}B^b_{j\alpha}]^2}. \tag{4.7}$$

From (4.6), (4.7), also recalling (2.5) we have

$$m^2\|H\|^2 - m^2\rho^{\perp} \geq \frac{2m}{m-1}\sum_{r=m+1}^{n-m}\sum_{1\leq i<j\leq m}[B^r_{ii}B^r_{jj}-(B^r_{ij})^2].$$

Using (4.5) and the foregoing equations we get

$$\rho^{\perp} - \|H\|^2 \leq \frac{\bar{c}}{3}\{2+\frac{4(trP-m)(trP-cos^2\theta)}{m(m-1)}\} - 2\rho, \tag{4.8}$$

which proves the desired inequality (4.1). □

Corollary 4.1 *Let N^m represent an invariant submanifold of dimension m in locally decomposable Golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\rho^{\perp} \leq \|H\|^2 - 2\rho + \frac{\bar{c}}{3}\left(2+\frac{4(trP-m)(trP-1)}{m(m-1)}\right).$$

Corollary 4.2 *Let N^m represent an anti-invariant submanifold of dimension m in locally decomposable golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\rho^{\perp} \leq \|H\|^2 - 2\rho + \frac{\bar{c}}{3}\left(2+\frac{4(trP-m)trP}{m(m-1)}\right).$$

Remark 4.1 The equality cases in the foregoing corollaries hold if and only if for some real functions f_1, f_2, f_3 and f^* the shape operator A takes the form (4.2), (4.3) and (4.4).

Chen's first inequality: Before proving the Chen inequality on Locally decomposable Golden Riemannian manifolds with constant G-sectional curvature we recall the following lemma from [4].

Lemma 4.1 *Let $b_1, ..., b_n, c$ be n+1 ,$n \geq 2$, real numbers such that*

$$\left(\sum_{i=1}^{n} b_i\right) = (n-1)\left((\sum_{i=1}^{n} b_i^2 + c\right),$$

then $2b_1b_2 \geq c$ and the equality case holds if and only if $b_1 + b_2 = b_3 = ... = b_n$.

We also use the following notations we denote $G(\hat{\Psi}e_1, e_1)$ and $G(\hat{\Psi}e_2, e_2)$ by $\hat{\Psi}_1(\pi)$ and $\hat{\Psi}_2(\pi)$ respectively.

Theorem 4.2 *Let N^m represent a slant-submanifold of dimension m in locally decomposable golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\delta_N \leq \frac{m^2(m-2)}{m-1}\|H\|^2 + \frac{\bar{c}}{6}\{m(m-1) - 2(trP - m)(trP - cos^2\theta) \\ -2 + 2\hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] + 4cos^2\theta\}.$$

Proof: Using (4.5), we put

$$\varepsilon = 2\tau - \frac{m^2(m-2)\|H\|^2}{m-1} - \frac{\bar{c}}{3}\{m(m-1) \\ -2(trP - m)(trP - cos^2\theta)\}. \tag{4.9}$$

Again (3.5) yields

$$k(\pi) = \frac{\bar{c}}{3}\{1 - \hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] - 2cos^2\theta\} \\ + \sum_{r=m+1}^{n} B_{11}^r B_{22}^r - (B_{12}^r)^2.$$

The forgoing equation can be rewritten as

$$\begin{aligned} k(\pi) &= \frac{\bar{c}}{3}\{1 - \hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] - 2cos^2\theta\} \\ &+ \sum_{r=m+2}^{n} B_{11}^r B_{22}^r - \sum_{r=m+1}^{n}(B_{12}^r)^2 + B_{11}^{n+1}B_{22}^{n+1}. \end{aligned}$$

Using lemma 4.2 we get

$$\begin{aligned} k(\pi) &\geq \frac{\bar{c}}{3}\{1 - \hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] - 2cos^2\theta\} \\ &+ \frac{1}{2}\sum_{i\neq j}(B_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=m+1}^{n}\sum_{i\neq j>2}(B_{ij}^r)^2 + \frac{1}{2}\sum_{r=m+2}^{n}(B_{11}^r + B_{22}^r)^2 \\ &+ \sum_{j>2}[(B_{1j}^{n+1})^2 + (B_{2j}^{n+1})^2] + \frac{\varepsilon}{2}, \end{aligned}$$

which implies

$$\begin{aligned} k(\pi) &\geq \tau - \frac{m^2(m-2)}{2(m-1)}\|H\|^2 \\ &- \frac{\bar{c}}{6}\{m(m-1) - 2(trP - m)(trP - cos^2\theta)\} \\ &+ \frac{\bar{c}}{3}\{1 - \hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] - 2cos^2\theta\}. \end{aligned}$$

On further computation we get

$$\begin{aligned} \tau - inf(k(\pi)) &\leq \frac{m^2(m-2)}{2(m-1)}\|H\|^2 + \frac{\bar{c}}{6}\{m(m-1) \\ &- 2(trP - m)(trP - cos^2\theta)\} \\ &- \frac{\bar{c}}{3}\{1 - \hat{\Psi}_1(\pi)[1 + 2cos^2\theta + 2\hat{\Psi}_2(\pi)] - 2cos^2\theta\} \end{aligned}$$

,

which proves the desired inequality. □

Corollary 4.3 *Let N^m represent an invariant submanifold of dimension m in locally decomposable golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\begin{aligned} \delta_N \leq \frac{m^2(m-2)}{m-1}\|H\|^2 + \frac{\bar{c}}{6}\{m(m-1) - 2(tr\phi - m)(tr\phi + 1) \\ +2 + 2\hat{\Psi}_1(\pi)[3 + 2\hat{\Psi}_2(\pi)]\}. \end{aligned}$$

Corollary 4.4 *Let N^m represent an anti-invariant submanifold of dimension m in locally decomposable golden Riemannian manifold $\bar{M}$ of constant G-sectional curvature $\bar{c}$ then we have*

$$\begin{aligned} \delta_N \leq \frac{m^2(m-2)}{m-1}\|H\|^2 + \frac{\bar{c}}{6}\{m(m-1) - 2(tr\phi - m)(tr\phi) \\ -2 + 2\hat{\Psi}_1(\pi)[1 + 2\hat{\Psi}_2(\pi)]\}. \end{aligned}$$

Conclusion. The notion of Golden Reimannan manifolds with constant G-sectional curvature was introduced by Sahin [24]. We obtained the DDVV-Conjecture and Chen's first inequality. The optimal inequalities for δ-casorati curvature, Chen's second delta invariant can be obtained.

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References

1. Alegre, P., Carriazo, A., Kim, Y. H., Yoon, D. W., *B. Y. Chen's inequality for submanifolds of generalized space forms.* Indian J. Pure Appl. Math., 38, 185-201, (2007).
2. Arslan, K., Ezentas, R., Mihai, I., Murathan, C., Özgür, C., *B. Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds.* Bull. Inst. Math. Acad. Sin., 29, 231-242, (2001).
3. Chen, B. Y., *Some new obstructions to minimal and Lagrangian isometric immersions.* Japan. J. Math. 26, 105-127, (2000).
4. Chen, B. Y., *Some pinching and classification theorems for minimal submanifolds.* Arch. Math. 60, 568-578, (1993).
5. Chen, B. Y., *A Riemannian invariant and its applications to submanifold theory.* Results in Mathematics. 27, 17-26, (1995).

6. Chen, B. Y., *A general inequality for submanifolds in complex-space-forms and its applications.* Arch. Math., 67, 519-528, (1996).

7. Cioroboiu, D., Oiaga, A., *B.-Y. Chen inequalities for slant submanifolds in Sasakian space forms.* Rend. Circ. Mat. Palermo. 52, 367-381, (2003).

8. Crasmareanu, M., Hretcanu, C. E., *Golden differential geometry.* Solitons Fract. 38(5), 1229–1238, (2008).

9. Defever, F., Mihai, I., Verstraelen, L., *B.Y. Chen's inequality for C-totally real submanifolds of Sasakian space forms.* Boll. Unione Mat. Ital, B. 11, 365-374, (1997).

10. De Smet, P. J., Dillen, F., Verstraelen, L., Vrancken, L., *A pointwise inequality in submanifold theory.* Arch. Math. (Brno). 35, 115–128, (1999).

11. Guadalupe, I. V., Rodriguez, L., *Normal curvature of surfaces in space forms.* Pacific J. Math. 106, 95–103, (1983).

12. Hretcanu, C. E., Crasmareanu, M. C., *Applications of the Golden ratio on Riemannian Manifolds.* Turk. J. Math. 33, 179–191, (2009).

13. Hretcanu, C. E., Blaga, A. M., *Slant and semi-slant submanifolds in metallic Riemannian manifolds,* J. Funct. Spaces Article ID 2864263, p 13 (2018).

14. Lone, M. A., *Basic inequalities for submanifolds of quaternionic space forms with a quarter-symmetric connection,*J. Geom. Phys. 159 (2021), 103927, DOI: https://doi.org/10.1016/j.geomphys.2020.103927.

15. Mihai, I., *B. Y. Chen's inequality for C-totally real submanifolds of Sasakian space forms.* Bollettino della unione matematica italiana. 11, 365-374, (1997).

16. Mihai, I., Al-Solamy, F. R., Shahid, M. H., *On Ricci curvature of a quaternion CR-submanifold in a quaternion space form.* Rad. Mat. 12(1), 91–98, (2003).

17. Mihai, I., *On generalized Wintgen inequality.* Riemannian geometry and applications–Proceedings RIGA 2011, pp. 203–208, Ed. Univ. Bucure¸sti, Bucharest (2011).

18. Mihai, I., *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms.* Nonlinear Anal. 95, 714–720 (2014).

19. Mihai, I., *On the generalized Wintgen inequality for submanifolds in Sasakian space forms.* Riemannian Geometry and Applications–Proceedings RIGA 2014, pp. 153–158, Editura Univ. Bucur., Bucharest (2014).

20. Murathan, C., Sahin, B., *A study of Wintgen like inequality for submanifolds in statistical warped product manifolds* J. Geom. 109 (2), Art. 30,18 pp (2018).

21. Ozkan, M., *Prolongations of golden structures to tangent bundles,* Differ. Geom. Dyn. Syst. 16(2014), 227–238.

22. Ozkan, M., Peltek, B., *A new structure on manifolds: silver structure,*Int. Electron. J. Geom. 9(2)(2016), 59–69.

23. Rouxel, B., *Sur une Famille des A-Surfaces dun Espace Euclidien E4.* Osterreischer Mathematiker Kongress, Insbruck (1981)

24. Sahin, F., Sahin, B., Erdogan, F. E., *Golden Riemannian Manifolds Having Constant Sectional Curvatures and their Submanifolds.* Mediterr. J. Math. (2022) 19:171 https://doi.org/10.1007/s00009-022-02094-3 1660-5446/22/040001-16 published online June 23, 2022.

25. Sahin, B., Akyol, M., *Golden maps between Golden Riemannian manifolds and constancy of certain maps.* Math. Commun. 19(2), 333–342, (2014).

26. Siddiqui, A. N., Ahmad, K., *Generalized Wintgen inequality for totally real submanifolds in LCS-manifolds.* Balkan J. Geom. Appl. 24 (2), 53–62, (2019).

27. Siddiqi, M. D., *Generic submanifolds of Golden Riemannian manifolds.*Uzbek Math. J. 4, 123–131, (2019).

28. Vilcu, G. E., *B. Y. Chen inequalities for slant submanifolds in quaternionic space forms.* Turk. j. Math. 34, 115-128, (2010).

29. Vilcu, G. E., *On Chen invariants and inequalities in quaternionic geometry.* J. Inequal. Appl. 66 (2013).

30. Wintgen, P., *On Chen-Willmore's inequality (in French),* C. R. Acad. Sci. Paris Ser. A-B. 288, A993-A995, (1979).

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