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Studies on λ-statistical convergence of sequences in generalized probabilistic metric spaces

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ABSTRACT: In this paper, we introduce the notion of λ-statistical convergence of sequences in probabilistic G-metric spaces. We study some basic properties of λ-statistical convergence of sequences. Also introducing the notion of λ-statistically Cauchy sequences, we study its relationship with λ-statistical convergence in probabilistic G-metric spaces. Further, introducing the notion of λ-statistically pre-Cauchy sequences in probabilistic G-metric spaces, we examine the connections among all these new concepts and establish some basic facts.

Key Words: Probabilistic G-metric spaces, λ-statistical convergence, λ-statistical Cauchy, λ-statistical pre-Cauchy sequences.

Contents

1. Introduction

In 1942, Menger [10] proposed a probabilistic generalization of the notion of distance between two points. He proposed to use a distribution function F_{xy} to define the distance between two points x, y instead of a real number. The value of the function F_{xy} at any $t > 0$, that is $F_{xy}(t)$ can be interpreted as the probability that the distance between x and y is less than t. Since Menger, Schwiezer and Sklar [19,20], Tardiff [22], Thorp [23] and many others have studied and advanced the theory of probabilistic metric spaces. We recommend the book [18] written by Schwiezer and Sklar on probabilistic metric spaces for interested readers. In 2014, Zhou et al. [24] proposed a generalization of Menger probabilistic metric spaces, namely Menger probabilistic G-metric spaces. The concept of Menger probabilistic G-metric spaces was generalized from Menger probabilistic metric spaces in a similar way that the concept of G-metric spaces was generalized from the notion of metric spaces. The notion of the G-metric spaces was introduced by Mustafa and Sims [15] as follows:

Definition 1.1 *[15] Let X be a non-empty set and $\mathcal{G} : X \times X \times X \to \mathbb{R}^+$ be a mapping satisfying the following conditions:*

1. $\mathcal{G}(a, b, c) = 0$ *if* $a = b = c\ \forall a, b, c \in X$;
2. $\mathcal{G}(a, a, b) > 0\ \forall\ a, b \in X$ *and* $a \neq b$;
3. $\mathcal{G}(a, a, b) \leq \mathcal{G}(a, b, c)\ \forall\ a, b, c \in X$ *and* $b \neq c$;
4. $\mathcal{G}(a, b, c) = \mathcal{G}(a, c, b) = \mathcal{G}(b, c, a) = \ldots$ *(symmetry in* $a, b, c \in X$*)*;
5. $\mathcal{G}(a, b, c) \leq \mathcal{G}(a, w, w) + \mathcal{G}(w, b, c)\ \forall\ a, b, c, w \in X$.

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Then the pair $(X, \mathcal{G})$ is called a G-metric space and $\mathcal{G}$ is said to be a generalized metric or a G-metric on X. We mention that we write G-metric because of the word "generalized" not for the function "$\mathcal{G}$". If a, b and c are three points in the euclidean plan $\mathbb{R}^2$, then we can interpret $\mathcal{G}(a, b, c)$ as the perimeter of the triangle with vertices at a, b and c. Some fixed point related works on G-metric spaces can be found in [13,14,16] and references therein. For some recent advancements in the direction of probabilistic G-metric spaces see [1,8,25], and the references therein.

In 1951, Fast [4] and Schoenberg [17] independently introduced the notion of statistical convergence of sequences of real numbers. The notion of statistical convergence is a generalization of the notion of usual convergence, that is, usual convergence sequences are statistical convergence but statistical convergence sequences may not be usually convergent. Later, in order to provide a Cauchy-like criterion for the statistical convergent sequences of real numbers, Connor et al. [3] introduced the notion of pre-Cauchy sequences of real numbers and established a relationship between the notions of statistical convergence and statistically pre-Cauchy sequences of real numbers. Further, the notion of λ-statistical convergence of sequences of real numbers was introduced by Mursaleen [11]. If $\lambda_n = n$, then the notion of statistical convergence of sequences of real numbers and the notion of λ-statistical convergence sequences of real numbers are equivalent. One can view [7,6,12] and a number of other references therein to learn about the further developments.

In [21], Şençimen et al. extended the notion of strong convergence of sequences to the notion of strong statistical convergence of sequences in a similar way that the notion of usual convergence sequences of real numbers was generalized to the notion of statistical convergence sequences of real numbers. Lately, Malik and Das [9] studied the notion of strong λ-statistical convergence of sequences (a generalization of the notion of strong statistical convergence of sequences), while Ghosh and Das [7] studied the notions of strongly λ-statistically pre-Cauchy sequences and strongly Vallée-Poussin pre-Cauchy sequences in probabilistic metric spaces. Although probabilistic G-metric space is a generalized version of probabilistic metric spaces, there have been no thorough studies of these notions (in [1], statistical convergence and statistical Cauchy sequences have been studied in this realm). In point of view of recent advancements of probabilistic G-metric spaces and the gap we just mentioned, we propose to study these notions of sequences in probabilistic G-metric spaces.

In this article, we introduce the notion of λ-statistical convergence of sequences in probabilistic G-metric spaces. Also, we introduce the notion of λ-statistically Cauchy and λ-statistically pre-Cauchy sequences in probabilistic G-metric spaces. We show that λ-statistically convergent sequences are λ-statistically Cauchy and λ-statistically Cauchy sequences are λ-statistically pre-Cauchy. Besides, we provide some examples to show that λ-statistically pre-Cauchy sequences may not be λ-statistically Cauchy, and λ-statistically Cauchy sequences may not be λ-statistically convergent.

2. Preliminaries

From now on, throughout the article, unless otherwise specified, $\lambda = (\lambda_n)$ will be used to denote a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_1 = 1$, $\lambda_{n+1} \le \lambda_n + 1$, $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, we write $I_n = [n - \lambda_n + 1, n]$.

First, we familiarize reader with the basic concepts of statistical convergence and λ-statistical convergence of sequences of real numbers.The notion of asymptotic density of the subsets of the set of all natural numbers $\mathbb{N}$ plays a central role in the concept of statistical convergence of sequences. We recall that a set $A \subset \mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

exists, where for all $n \in \mathbb{N}$, $A(n) = \{k \in A : k \le n\}$.

Definition 2.1 *[5] A sequence (x_k) of real numbers is said to be statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$, $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \ge \varepsilon\}$.*

Definition 2.2 *[3] A sequence (x_k) of real numbers is said to be statistically pre-Cauchy if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^2} |\{(j, k) \in [1, n] \times [1, n] : |x_j - x_k| \ge \varepsilon\}| = 0.$$

Definition 2.3 *[9] A set $A \subset \mathbb{N}$ is said to have λ-asymptotic density $d_\lambda(A)$ if*

$$d_\lambda(A) = \lim_{n\to\infty} \frac{|A(n)|}{\lambda_n}$$

exists, where for all $n \in \mathbb{N}$, $A(n) = \{k \in A : k \in I_n\}$.

Definition 2.4 *[11] A sequence (x_k) of real numbers is said to be λ-statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0,$$

or,

$$d_\lambda\left(\{k \in \mathbb{N} : |x_k - l| < \varepsilon\}\right) = 1.$$

Now we revisit Menger probabilistic G-metric spaces (in brief PGM-spaces).

Definition 2.5 *[18] A non-decreasing real valued function h defined on $[0, \infty]$ that satisfies $h(0) = 0$ and $h(\infty) = 1$ and is left continuous on $(0, \infty)$ is called a distance distribution function (d.d.f. in short).*

The set of all distance distribution functions is denoted by $\mathcal{D}^+$.

Definition 2.6 *[18] Let τ be a function from the closed unit square $[0, 1] \times [0, 1]$ into the closed unit interval $[0, 1]$ satisfying*

1. $\tau(a, b) = \tau(b, a)$, $\forall a, b \in [0, 1]$;
2. $\tau(a, b) \leq \tau(c, d)$ *whenever* $a \leq c$, $b \leq d$;
3. $\tau(a, 1) = a$, $\forall a \in [0, 1]$;
4. $\tau(\tau(a, b), c) = \tau(a, \tau(b, c))$ *for all* $a, b, c \in [0, 1]$.

Then τ is called a triangular norm (in short, a t-norm).

Definition 2.7 *[18] Let τ be a binary operation on $[0, 1]$ that is nondecreasing in each place. Then τ is left continuous if*

$$\tau(x, y) = sup\{\tau(u, v) : 0 < u < x, 0 < v < y\}$$

for all $x, y \in (0, 1]$; and τ is right continuous if

$$\tau(x, y) = inf\{\tau(u, v) : x < u < 1, y < v < 1\}$$

for all $x, y \in [0, 1)$.

τ is said to be continuous if it is both left and right continuous.

From now on, we write the value of $\mathcal{G}$ at (a, b, c) as $\mathcal{G}_{(a,b,c)}$ where $\mathcal{G} : X \times X \times X \longrightarrow \mathcal{D}^+$ and $a, b, c \in X$, and the value of $\mathcal{G}_{(a,b,c)}$ at t as $\mathcal{G}_{(a,b,c)}(t)$.

Definition 2.8 *[24] Let X be a nonempty set, $\mathcal{G}$ be a function from $X \times X \times X$ into $\mathcal{D}^+$ and τ be a continuous t-norm such that for all $a, b, c \in X$ we have*

1. $\mathcal{G}_{(a,b,c)}(t) = 1$ $\forall$ $a, b, c \in X$ *and* $t > 0$ *if and only if* $a = b = c$;
2. $\mathcal{G}_{(a,a,b)}(t) \geq \mathcal{G}_{(a,b,c)}(t)$ $\forall a, b$ *and* $c \neq b$, $t > 0$;
3. $\mathcal{G}_{(a,b,c)}(t) = \mathcal{G}_{(b,a,c)}(t) = \mathcal{G}_{(c,a,b)}(t) = \ldots$ *(symmetry in* $a, b, c \in X$*)*;
4. $\mathcal{G}_{(a,b,c)}(u + v) \geq \tau\left(\mathcal{G}_{(a,w,w)}(u), \mathcal{G}_{(w,b,c)}(v)\right)$ $\forall$ $a, b, c, w \in X$ *and* $u, v \geq 0$.

Then the triplet $(X, \mathcal{G}, \tau)$ is called a Menger probabilistic G-metric space (in short a PGM-space).

Example 2.1 [24] Let $(X, \mathcal{F}, \tau)$ be a PM space and $\mathcal{G} : X \times X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$\mathcal{G}_{(a,b,c)}(t) = \min\{\mathcal{F}_{(a,b)}(t), \mathcal{F}_{(b,c)}(t), \mathcal{F}_{(c,a)}(t)\}.$$

Then $(X, \mathcal{G}, \tau)$ is a PGM space.

Definition 2.9 *[24] Let $(X, \mathcal{G}, \tau)$ be a PGM space and a_0 be an element in X. Then for $\varepsilon > 0$ and $0 < \delta < 1$ the (ε, δ)-neighbourhood of a_0 is denoted by $\mathcal{N}_{a_0}(\varepsilon, \delta)$ and is defined as*

$$\mathcal{N}_{a_0}(\varepsilon, \delta) = \left\{b \in X : \mathcal{G}_{(a_0,b,b)}(\varepsilon) > 1 - \delta,\ \mathcal{G}_{(b,a_0,a_0)}(\varepsilon) > 1 - \delta\right\}.$$

One can interpret $\mathcal{N}_{a_0}(\varepsilon, \delta)$ as the set of all points $b \in X$ for which the probability of the distance from a_0 to b being less than ε is greater than $1 - \delta$.

Theorem 2.1 *[24] Let $(X, \mathcal{G}, \tau)$ be a PGM space.*

1. *If $\varepsilon_1 \leq \varepsilon_2$ and $\delta_1 \leq \delta_2$, then $\mathcal{N}_L(\varepsilon_1, \delta_1) \subset \mathcal{N}_L(\varepsilon_2, \delta_2)$.*
2. *Then $(X, \mathcal{G}, \tau)$ is a Hausdorff space in the topology induced by the family $\{\mathcal{N}_{a_0}(\varepsilon, \delta)\}$ of (ε, δ)-neighbourhood of a_0.*

Definition 2.10 *[24] Let $(X, \mathcal{G}, \tau)$ be a PGM space. A sequence (x_k) in X is said to be convergent to $l \in X$ if, for every $\varepsilon > 0$ and $0 < \delta < 1$, $\exists$ a natural number $k_{\varepsilon\delta}$ such that*

$$x_k \in \mathcal{N}_l(\varepsilon, \delta), \qquad \text{for all } k \geq k_{\varepsilon\delta}.$$

In this case, we write $\mathcal{G}\text{-}\lim_{k\to\infty} x_k = l$ or $x_k \xrightarrow{\mathcal{G}} l$.

Definition 2.11 *[24] Let $(X, \mathcal{G}, \tau)$ be a PGM space. A sequence (x_k) in X is said to be Cauchy if, for every $\varepsilon > 0$ and $0 < \delta < 1$, $\exists$ a natural number $k_{\varepsilon\delta}$ such that*

$$\mathcal{G}_{(x_l,x_m,x_n)}(\varepsilon) > 1 - \delta, \qquad \text{whenever } l, m, n \geq k_{\varepsilon\delta}.$$

In [1], Abazari provided the definition of statistical convergence of sequences in probabilistic g-metric spaces of order l (if we set $l = 2$, then we get the definition of statistical convergence of sequences in probabilistic G-metric spaces). However, following the definition of convergence of sequences, we reintroduce the definition of statistical convergence of sequences in probabilistic G-metric spaces differently, as follows:

Definition 2.12 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. A sequence (x_k) in X is said to be statistically convergent to $l \in X$ if, for any $\varepsilon > 0$ and $0 < \delta < 1$*

$$d(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_l(\varepsilon, \delta)\}) = 0, \qquad \text{or} \qquad d(\{k \in \mathbb{N} : x_k \in \mathcal{N}_l(\varepsilon, \delta)\}) = 1.$$

In this case, we write $st^{\mathcal{G}}\text{-}\lim_{k\to\infty} x_k = l$.

3. Main Results

In this section, we introduce one notion of convergence of sequences and two different notions of Cauchy-like sequences in probabilistic G-metric spaces. And we study the interrelationships among these new concepts. This whole section is divided into three parts for the convenience.

3.1. λ-statistical convergence

Definition 3.1 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. A sequence (x_k) in X is said to be λ-statistically convergent to $x \in X$ if, for any $\varepsilon > 0$ and $0 < \delta < 1$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}| = 1$$

or,

$$d_\lambda(\{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}) = 1.$$

In this case, we write $st_\lambda^{\mathcal{G}}\text{-}\lim_{k\to\infty} x_k = x$, or, $x_k \xrightarrow{st_\lambda^{\mathcal{G}}} x$.

Remark 3.1 In the above definition (Definition 3.1), if we set $\lambda_n = n$, then we have the notion of statistical convergence of sequences in $(X, \mathcal{G}, \tau)$. In fact, if a sequence (x_k) in X is λ-statistically convergent to $x \in X$, then (x_k) is statistically convergent to $x \in X$

Proposition 3.1 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. A statistically convergent sequence in X is λ-statistically convergent if and only if $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$.*

Proof: Let $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$ and the sequence (x_k) be statistically convergent to x in X. Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Then, we have

$$\begin{aligned} & \frac{1}{n}\left|\left\{j \in [1,n] : \mathcal{G}_{(x_j,x,x)}(\varepsilon) \leq 1-\delta \vee \mathcal{G}_{(x,x_j,x_j)}(\varepsilon) \leq 1-\delta\right\}\right| \\ \geq\ & \frac{1}{n}\left|\left\{j \in I_n : \mathcal{G}_{(x_j,x,x)}(\varepsilon) \leq 1-\delta \vee \mathcal{G}_{(x,x_j,x_j)}(\varepsilon) \leq 1-\delta\right\}\right| \\ =\ & \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{j \in I_n : \mathcal{G}_{(x_j,x,x)}(\varepsilon) \leq 1-\delta \vee \mathcal{G}_{(x,x_j,x_j)}(\varepsilon) \leq 1-\delta\right\}\right|. \end{aligned}$$

Now $\{\lambda_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of positive numbers such that $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_n \leq n$ for all n. Thus the quotient $\frac{\lambda_n}{n}$ is non-negative and bounded above by 1. So, we have $0 < \liminf_{n\to\infty} \frac{\lambda_n}{n} \leq 1$ and since

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{j \in [1,n] : \mathcal{G}_{(x_j,x,x)}(\varepsilon) \leq 1-\delta \vee \mathcal{G}_{(x,x_j,x_j)}(\varepsilon) \leq 1-\delta\right\}\right| = 0,$$

so

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\left|\left\{j \in I_n : \mathcal{G}_{(x_j,x,x)}(\varepsilon) \leq 1-\delta \vee \mathcal{G}_{(x,x_j,x_j)}(\varepsilon) \leq 1-\delta\right\}\right| = 0,$$

that is,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\left|\left\{j \in I_n : \mathcal{G}_{(x_j,x,x)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right| = 1.$$

Hence, (x_k) is λ-statistically convergent to x.

Conversely, if possible, let $\liminf_{n\to\infty} \frac{\lambda_n}{n} = 0$. Then there exists a subsequence $\left(\frac{\lambda_{n_k}}{n_k}\right)$ of $\left(\frac{\lambda_n}{n}\right)$ such that $\frac{\lambda_{n_k}}{n_k} < \frac{1}{k}$. Now consider the probabilistic G-metric space of the Example 3.2. Define a sequence (x_k) as follows:

$$x_j = \begin{cases} 1, & \text{if } j \in I_{n_k}, \quad k = 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

Then (x_j) is statistically convergent 0 but not λ-statistically convergent. □

Theorem 3.1 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. If (x_k) is a sequence in X, λ-statistically convergent to both x and y in X, then $x = y$.*

Proof: Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Since τ is continuous, there exists $0 < \delta_0 < 1$ such that $\tau(1-\delta_0, 1-\delta_0) > 1-\delta$. Let

$$A_n = A_n\left(\frac{\varepsilon}{2}, \delta_0\right) = \left\{k \in I_n : x_k \in \mathcal{N}_x\left(\frac{\varepsilon}{2}, \delta_0\right)\right\}$$

and

$$B_n = B_n\left(\frac{\varepsilon}{2}, \delta_0\right) = \left\{k \in I_n : x_k \in \mathcal{N}_y\left(\frac{\varepsilon}{2}, \delta_0\right)\right\}.$$

Then $\lim\limits_{n\to\infty} \frac{|A_n|}{\lambda_n} = 1$ and $\lim\limits_{n\to\infty} \frac{|B_n|}{\lambda_n} = 1$. Now for each $n \in \mathbb{N}$,

$$|A_n \cup B_n| = |A_n| + |B_n| - |A_n \cap B_n|$$

and

$$|A_n| \le |A_n \cup B_n| \le |I_n|.$$

Consequently, $\lim\limits_{n\to\infty} \frac{|A_n \cap B_n|}{\lambda_n} = 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$, $A_n \cap B_n \ne \emptyset$. Let $k \in A_n \cap B_n$. Then $x_k \in \mathcal{N}_x\left(\frac{\varepsilon}{2}, \delta_0\right)$ and $x_k \in \mathcal{N}_y\left(\frac{\varepsilon}{2}, \delta_0\right)$. Thus $\mathcal{G}_{(x_k,x,x)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$, $\mathcal{G}_{(x_k,y,y)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$, $\mathcal{G}_{(x,x_k,x_k)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$ and $\mathcal{G}_{(y,x_k,x_k)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$. Now

$$\mathcal{G}_{(x,y,y)}(\varepsilon) \ge \tau\left(\mathcal{G}_{(x,x_k,x_k)}\left(\frac{\varepsilon}{2}\right), \mathcal{G}_{(x_k,y,y)}\left(\frac{\varepsilon}{2}\right)\right) > \tau\left(1-\delta_0, 1-\delta_0\right) > 1-\delta.$$

Since $\delta > 0$ was arbitrary, we have $\mathcal{G}_{(x,y,y)}(\varepsilon) = 1$. Hence $x = y$. □

Theorem 3.2 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. Let (x_n), (y_n) and (z_n) be sequences in X and $x, y, z \in X$. If $st_\lambda^{\mathcal{G}}\text{-}\lim\limits_{n\to\infty} x_n = x$, $st_\lambda^{\mathcal{G}}\text{-}\lim\limits_{n\to\infty} y_n = y$, and $st_\lambda^{\mathcal{G}}\text{-}\lim\limits_{n\to\infty} z_n = z$, then, for any $\varepsilon > 0$, $(\mathcal{G}_{x_n,y_n,z_n}(\varepsilon))$ is λ-statistically convergent to $\mathcal{G}_{x,y,z}(\varepsilon)$.*

Proof: Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\varepsilon - 2\delta > 0$. Then

$$\begin{aligned}
& \mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon) \\
\ge\ & \mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon-\delta) \\
\ge\ & \tau\left(\mathcal{G}_{(x_n,x,x)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(x,y_n,z_n)}\left(\frac{3\varepsilon-4\delta}{3}\right)\right) \\
\ge\ & \tau\left(\mathcal{G}_{(x_n,x,x)}\left(\frac{\delta}{3}\right), \tau\left(\mathcal{G}_{(y_n,y,y)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(y,x,z_n)}\left(\varepsilon-\frac{5\delta}{3}\right)\right)\right) \\
\ge\ & \tau\left(\mathcal{G}_{(x_n,x,x)}\left(\frac{\delta}{3}\right), \tau\left(\mathcal{G}_{(y_n,y,y)}\left(\frac{\delta}{3}\right), \tau\left(\mathcal{G}_{(z_n,z,z)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(x,y,z)}\left(\varepsilon-2\delta\right)\right)\right)\right).
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \mathcal{G}_{(x,y,z)}(\varepsilon) \\
\ge\ & \mathcal{G}_{(x,y,z)}(\varepsilon-\delta) \\
\ge\ & \tau\left(\mathcal{G}_{(x,x_n,x_n)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(x_n,y,z)}\left(\frac{3\varepsilon-4\delta}{3}\right)\right) \\
\ge\ & \tau\left(\mathcal{G}_{(x,x_n,x_n)}\left(\frac{\delta}{3}\right), \tau\left(\mathcal{G}_{(y,y_n,y_n)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(y_n,x_n,z)}\left(\varepsilon-\frac{5\delta}{3}\right)\right)\right) \\
\ge\ & \tau\Bigg(\mathcal{G}_{(x,x_n,x_n)}\left(\frac{\delta}{3}\right), \tau\Bigg(\mathcal{G}_{(y,y_n,y_n)}\left(\frac{\delta}{3}\right), \\
& \qquad \tau\left(\mathcal{G}_{(z,z_n,z_n)}\left(\frac{\delta}{3}\right), \mathcal{G}_{(x_n,y_n,z_n)}\left(\varepsilon-2\delta\right)\right)\Bigg)\Bigg).
\end{aligned}$$

Since τ is continuous, it is statistically continuous [Theorem 2, [2]]. Thus $\mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon-2\delta) \geq \mathcal{G}_{(x,y,z)}(\varepsilon)$ for all $n \in A$, $d_\lambda(A) = 1$ and $\mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon) \geq \mathcal{G}_{(x,y,z)}(\varepsilon-2\delta)$ for all $n \in B$, $d_\lambda(B) = 1$. Set $C = A \cap B$. Then $d_\lambda(C) = 1$ and $\mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon-2\delta) \geq \mathcal{G}_{(x,y,z)}(\varepsilon)$ and $\mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon) \geq \mathcal{G}_{(x,y,z)}(\varepsilon-2\delta)$ for all $n \in C$. Thus by the left-continuity of $\mathcal{G}$, $st_\lambda\text{-}\lim_{n\to\infty} \mathcal{G}_{(x_n,y_n,z_n)}(\varepsilon) = \mathcal{G}_{(x,y,z)}(\varepsilon)$ for all $\varepsilon > 0$.

□

Theorem 3.3 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_k) be a sequence in X. If (x_k) is convergent to $x \in X$, then $st^{\mathcal{G}}_\lambda\text{-}\lim_{k\to\infty} x_k = x$.*

Proof: Let $\varepsilon > 0$ and $0 < \delta < 1$ be give. Then there is a natural number $n_{\varepsilon,\delta}$ such that $x_k \in \mathcal{N}_x(\varepsilon, \delta)$ for all $k \geq n_{\varepsilon,\delta}$. Thus $d_\lambda(\{k \in \mathbb{N} : x_k \in N_x(\varepsilon, \delta)\}) = 1$ and so $st^{\mathcal{G}}_\lambda\text{-}\lim_{k\to\infty} x_k = x$. □

Theorem 3.4 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_n) be sequence in X. Then $st^{\mathcal{G}}_\lambda\text{-}\lim_{n\to\infty} x_n = x$ if and only if there exists a subset $M = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $d_\lambda(M) = 1$ and $\mathcal{G}\text{-}\lim_{n\to\infty} x_{k_n} = x$.*

Proof: Let there exists a subset $M = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $d_\lambda(M) = 1$ and $\mathcal{G}\text{-}\lim_{n\to\infty} x_{k_n} = x$. Let $\varepsilon > 0$ and $0 < \delta < 1$. Then there exists a positive integer $n_{\varepsilon,\delta}$ such that $x_{k_n} \in \mathcal{N}_x(\varepsilon, \delta)$ whenever $n \geq n_{\varepsilon,\delta}$. Thus $\{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\} \supset \{k_n : n \geq n_{\varepsilon,\delta}\}$. Since the later set has λ-density 1, $d_\lambda(\{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}) = 1$. Hence, $st^{\mathcal{G}}_\lambda\text{-}\lim_{n\to\infty} x_n = x$.

Conversely, let $st^{\mathcal{G}}_\lambda\text{-}\lim_{n\to\infty} x_n = x$. Then for each $\varepsilon > 0$ and $0 < \delta < 1$,

$$d_\lambda(\{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}) = 1.$$

Set $A(\varepsilon, \delta) = \{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}$. Now for $\varepsilon_n = \frac{1}{n}$ and $\delta_n = \frac{1}{n}$ with $n \geq 2$, we have

$$\mathcal{N}_x(1/2, 1/2) \supset \mathcal{N}_x(1/3, 1/3) \supset ... \supset \mathcal{N}_x(1/n, 1/n) \supset \mathcal{N}_x(1/(n+1), 1/(n+1)) \supset ...$$

Consequently,

$$A(1/2, 1/2) \supset A(1/3, 1/3) \supset ... \supset A(1/n, 1/n) \supset A(1/(n+1), 1/(n+1)) \supset ...$$

Note that $d_\lambda(A(1/n, 1/n)) = 1$ for each $n(> 1) \in \mathbb{N}$. Set $y_1 = 1$. Since $d_\lambda(A(1/2, 1/2)) = 1$, there exists $y_2 \in A(1/2, 1/2)$ and $y_2 > y_1$ such that for all $n \geq y_2$, we have

$$\frac{|\{k \in I_n : k \in A(1/2, 1/2)\}|}{\lambda_n} > 1 - 1/2.$$

Since $d_\lambda(A(1/3, 1/3)) = 1$, there exists $y_3 \in A(1/3, 1/3)$ with $y_3 > y_2$ such that for all $n \geq y_3$,

$$\frac{|\{k \in I_n : k \in A(1/3, 1/3)\}|}{\lambda_n} > 1 - 1/3.$$

Again, since $d_\lambda(A(1/4, 1/4)) = 1$, there exists $y_4 \in A(1/4, 1/4)$ with $y_4 > y_3$ such that $\forall n \geq y_4$,

$$\frac{|\{k \in I_n : k \in A(1/4, 1/4)\}|}{\lambda_n} > 1 - 1/4.$$

Continuing these processes, we will get a strictly increasing sequence of positive integers (y_m) such that $y_m \in A(1/m, 1/m)$ and for all $n \geq m$, we have

$$\frac{|\{k \in I_n : k \in A(1/m, 1/m)\}|}{\lambda_n} > 1 - 1/m.$$

We now construct a set A as follows:

$$A = \{k \in \mathbb{N} : k \in [y_1, y_2]\} \bigcup \left\{ \bigcup_{m \in \mathbb{N}} \{k \in \mathbb{N} : k \in [y_m, y_{m+1}] \cap A(1/m, 1/m)\} \right\}.$$

Therefore for each $n \in \mathbb{N}$, $y_m \leq n < y_{m+1}$, we have

$$\frac{|\{k \in I_n : k \in A\}|}{\lambda_n} \geq \frac{|\{k \in I_n : k \in A(1/m, 1/m)\}|}{\lambda_n} \geq 1 - 1/m.$$

Thus $d_\lambda(A) = 1$. Let $\varepsilon > 0$ and $0 < \delta < 1$. We choose a large $l \in \mathbb{N}$ such that

$$\frac{1}{l} < \varepsilon \text{ and } \frac{1}{l} < \delta.$$

Let $n \geq y_l$, $n \in A$. Then there exists $r \in \mathbb{N}$ such that $y_r \leq n < y_{r+1}$ and $r > l$. Then $n \in A\left(\frac{1}{r}, \frac{1}{r}\right)$. Therefore

$$x_n \in \mathcal{N}_x\left(\frac{1}{r}, \frac{1}{r}\right) \subset \mathcal{N}_x\left(\frac{1}{l}, \frac{1}{l}\right) \subset \mathcal{N}_x(\varepsilon, \delta).$$

Thus $x_n \in \mathcal{N}_x(\varepsilon, \delta)$ for each $n \in A$ with $n \geq y_l$. Write $A = \{k_1 < k_2 < ...\}$. Therefore $\mathcal{G}\text{-}\lim_{n\to\infty} x_{k_n} = x$. This completes the proof. □

Theorem 3.5 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_k) be a sequence in X. Then $x_k \xrightarrow{st_\lambda^{\mathcal{G}}} x$ if and only if there exists a sequence (y_k) such that $x_k = y_k$ for λ-a.a.k. and $y_k \xrightarrow{\mathcal{G}} x$.*

Proof: Let $x_k \xrightarrow{st_\lambda^{\mathcal{G}}} x$. Then by Theorem 3.4, there exists a set

$$H = \{q_1 < q_2 < ... < q_n < ...\} \subset \mathbb{N}$$

such that $d_\lambda(H) = 1$ and $\mathcal{G}\text{-}\lim_{n\to\infty} x_{q_n} = x$.

Define a sequence (y_k) as follows:

$$y_k = \begin{cases} x_k, & \text{if } k = q_n \text{ for some } n \\ x, & \text{otherwise.} \end{cases}$$

Clearly, $y_k \xrightarrow{\mathcal{G}} x$ and $x_k = y_k$ for λ-a.a.k.

Conversely, let $x_k = y_k$ for λ-a.a.k. and $y_k \xrightarrow{\mathcal{G}} x$. Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Then for each $n \in \mathbb{N}$, we have

$$\{k \in I_n : x_k \notin \mathcal{N}_x(\varepsilon, \delta)\} \subset \{k \in I_n : x_k \neq y_k\} \cup \{k \in I_n : y_k \notin \mathcal{N}_x(\varepsilon, \delta)\}.$$

Since (y_k) is convergent to x, $\{k \in \mathbb{N} : y_k \notin \mathcal{N}_x(\varepsilon, \delta)\}$ is finite. Consequently,

$$d_\lambda(\{k \in \mathbb{N} : y_k \notin \mathcal{N}_x(\varepsilon, \delta)\}) = 0.$$

Thus

$$\begin{aligned} & d_\lambda(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_x(\varepsilon, \delta)\}) \\ & \leq d_\lambda(\{k \in \mathbb{N} : x_k \neq y_k\}) + d_\lambda(\{k \in \mathbb{N} : y_k \notin \mathcal{N}_x(\varepsilon, \delta)\}) = 0. \end{aligned}$$

Therefore $d_\lambda(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_x(\varepsilon, \delta)\}) = 0$, that is, $d_\lambda(\{k \in \mathbb{N} : x_k \in \mathcal{N}_x(\varepsilon, \delta)\}) = 1$. Hence the sequence (x_k) is λ-statistically convergent to x. □

3.2. λ-statistically Cauchy sequences

In this subsection, we introduce the notion of λ-statistically Cauchy sequences in PGM spaces. We establish a relationship between λ-statistically convergent sequences and λ-statistically Cauchy sequences in PGM spaces.

Definition 3.2 *Let (x_k) be a sequence in a PGM space $(X, \mathcal{G}, \tau)$. Then (x_k) is said to be λ-statistically Cauchy if, for every $\varepsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $N = N(\varepsilon, \delta)$ such that*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{j \in I_n : \mathcal{G}_{(x_j,x_N,x_N)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_N,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right| = 1.$$

or,

$$d_\lambda\left(\left\{j \in \mathbb{N} : \mathcal{G}_{(x_j,x_N,x_N)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_N,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right) = 1.$$

Theorem 3.6 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_k) be a sequence in X. If (x_k) is λ-statistically convergent to x, then (x_k) is λ-statistically Cauchy.*

Proof: Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Then by the continuity of τ, there exists $0 < \delta_0 < 1$ such that

$$\tau(1-\delta_0, 1-\delta_0) > 1-\delta.$$

Since $x_k \xrightarrow{st_\lambda^{\mathcal{G}}} x$, $d_\lambda\left(\left\{k \in \mathbb{N} : x_k \in \mathcal{N}_x\left(\frac{\varepsilon}{2}, \delta_0\right)\right\}\right) = 1$. Set $A = \left\{k \in \mathbb{N} : x_k \in \mathcal{N}_x\left(\frac{\varepsilon}{2}, \delta_0\right)\right\}$. Then $d_\lambda(A) = 1$. Let n_0 be an arbitrary but fixed element of A. Then $x_{n_0} \in \mathcal{N}_x\left(\frac{\varepsilon}{2}, \delta_0\right)$. Then for $k \in A$, we have

$$\begin{aligned} &\mathcal{G}_{(x_k,x_{n_0},x_{n_0})}(\varepsilon) \\ \geq\ &\tau\left(\mathcal{G}_{(x_k,x,x)}\left(\frac{\varepsilon}{2}\right), \mathcal{G}_{(x,x_{n_0},x_{n_0})}\left(\frac{\varepsilon}{2}\right)\right) \\ >\ &\tau\left(1-\delta_0, 1-\delta_0\right) > 1-\delta. \end{aligned}$$

Also,

$$\begin{aligned} &\mathcal{G}_{(x_{n_0},x_k,x_k)}(\varepsilon) \\ \geq\ &\tau\left(\mathcal{G}_{(x_{n_0},x,x)}\left(\frac{\varepsilon}{2}\right), \mathcal{G}_{(x,x_k,x_k)}\left(\frac{\varepsilon}{2}\right)\right) \\ >\ &\tau\left(1-\delta_0, 1-\delta_0\right) > 1-\delta. \end{aligned}$$

Thus,

$$A \subset \left\{k \in \mathbb{N} : \mathcal{G}_{(x_k,x_{n_0},x_{n_0})}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_{n_0},x_k,x_k)}(\varepsilon) > 1-\delta\right\}.$$

Consequently,

$$d_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{G}_{(x_k,x_{n_0},x_{n_0})}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_{n_0},x_k,x_k)}(\varepsilon) > 1-\delta\right\}\right) = 1.$$

Hence, the sequence (x_k) is λ-statistically Cauchy. □

Theorem 3.7 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_k) be a sequence in X. If x is λ-statistically Cauchy, then for $\varepsilon > 0$ and $0 < \delta < 1$, there is a set $M_{\varepsilon,\delta} \subset \mathbb{N}$ with $d_\lambda\left(M_{\varepsilon,\delta}\right) = 0$ such that $\mathcal{G}_{(x_k,x_j,x_p)}(\varepsilon) > 1-\delta$ for all $k, j, p \notin M_{\varepsilon,\delta}$.*

Proof: Let (x_k) be λ-statistically Cauchy. Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Then by the continuity of τ, there exists $0 < \delta_0 < 1$ such that, $\tau\left(1-\delta_0, \tau\left(1-\delta_0, 1-\delta_0\right)\right) > 1-\delta$. Since (x_k) is λ-statistically Cauchy, there is an $n_{\varepsilon,\delta_0} \in \mathbb{N}$ such that

$$d_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{G}_{(x_k,x_{n_{\varepsilon,\delta_0}},x_{n_{\varepsilon,\delta_0}})}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0, \mathcal{G}_{(x_{n_{\varepsilon,\delta_0}},x_k,x_k)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0\right\}\right) = 1.$$

Set $M = \left\{k \in \mathbb{N} : \mathcal{G}_{(x_k, x_{n_{\varepsilon,\delta_0}}, x_{n_{\varepsilon,\delta_0}})}\left(\frac{\varepsilon}{2}\right) \le 1-\delta_0 \vee \mathcal{G}_{(x_{n_{\varepsilon,\delta_0}}, x_k, x_k)}\left(\frac{\varepsilon}{2}\right) \le 1-\delta_0\right\}$. Then $d_\lambda(M) = 0$. For $j, p, q \notin M$, we have

$$\begin{aligned}
& \mathcal{G}_{(x_j,x_p,x_q)}(\varepsilon) \\
\ge\ & \tau\left(\mathcal{G}_{(x_j,x_{n_{\varepsilon,\delta_0}},x_{n_{\varepsilon,\delta_0}})}\left(\frac{\varepsilon}{3}\right), \mathcal{G}_{(x_{n_{\varepsilon,\delta_0}},x_p,x_q)}\left(\frac{2\varepsilon}{3}\right)\right) \\
\ge\ & \tau\left(\mathcal{G}_{(x_j,x_{n_{\varepsilon,\delta_0}},x_{n_{\varepsilon,\delta_0}})}\left(\frac{\varepsilon}{3}\right), \tau\left(\mathcal{G}_{(x_p,x_{n_{\varepsilon,\delta_0}},x_{n_{\varepsilon,\delta_0}})}\left(\frac{\varepsilon}{3}\right), \mathcal{G}_{(x_{n_{\varepsilon,\delta_0}},x_{n_{\varepsilon,\delta_0}},x_q)}\left(\frac{\varepsilon}{3}\right)\right)\right) \\
>\ & \tau\left(1-\delta_0, \tau\left(1-\delta_0, 1-\delta_0\right)\right) \\
>\ & 1-\delta.
\end{aligned}$$

Note that M depends on both ε and δ. Hence for every $\varepsilon > 0$ and $0 < \delta < 1$, there is a set $M_{\varepsilon,\delta} \subset \mathbb{N}$ with $d_\lambda\left(M_{\varepsilon,\delta}\right) = 0$ such that $\mathcal{G}_{(x_j,x_p,x_q)}(\varepsilon) > 1-\delta$ for every $j, p, q \notin M_{\varepsilon,\delta}$. □

Corollary 3.1 *Let $(X, \mathcal{G}, \tau)$ be a PGM space and (x_k) be a sequence in X. If (x_k) is λ-statistically Cauchy, then for each $\varepsilon > 0$ and $0 < \delta < 1$, there is a set $P_{\varepsilon,\delta} \subset \mathbb{N}$ with $d_\lambda\left(P_{\varepsilon,\delta}\right) = 1$ such that $\mathcal{G}_{(x_k,x_j,x_p)}(\varepsilon) > 1-\delta$ for all $k, j, p \in P_{\varepsilon,\delta}$.*

In the next example, we show that λ-statistically Cauchy sequences may not be λ-statistically convergent in some PGM spaces. In fact, it is sufficient to show tha a statistically Cauchy sequence may not be statistically convergent because if every λ-statistically Cauchy sequence is λ-statistically convergent for every λ, then in particular, for $\lambda_n = n$, every statistically Cauchy sequence is statistically convergent.

Example 3.1 Consider $\lambda_n = n$. Let the distribution function H be defined as follows:

$$H(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1, & \text{if } t > 0. \end{cases}$$

and the distribution function D be defined as follows:

$$D(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1 - e^{-t}, & \text{if } t > 0. \end{cases}$$

Also, we make the convention that $D(t/0) = D(\infty) = 1$ for $t > 0$ and $D(0/0) = 0$.

For $t > 0$, we define a function $\mathcal{G} : (0,1) \times (0,1) \times (0,1) \to \mathbb{R}^+$ by

$$\mathcal{G}_{(x,y,z)}(t) = \begin{cases} H(t), & \text{if } x = y = z \\ D\left(\frac{t}{|x-y|+|y-z|+|z-x|}\right), & \text{otherwise.} \end{cases}$$

Then $((0,1), \mathcal{G}, T)$ becomes a PGM space, where T is the minimum triangle function [Example 1.6, [24]]. Define a sequence (x_k) as follows:

$$x_k = \begin{cases} k, & \text{if } k \text{ is square} \\ 1/4k, & \text{otherwise.} \end{cases}$$

Since no subsequence of (x_k) is convergent, (x_k) is not statistically convergent. Now, we show that (x_k) is statistically Cauchy. Let $t > 0$ and $0 < \delta < 1$ be given. Choose a positive integer N such that $e^{-N} < \delta$. Let j be a non-square positive integer. Then $\mathcal{G}_{(j,N,N)}(t) = 1 - e^{-t/2|x_j - x_N|} = 1 - e^{-t/2|1/j - 1/N|} \ge 1 - e^{-N} > 1 - \delta$. Likewise, $\mathcal{G}_{(N,j,j)}(t) > 1 - \delta$. Thus

$$\left\{j \in \mathbb{N} : \mathcal{G}_{(x_j,x_N,x_N)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_N,x_j,x_j)}(\varepsilon) > 1-\delta\right\} \supset \left\{j \in \mathbb{N} : j \ \text{ is non-square}\right\}.$$

Consequently,

$$d\left(\left\{j \in \mathbb{N} : \mathcal{G}_{(x_j,x_N,x_N)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_N,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right) = 1.$$

Hence (x_k) is statistically Cauchy.

3.3. λ-statistical pre-Cauchy sequences

In this subsection, we introduce the notion of λ-statistically pre-Cauchy. We establish some relationships among λ-statistically convergent sequences, λ-statistically Cauchy sequences and λ-statistically pre-Cauchy sequences in PGM spaces.

Definition 3.3 *A sequence (x_k) in a PGM space $(X, \mathcal{G}, \tau)$ is said to be λ-statistically pre-Cauchy if for every $\varepsilon > 0$ and $0 < \delta < 1$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : \mathcal{G}_{(x_j,x_k,x_k)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right| = 1.$$

Note 1 *In the above Definition 3.3, if we replace λ_n by n for each $n \in \mathbb{N}$, then we say the sequence is statistically pre-Cauchy in X. Note that, in this case, $I_n = [1, n]$.*

Theorem 3.8 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. If a sequence (x_k) is λ-statistically Cauchy, then it is λ-statistically pre-Cauchy in X.*

Proof: Let the sequence (x_k) be λ-statistically Cauchy in X. Let $\varepsilon > 0$ and $0 < \delta < 1$ be given. Since τ is continuous, there is $0 < \delta_0 < 1$ such that $\tau(1-\delta_0, 1-\delta_0) > 1-\delta$. On the other hand, there exists a positive integer N such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{j \in I_n : \mathcal{G}_{(x_j,x_N,x_N)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0, \mathcal{G}_{(x_N,x_j,x_j)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0\right\}\right| = 1.$$

Set $A_n = \left\{j \in I_n : \mathcal{G}_{(x_j,x_N,x_N)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0, \mathcal{G}_{(x_N,x_j,x_j)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0\right\}$. Let $j,k \in A_n$. Then $\mathcal{G}_{(x_k,x_N,x_N)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$, $\mathcal{G}_{(x_j,x_N,x_N)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$, $\mathcal{G}_{(x_N,x_k,x_k)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$ and $\mathcal{G}_{(x_N,x_j,x_j)}\left(\frac{\varepsilon}{2}\right) > 1-\delta_0$. Thus,

$$\begin{aligned} & \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) \\ \geq\ & \tau\left(\mathcal{G}_{(x_N,x_j,x_j)}\left(\frac{\varepsilon}{2}\right) \mathcal{G}_{(x_k,x_N,x_N)}\left(\frac{\varepsilon}{2}\right)\right) \\ >\ & \tau(1-\delta_0, 1-\delta_0) \\ >\ & 1-\delta. \end{aligned}$$

Similarly,

$$\mathcal{G}_{(x_j,x_k,x_k)}(\varepsilon) > 1-\delta.$$

Hence

$$A_n \times A_n \subset \left\{(j,k) \in I_n \times I_n : \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_j,x_k,x_k)}(\varepsilon) > 1-\delta\right\}.$$

This implies

$$\begin{aligned} & \left[\frac{|A_n|}{\lambda_n}\right]^2 \\ \leq\ & \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_j,x_k,x_k)}(\varepsilon) > 1-\delta\right\}\right| \\ \leq\ & 1. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{|A_n|}{\lambda_n} = 1$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : \mathcal{G}_{(x_k,x_j,x_j)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_j,x_k,x_k)}(\varepsilon) > 1-\delta\right\}\right| = 1.$$

Hence, the sequence (x_k) is λ-statistically pre-Cauchy in X. $\square$

Corollary 3.2 *Let $(X, \mathcal{G}, \tau)$ be a PGM space. If a sequence (x_k) is λ-statistically convergent, then it is λ-statistically pre-Cauchy in X.*

Remark 3.2 We claim that there exists a λ-statistically pre-Cauchy sequence in some PGM space which is not λ-statistically Cauchy. To justify our claim, we consider the following example:

Example 3.2 In this example, we let $\lambda_n = n$. Let the distribution function H be defined as follows:

$$H(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1, & \text{if } t > 0. \end{cases}$$

and the distribution function D be defined as follows:

$$D(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1 - e^{-t}, & \text{if } t > 0. \end{cases}$$

Also, we make the convention that $D(t/0) = D(\infty) = 1$ for $t > 0$ and $D(0/0) = 0$.

For $t > 0$, we define a function $\mathcal{G} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$\mathcal{G}_{(x,y,z)}(t) = \begin{cases} H(t), & \text{if } x = y = z \\ D\left(\frac{t}{|x-y|+|y-z|+|z-x|}\right), & \text{otherwise.} \end{cases}$$

Then $(\mathbb{R}, \mathcal{G}, T)$ becomes a PGM space, where T is the minimum triangle function [Example 1.6, [24]]. Now for $k \in \mathbb{N}$, we define $x_k = n$, where $(n-1)! < k \le n!$. Then (x_k) is a sequence in $\mathbb{R}$. We show that (x_k) is statistically pre-Cauchy but not statistically Cauchy. Let $t > 0$ and $0 < \delta < 1$ be given. Then for $24 < (m-1)! < n \le m!$, we have,

$$\begin{aligned} & \left|\left\{(j,k) : \mathcal{G}_{(x_j,x_k,x_k)}(t) > 1-\delta, \mathcal{G}_{(x_k,x_j,x_j)}(t) > 1-\delta, j,k \le n\right\}\right| \\ \ge \; & [n - (m-1)!]^2. \\ \Longrightarrow \; & \\ & \lim_{n\to\infty} \frac{1}{n^2} \left|\left\{(j,k) : \mathcal{G}_{(x_j,x_k,x_k)}(t) > 1-\delta, \mathcal{G}_{(x_k,x_j,x_j)}(t) > 1-\delta, j,k \le n\right\}\right| \\ \ge \; & \lim_{n\to\infty} \frac{1}{n^2}[n - (m-1)!]^2 = 1. \end{aligned}$$

Therefore, (x_k) is statistically pre-Cauchy. Let $t > 0$ and $0 < \delta < 1/2$ be given. Since $e^{-t/2n} \to 1$ as $n \to \infty$, there exists a positive integer n_0 such that for all $n > n_0$, we have $e^{-t/2n} > 1 - \delta$, or, $1 - e^{-t/2n} < \delta$, or, $1 - e^{-t/2n} \not> 1 - \delta$. Let $(m_1 - 1)! < n_1 \le m_1!$ be an arbitrary but fixed positive integer. Then for all positive integer m, $m > n_0 + m_1$ and for $(m-1)! < n \le m!$, we have

$$\begin{aligned} & \mathcal{G}_{(x_k,x_{n_1},x_{n_1})}(t) \\ = \; & D\left(\frac{t}{2\left|x_k - x_{n_1}\right|}\right) \\ = \; & 1 - e^{-t/2(m-m_1)} \\ \not> \; & 1 - \delta, \end{aligned}$$

for all $k \le n$. Thus

$$\left|\left\{k \le n : \mathcal{G}_{(x_k,x_{n_1},x_{n_1})}(t) \le 1 - \delta\right\}\right| \ge n - (n_0 + m_1)!.$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \mathcal{G}_{(x_k,x_{n_1},x_{n_1})}(t) \le 1 - \delta\right\}\right| = 1.$$

This proves that there exists no such positive integer N for which

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{j \in [1,n] : \mathcal{G}_{(x_j,x_N,x_N)}(\varepsilon) > 1-\delta, \mathcal{G}_{(x_N,x_j,x_j)}(\varepsilon) > 1-\delta\right\}\right| = 1.$$

Consequently, (x_k) is not statistically Cauchy.

This example also shows that λ-statistically pre-Cauchy sequences may not be λ-statistically convergent.

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