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Existence of a positive solution for a second order Sturm-Liouville boundary value problem on the half-line with a nonlinear derivative dependence via variational methods

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ABSTRACT: In this paper, a class of second order Sturm-Liouville boundary value problem on the half-line with dependence derivative is considered. The existence of a positive classical solution is proved by using variational methods and iterative methods.

Key Words: Sturm-Liouville BVPs, half-line, variational methods, iterative methods, nonlinear derivative dependence, critical point theory.

Contents

1. Introduction

Our aim in this paper is to apply critical point theory and iterative methods to prove the existence of a positive classical solution for a second order Sturm-Liouville boundary value problem with derivative dependence. We consider the following problem

$$\begin{cases} -\big(\rho(t)x'(t)\big)' + s(t)x(t) = f(t, x(t), x'(t)), & t \in [0, +\infty), \\ \alpha x'(0) - \beta x(0) = A, \quad x(+\infty) = 0, \end{cases} \tag{1.1}$$

where: $f \in C([0, +\infty) \times [0, +\infty) \times \mathbb{R}, [0, +\infty))$ and $f(t, 0, \xi) \not\equiv 0$ for $t \in [0, +\infty)$, $\xi \in \mathbb{R}$, ρ, $s \in L^\infty[0, +\infty)$ with $\operatorname*{ess\,inf}_{[0,+\infty)} \rho > 0$, $\operatorname*{ess\,inf}_{[0,+\infty)} s > 0$, $0 < \rho(0) < +\infty$, $A \leq 0$, and $\alpha, \beta > 0$.

Recently, these types of problems have been extensively studied in the case when there is no the presence of the derivative ([1], [2], [3], [4], [5], [7], [9], [12]) by using fixed point theorems and variational methods. In our case, we can not use critical point theory because the problem (1.1) is not of variational type. De Figueiredo, Girardi and Matzeu in [8] studied the existence of solutions for semilinear elliptic equations with dependence on the gradient by using iterative methods, and associating a family of semilinear elliptic problems with no dependence on the gradient of the solution.

In order to use variational methods, we consider a family of boundary value problems with no dependence on the derivative of the solution, that is, for each $\omega \in X$, we consider the following problem

$$\begin{cases} -\big(\rho(t)x'(t)\big)' + s(t)x(t) = f(t, x(t), \omega'(t)), & t \in [0, +\infty), \\ \alpha x'(0) - \beta x(0) = A, \quad x(+\infty) = 0. \end{cases} \tag{1.2}$$

Now we can treat Problem (1.2) by Variational Methods and the existence of a solution for the initial problem can be obtained by iterative methods.

In this paper, we ask new assumptions on the nonlinearity f which are different from those in ([1], [3], [7], [9], [12]). Let the assumptions:

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(C_1) there exist $\mu > 2$, $h \in C([0,+\infty) \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$ and $l \in C([0,+\infty),[0,+\infty))$ with $\min_{t\in[0,+\infty)} l(t) > 0$ such that

$$f(t,x,\xi) = l(t)\Phi_\mu(x) + h(t,x,\xi),$$

where $\Phi_\mu(x) := |x|^{\mu-2}x$.

(C_2) there exist $c \in L^1([0,+\infty),[0,+\infty))$ and $d \in C([0,+\infty),[0,+\infty))$ such that

$$h(t,x,\xi) \leq c(t) + d(t)x.$$

(C_3)

$$\frac{\|d\|_\infty}{\operatorname{ess\,inf}_{[0,+\infty)} s} < \frac{\mu-2}{\mu}.$$

(C_4) There exists $R_0 > 0$ such that

$$\frac{1}{2}\left(1 - \frac{\|d\|_\infty}{\operatorname{ess\,inf}_{[0,+\infty)} s}\right) R_0^2 > \frac{\|l\|_\infty M^\mu R_0^\mu}{\mu} + \|c\|_{L^1} M R_0 + \frac{\rho(0)A^2}{2\beta\alpha}.$$

(C_5) There exist positive constants L_1 and L_2 such that

$$\begin{aligned}
|f(t,x,\xi) - f(t,y,\xi)| &\leq L_1|x-y|, \quad \forall t \in [0,+\infty), x,y \in [0,MR_0), \xi \in \mathbb{R},\\
|f(t,x,\xi) - f(t,x,\xi')| &\leq L_2|\xi-\xi'|, \quad \forall t \in [0,+\infty), x \in [0,MR_0), \xi,\xi' \in \mathbb{R}.
\end{aligned}$$

2. Related lemmas

Let X be the Sobolev space $W^{1,2}([0,+\infty))$ endowed with the norm

$$\|x\|_X = \left(\int_0^{+\infty} \left(\rho(t)|x'(t)|^2 + s(t)|x(t)|^2\right)\mathrm{d}t\right)^{1/2},$$

which is equivalent to the usual norm.
Let $p : [0,+\infty) \longrightarrow (0,+\infty)$ be bounded, continuous, differentiable function with

$$M = \max\left(\frac{\|p\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}}, \frac{\|p'\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}\right) < +\infty.$$

We consider the following spaces

$$C_{l,p}[0,+\infty) = \{x \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p(t)x(t) \text{ exists}\}$$

endowed with the norm

$$\|x\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|x(t)|.$$

Lemma 2.1 *X embeds continuously into $C_{l,p}[0,+\infty)$.*

Proof: For $x \in X$, we have

$$\begin{aligned}
|p(t)x(t)| &= \left| \int_t^{+\infty} (px)'(\theta) d\theta \right| \\
&\leq \left| \int_t^{+\infty} p'(\theta)x(\theta) d\theta \right| + \left| \int_t^{+\infty} p(\theta)x'(\theta) d\theta \right| \\
&\leq \left(\int_0^{+\infty} p'^2(\theta) d\theta \right)^{\frac{1}{2}} \left(\int_0^{+\infty} x^2(\theta) d\theta \right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^{+\infty} p^2(\theta) d\theta \right)^{\frac{1}{2}} \left(\int_0^{+\infty} x'^2(\theta) d\theta \right)^{\frac{1}{2}} \\
&\leq \frac{1}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}} \left(\int_0^{+\infty} p'^2(\theta) d\theta \right)^{\frac{1}{2}} \left(\int_0^{+\infty} s(\theta)x^2(\theta) d\theta \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}} \left(\int_0^{+\infty} p^2(\theta) d\theta \right)^{\frac{1}{2}} \left(\int_0^{+\infty} \rho(\theta)x'^2(\theta) d\theta \right)^{\frac{1}{2}} \\
&\leq \frac{\|p'\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}} \|x\|_X + \frac{\|p\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}} \|x\|_X \\
&\leq \max\left(\frac{\|p'\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}, \frac{\|p\|_{L^2}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}} \right) \|x\|_X \\
&\leq M\|x\|_X.
\end{aligned}$$

Then $\|x\|_{\infty,p} \leq M\|x\|_X$. □

The following compactness embedding is an important result.

Lemma 2.2 *The embedding $X \hookrightarrow C_{l,p}[0,+\infty)$ is compact.*

To prove that X embeds compactly into $C_{l,p}[0,+\infty)$, we need the following Corduneanu compactness criterion.

Lemma 2.3 ([6]) *Let $D \subset C_{l,p}([0,+\infty), \mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $[0,+\infty)$, i.e., for all $J \subset [0,+\infty)$ compact, for all $\varepsilon > 0$, there exists $\delta > 0$, for all $t_1, t_2 \in J : |t_1 - t_2| < \delta$ implies $|p(t_1)x(t_1) - p(t_2)x(t_2)| \leq \varepsilon$, for all $x \in D$,
(b) D is equiconvergent at $+\infty$ i.e., for all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that for all $t : t \geq T(\varepsilon)$ implies $|p(t)x(t) - (px)(+\infty)| \leq \varepsilon$, for all $x \in D$.

Proof: [Proof of Lemma 2.2] Let $D \subset X$ be a bounded set. Then it is bounded in $C_{l,p}[0,+\infty)$ by Lemma 2.1. Let $M_1 > 0$ be such that for all $x \in D$, $\|x\|_X \leq M_1$. We will apply Lemma 2.3.
(a) D is equicontinuous on every compact interval of $[0,+\infty)$. Let $x \in D$ and $t_1, t_2 \in J \subset [0,+\infty)$ where

J is a compact sub-interval of $[0,+\infty)$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|p(t_1)x(t_1)-p(t_2)x(t_2)| &= \left|\int_{t_2}^{t_1}(px)'(\theta)d\theta\right| \\
&= \left|\int_{t_2}^{t_1}p'(\theta)x(\theta)+x'(\theta)p(\theta)d\theta\right| \\
&\le \left(\int_{t_2}^{t_1}p'^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{t_2}^{t_1}x^2(\theta)d\theta\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_2}^{t_1}p^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{t_2}^{t_1}x'^2(\theta)d\theta\right)^{\frac{1}{2}} \\
&\le \max\left[\frac{\left(\int_{t_2}^{t_1}p'^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}, \frac{\left(\int_{t_2}^{t_1}p^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}}\right]\|x\|_X \\
&\le M_1\max\left[\frac{\left(\int_{t_2}^{t_1}p'^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}, \frac{\left(\int_{t_2}^{t_1}p^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}}\right] \longrightarrow 0,
\end{aligned}$$

as $|t_1-t_2|\to 0$.
(b) D is equiconvergent at $+\infty$. For $t\in[0,+\infty)$ and $x\in D$, by the Cauchy-Schwarz inequality and since $p\in L^2([0,+\infty))$ then $p(+\infty)=0$. We have

$$\begin{aligned}
|(px)(t)-(px)(+\infty)| &= |(px)(t)-p(+\infty)x(+\infty)| \\
&= |(px)(t)| \\
&= \left|\int_{t}^{+\infty}(px)'(\theta)d\theta\right| \\
&= \left|\int_{t}^{+\infty}p'(\theta)x(\theta)+x'(\theta)p(\theta)d\theta\right| \\
&\le \max\left[\frac{\left(\int_{t}^{+\infty}p'^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}, \frac{\left(\int_{t}^{+\infty}p^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}}\right]\|x\|_X \\
&\le M_1\max\left[\frac{\left(\int_{t}^{+\infty}p'^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} s)^{1/2}}, \frac{\left(\int_{t}^{+\infty}p^2(\theta)d\theta\right)^{\frac{1}{2}}}{(\operatorname{ess\,inf}_{[0,+\infty)} \rho)^{1/2}}\right] \longrightarrow 0,
\end{aligned}$$

as $t\to+\infty$. □

Definition 2.1 A function $x\in X$ is said to be a classical solution of BVP (1.1) if x satisfies the equation in (1.1) for all $t\in[0,+\infty)$ and the boundary conditions of (1.1). Moreover, x is said to be a positive classical solution of BVP (1.1) if $x(t)>0$ for every $t\in[0,+\infty)$.

We need also the following lemma where the proof is easy.

Lemma 2.4 *For $x \in X$, let $x^{\pm} = \max\{\pm x, 0\}$. Then the following six properties hold:*

(i) $x \in X$ *implies* $x^+, x^- \in X$*;*

(ii) $x = x^+ - x^-$*;*

(ii) $\|x^+\|_X \leq \|x\|_X$*;*

(iv) if (x_n) *uniformly converges to* x *in* $C_{l,p}[0,+\infty)$*, then* (x_n^+) *converges to* x^+ *in* $C_{l,p}[0,+\infty)$*;*

(v) $x^+(t)x^-(t) = 0$, $(x^+)'(t)(x^-)'(t) = 0$ *for* $t \in [0,+\infty)$*;*

(vi) $\Phi_p(x)x^+ = |x^+|^p$, $\Phi_p(x)x^- = -|x^-|^p$.

Lemma 2.5 *Let* $\omega \in X$ *be fixed. If* $x \in C([0,+\infty))$ *is a classical solution of the BVP*

$$\begin{cases} -\big(\rho(t)x'(t)\big)' + s(t)x(t) = f(t, x^+(t), \omega'(t)), & t \in [0,+\infty), \\ \alpha x'(0) - \beta x(0) = A, \quad x(+\infty) = 0, \end{cases} \tag{2.1}$$

then x *is a positive classical solution of BVP (1.2), i,e.,* $x(t) > 0$ *for any* $t \in [0,+\infty)$.

Proof: If $x \in C([0,+\infty))$ is a classical solution of BVP (1.2), by Lemma 2.1 in [11], we have

$$\begin{aligned}
0 &= \int_0^{+\infty} \Big[\big(\rho(t)x'(t)\big)' - s(t)x(t) + f(t, x^+(t), \omega'(t))\Big]\, x^-(t)\mathrm{d}t \\
&= \int_0^{+\infty} \big(\rho(t)x'(t)\big)' x^-(t)\mathrm{d}t - \int_0^{+\infty} s(t)x(t)x^-(t)\mathrm{d}t + \int_0^{+\infty} f(t, x^+(t), \omega'(t))x^-(t)\mathrm{d}t \\
&= \big[\rho(t)x'(t)x^-(t)\big]_0^{+\infty} - \int_0^{+\infty} \rho(t)x'(t)(x^-)'(t)\mathrm{d}t + \int_0^{+\infty} s(t)\big(x^-(t)\big)^2\mathrm{d}t \\
&\quad + \int_0^{+\infty} f(t, x^+(t), \omega'(t))x^-(t)\mathrm{d}t \\
&= -\rho(0)x'(0)x^-(0) + \int_0^{+\infty} \rho(t)\big((x^-)'(t)\big)^2\mathrm{d}t + \int_0^{+\infty} s(t)\big(x^-(t)\big)^2\mathrm{d}t \\
&\quad + \int_0^{+\infty} f(t, x^+(t), \omega'(t))x^-(t)\mathrm{d}t \\
&= -\rho(0)\frac{A + \beta x(0)}{\alpha}x^-(0) + \int_0^{+\infty} \Big(\rho(t)\big((x^-)'(t)\big)^2 + s(t)\big(x^-(t)\big)^2\Big)\,\mathrm{d}t \\
&\quad + \int_0^{+\infty} f(t, x^+(t), \omega'(t))x^-(t)\mathrm{d}t \\
&= -\rho(0)\frac{Ax^-(0) - \beta\big(x^-(0)\big)^2}{\alpha} + \|x^-\|_X^2 + \int_0^{+\infty} f(t, x^+(t), \omega'(t))x^-(t)\mathrm{d}t \\
&\geq +\rho(0)\frac{-Ax^-(0) + \beta\big(x^-(0)\big)^2}{\alpha} + \|x^-\|_X^2 \\
&\geq \|x^-\|_X^2.
\end{aligned}$$

Then $x^-(t) = 0$ for $t \in [0,+\infty)$, that means that $x(t) \geq 0$ for $t \in [0,+\infty)$. If $x(t) \equiv 0$ for $t \in [0,+\infty)$, we have the fact that $f(t, 0, \xi) \not\equiv 0$ for $t \in [0,+\infty)$ and $\xi \in \mathbb{R}$, which it is a contradiction. □

Remark 2.1 To prove the existence of a positive classical solution of BVP (1.2), it suffices to get a classical solution of BVP (2.1) by using Lemma 2.5.

We need also the following Lemma.

Lemma 2.6 (See [13], Theorem 38.A) *For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \varnothing$, $\min_{u\in M} F(u) = \alpha$ has a solution in case the following hold:*

(i) X is a real reflexive Banach space;

(ii) M is bounded and weak sequentially closed, i.e., by definition, for each sequence (u_n) in M such that $u_n \rightharpoonup u$ as $n \longrightarrow +\infty$, we always have $u \in M$;

(iii) F is weak sequentially lower semi-continuous on M.

3. Main results

For each $\omega \in X$ fixed, the functional corresponding to (2.1) is

$$\begin{aligned}\varphi_\omega(x) \quad &:= \quad \frac{\|x\|_X^2}{2} + \frac{\alpha\rho(0)}{2\beta}\left(\frac{A+\beta x(0)}{\alpha}\right)^2 \\ &\qquad - \int_0^{+\infty} \left[F(t, x^+(t), \omega'(t)) - f(t, 0, \omega'(t))x^-(t)\right] \mathrm{d}t, \end{aligned} \tag{3.1}$$

where

$$F(t, x, \xi) = \int_0^x f(t, u, \xi)\mathrm{d}u.$$

It is easy to see that the functional φ_ω is Gâteaux differentiable and the Gâteaux derivative at a point $x \in X$ is

$$\begin{aligned}\langle \varphi'_\omega(x), v\rangle \quad &= \quad \int_0^{+\infty} [\rho(t)x'(t)v'(t) + s(t)x(t)v(t)]\,\mathrm{d}t + \rho(0)\frac{A+\beta x(0)}{\alpha}v(0) \\ &\qquad - \int_0^{+\infty} f(t, x^+(t), \omega'(t))v(t)\mathrm{d}t. \end{aligned} \tag{3.2}$$

for all $v \in X$. By [12], $\varphi'_\omega : X \longrightarrow X^*$ is continuous and the critical point of the functional φ_ω is the solution of BVP (2.1).

Theorem 3.1 *Assume that $(C1)-(C4)$ hold, then for each $\omega \in X$, Problem (1.2) has at least one positive classical solution x_0 with $\|x_0\|_X < R_0$.*

Proof: We will show that there exists an $R > 0$ such that the functional φ_ω has a local minimum $x_0 \in B_R := \{x \in X : \|x\|_X < R\}$.
Firstly: X is a reflexive Banach space.
Secondly: We will show that $\overline{B}_R$ is weak sequentially closed.
Let $R > 0$ which will be determined later and let $(u_n) \subset \overline{B}_R$ be a sequence such that (u_n) is weakly convergent to u in X. By the Mazur theorem [10], there exists a sequence of convex combinations (v_n), such that $v_n \to u$ in X. Since $\overline{B}_R$ is a closed convex set, $(v_n) \subset \overline{B}_R$ and $u \in \overline{B}_R$.
Thirdly: Let $\omega \in X$ be fixed. We will show that φ_ω is sequentially weak lower semi-continuous (w.l.s.c) on $\overline{B}_R$.
For this reason, we use the following decomposition $\varphi_\omega(x) = \varphi^1_\omega(x) + \varphi^2_\omega(x)$, where

$$\varphi^1_\omega(x) = \frac{1}{2}\int_0^{+\infty} \left[\rho(t)(x'(t))^2 + s(t)(x(t))^2\right] \mathrm{d}t$$

and

$$\varphi^2_\omega(x) \quad = \quad \frac{1}{2}\int_0^{+\infty} \left[F(t, x^+(t), \omega'(t)) - f(t, 0, \omega'(t))x^-(t)\right] \mathrm{d}t + \frac{\alpha\rho(0)}{2\beta}\left(\frac{A+\beta x(0)}{\alpha}\right)^2.$$

As $x_n \rightharpoonup x$ on X, we have (x_n) converges to x in $C_{l,p}[0,+\infty)$. So φ_ω^2 is weak sequentially continuous. It is easy to see that φ_ω^1 is continuous and by using the convexity of φ_ω^1, we have that φ_ω^1 is sequentially weak lower semi-continuous. Therefore, φ_ω is sequentially weak lower semi-continuous on $\overline{B}_R$. So our claim follows from Lemma 2.6. Without loss of generality, we assume that $\varphi_\omega(x_0) = \min_{x\in\overline{B}_R} \varphi_\omega(x)$.
Fourthly: Now we will show that

$$\varphi_\omega(x_0) < \inf_{x\in\partial B_R} \varphi_\omega(x). \tag{3.3}$$

For any $x \in \partial B_R$, we have by (3.1), $(C1)$ and Lemma 2.1,

$$\begin{aligned}
\varphi_\omega(x) &\geq \frac{R^2}{2} - \int_0^{+\infty} F(t, x^+(t), \omega'(t))\mathrm{d}t \\
&\geq \frac{R^2}{2} - \int_0^{+\infty} \left[\frac{l(t)|x^+(t)|^\mu}{\mu} + c(t)x^+(t) + \frac{d(t)(x^+(t))^2}{2}\right]\mathrm{d}t \\
&\geq \frac{R^2}{2} - \frac{\|l\|_\infty}{\mu}\|x\|_{\infty,p}^\mu - \|c\|_{L^1}\|x\|_{\infty,p} - \frac{1}{2}\|d\|_\infty \int_0^{+\infty} (x^+(t))^2\mathrm{d}t \\
&\geq \frac{R^2}{2} - \frac{\|l\|_\infty}{\mu}\|x\|_{\infty,p}^\mu - \|c\|_{L^1}\|x\|_\infty - \frac{\|d\|_\infty}{2\operatorname{ess\,inf}_{[0,+\infty)} s}\|x\|_X^2 \\
&\geq \frac{R^2}{2} - \frac{\|l\|_\infty}{\mu}M^\mu\|x\|_X^\mu - \|c\|_{L^1}M\|x\|_X - \frac{\|d\|_\infty}{2\operatorname{ess\,inf}_{[0,+\infty)} s}\|x\|_X^2 \\
&= \frac{R^2}{2} - \frac{\|l\|_\infty}{\mu}M^\mu R^\mu - \|c\|_{L^1}MR - \frac{\|d\|_\infty}{2\operatorname{ess\,inf}_{[0,+\infty)} s}R^2.
\end{aligned}$$

So

$$\inf_{x\in\partial B_R} \varphi_\omega(x) \geq \frac{R^2}{2} - \frac{\|l\|_\infty}{\mu}M^\mu R^\mu - \|c\|_{L^1}MR - \frac{\|d\|_\infty}{2\operatorname{ess\,inf}_{[0,+\infty)} s}R^2.$$

Note that

$$\varphi_\omega(x_0) \leq \varphi_\omega(0) = \frac{\rho(0)A^2}{2\beta\alpha},$$

so by the condition $(C4)$, we have, there exists an R_0 such that $\varphi_\omega(x) > \varphi_\omega(0) \geq \varphi_\omega(x_0)$, for any $x \in \partial B_{R_0}$. Then (3.3) holds and $x_0 \in B_{R_0}$, *i.e.*,

$$\|x_0\|_X < R_0. \tag{3.4}$$

□

Theorem 3.2 *Assume that $(C1) - (C5)$ hold, then problem (3.1) has at least one positive classical solution provided that*

$$0 < \frac{\dfrac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)} \sqrt{\rho s}}}{1 - \dfrac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)} s} - \dfrac{\beta\rho(0)C_1 M}{\alpha}} < 1,$$

where $C_1 = \sup_{[0,+\infty)} \frac{1}{p}$.

Proof: We construct a sequence $(x_n) \subset X$ as solutions of the following problem

$$(P_n) : \begin{cases} -(\rho(t)x_n'(t))' + s(t)x_n(t) = f(t, x_n^+(t), (x_{n-1})'(t)), & \text{for } t \in [0, +\infty) \\ \alpha x_n'(0) - \beta x_n(0) = A, \quad x_n(+\infty) = 0, \end{cases}$$

obtained by the minimization principal in Theorem 3.1. We start with an arbitrary $x_0 \in X$. It follows from (3.4) and Lemma 2.1 that

$$\|x_n\|_X < MR_0. \tag{3.5}$$

We multiply the two sides in (P_n) and (P_{n+1}) by $x_{n+1}(t) - x_n(t)$ and we integrate over $(0, +\infty)$, we obtain

$$\begin{aligned}\int_0^{+\infty} (\rho(t)x_n'(t))(x_{n+1}'(t) - x_n'(t))\mathrm{d}t + \int_0^{+\infty} s(t)x_n(t)(x_{n+1}(t) - x_n(t))\mathrm{d}t \\ +\rho(0)\frac{A + \beta x_n(0)}{\alpha}(x_{n+1}(0) - x_n(0)) = \int_0^{+\infty} f(t, x_n^+(t), (x_{n-1})'(t))(x_{n+1}(t) - x_n(t))\mathrm{d}t,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_0^{+\infty} (\rho(t)x_{n+1}'(t))(x_{n+1}'(t) - x_n'(t))\mathrm{d}t + \int_0^{+\infty} s(t)x_{n+1}(t)(x_{n+1}(t) - x_n(t))\mathrm{d}t \\ +\rho(0)\frac{A + \beta x_{n+1}(0)}{\alpha}(x_{n+1}(0) - x_n(0)) = \int_0^{+\infty} f(t, x_{n+1}^+(t), (x_n)'(t))(x_{n+1}(t) - x_n(t))\mathrm{d}t.\end{aligned} \tag{3.7}$$

Subtracting (3.7) from (3.6), we obtain that

$$\begin{aligned}&\int_0^{+\infty} \rho(t)(x_{n+1}'(t) - x_n'(t))(x_{n+1}'(t) - x_n'(t))\mathrm{d}t \\ &\quad + \int_0^{+\infty} s(t)(x_{n+1}(t) - x_n(t))(x_{n+1}(t) - x_n(t))\mathrm{d}t - \frac{\beta\rho(0)}{\alpha}(x_{n+1}(0) - x_n(0))^2 \\ &\quad\quad = \int_0^{+\infty} \left(f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_{n-1})'(t))\right)(x_{n+1}(t) - x_n(t))\mathrm{d}t.\end{aligned}$$

Then,

$$\begin{aligned}&\int_0^{+\infty} \left[\rho(t)(x_{n+1}'(t) - x_n'(t))^2 + s(t)(x_{n+1}(t) - x_n(t))^2\right]\mathrm{d}t - \frac{\beta\rho(0)}{\alpha}(x_{n+1}(0) - x_n(0))^2 \\ &\quad = \int_0^{+\infty} \left(f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_{n-1})'(t))\right)(x_{n+1}(t) - x_n(t))\mathrm{d}t.\end{aligned}$$

So

$$\begin{aligned}\|x_{n+1} - x_n\|_X^2 &= \int_0^{+\infty} \left(f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_{n-1})'(t))\right)(x_{n+1}(t) - x_n(t))\mathrm{d}t \\ &\quad + \frac{\beta\rho(0)}{\alpha}(x_{n+1}(0) - x_n(0))^2.\end{aligned}$$

We have

$$\begin{aligned}&\int_0^{+\infty} \left[f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_{n-1})'(t))\right](x_{n+1}(t) - x_n(t))\mathrm{d}t \\ &\quad = \int_0^{+\infty} \big[f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_n)'(t)) + f(t, x_n^+(t), (x_n)'(t)) \\ &\quad\quad\quad - f(t, x_n^+(t), (x_{n-1})'(t))\big](x_{n+1}(t) - x_n(t))\mathrm{d}t \\ &\quad \le \int_0^{+\infty} |f(t, x_{n+1}^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_n)'(t))||x_{n+1}(t) - x_n(t)|\mathrm{d}t \\ &\quad\quad\quad + \int_0^{+\infty} |f(t, x_n^+(t), (x_n)'(t)) - f(t, x_n^+(t), (x_{n-1})'(t))||x_{n+1}(t) - x_n(t)|\mathrm{d}t.\end{aligned}$$

It follows from the assumption $(C5)$ that

$$\begin{aligned}
&\int_0^{+\infty}\left[f(t,x_{n+1}^+(t),(x_n)'(t))-f(t,x_n^+(t),(x_{n-1})'(t))\right](x_{n+1}(t)-x_n(t))\mathrm{d}t\\
&\leq \int_0^{+\infty}L_1(x_{n+1}(t)-x_n(t))^2\mathrm{d}t+L_2\int_0^{+\infty}|x_n'(t)-x_{n-1}'(t)||x_{n+1}(t)-x_n(t)|\mathrm{d}t.\\
&\leq \int_0^{+\infty}\frac{1}{s(t)}L_1 s(t)(x_{n+1}(t)-x_n(t))^2\mathrm{d}t\\
&\quad +L_2\int_0^{+\infty}\frac{1}{\sqrt{\rho(t)s(t)}}\sqrt{\rho(t)}|x_n'(t)-x_{n-1}'(t)|\sqrt{s(t)}|x_{n+1}(t)-x_n(t)|\mathrm{d}t.\\
&\leq \frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}\|x_{n+1}-x_n\|_X^2\\
&\quad +\frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\left(\int_0^{+\infty}\rho(t)|x_n'(t)-x_{n-1}'(t)|^2\mathrm{d}t\right)^{1/2}\left(\int_0^{+\infty}s(t)|x_{n+1}(t)-x_n(t)|^2\mathrm{d}t\right)^{1/2}.\\
&\leq \frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}\|x_{n+1}-x_n\|_X^2+\frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\|x_n-x_{n-1}\|_X\|x_{n+1}-x_n\|_X,
\end{aligned}$$

which gives

$$\begin{aligned}
\|x_{n+1}-x_n\|_X^2 \leq\ & \frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}\|x_{n+1}-x_n\|_X^2+\frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\|x_n-x_{n-1}\|_X\|x_{n+1}-x_n\|_X\\
&+\frac{\beta\rho(0)}{\alpha}(x_{n+1}(0)-x_n(0))^2.
\end{aligned}$$

$$\begin{aligned}
\left(1-\frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}\right)\|x_{n+1}-x_n\|_X^2 \leq\ & \frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\|x_n-x_{n-1}\|_X\|x_{n+1}-x_n\|_X\\
&+\frac{\beta\rho(0)}{\alpha}(x_{n+1}(0)-x_n(0))^2.
\end{aligned}$$

We have

$$|x_{n+1}(t_1)-x_n(t_1)|\leq\frac{1}{p(t)}p(t)|x_{n+1}(t_1)-x_n(t_1)|\leq C_1\|x_{n+1}-x_n\|_{\infty,p}\leq C_1M\|x_{n+1}-x_n\|_X,$$

where $C_1=\sup_{[0,+\infty)}\frac{1}{p}$. Then

$$\begin{aligned}
\left(1-\frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}\right)\|x_{n+1}-x_n\|_X^2 \leq\ & \frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\|x_n-x_{n-1}\|_X\|x_{n+1}-x_n\|_X\\
&+\frac{\beta\rho(0)C_1M}{\alpha}\|x_{n+1}-x_n\|_X^2.
\end{aligned}$$

That is,

$$\left(1-\frac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}-\frac{\beta\rho(0)C_1M}{\alpha}\right)\|x_{n+1}-x_n\|_X^2\leq\frac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}\|x_n-x_{n-1}\|_X\|x_{n+1}-x_n\|_X.$$

So

$$\|x_{n+1}-x_n\|_X\leq\frac{\dfrac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)}\sqrt{\rho s}}}{1-\dfrac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)}s}-\dfrac{\beta\rho(0)C_1M}{\alpha}}\|x_n-x_{n-1}\|_X.$$

Since $0 < \dfrac{\dfrac{L_2}{\operatorname{ess\,inf}_{[0,+\infty)} \sqrt{\rho s}}}{1 - \dfrac{L_1}{\operatorname{ess\,inf}_{[0,+\infty)} s} - \dfrac{\beta\rho(0)C_1 M}{\alpha}} < 1$, (x_n) is a Cauchy sequence in the reflexive Banach space X, so the sequence (x_n) converges strongly in X to some x which is a weak solution of Problem (2.1). Since the nonlinear term f is continuous, then a weak solution of our problem is a classical solution. □

References

1. R. P. Agarwal, Huei-Lin Hong and Cheh-Chih Yeh., *The existence of positive solutions for the Sturm-Liouville boundary value problems.* Comp. Math. Appl. 35, 89-96, (1998).
2. V. Anuradha, D. D. Hai and R. Shivaji., *Existence results for superlinear semi-positive BVP's*, Proc. Amer. Math. Soc. 124, 757-763, (1996).
3. D. Averna and G. Bonanno., *Three solutions for a quasilinear two point boundary value problem involving the one-dimensional p-Laplacian*, Proc. Edinburgh Math. Soc. 47, 257-270, (2004).
4. G. Bonanno and R. Livrea., *Multiplicity theorems for the Dirichlet problem involving the p-Laplacian*, Nonlinear Anal. 54, 1-7, (2003).
5. G. Bonanno., *Existence of three solutions for a two point boundary value problem*, Appl. Math. Letters. 13, 53-57, (2000).
6. Corduneanu, C., *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.
7. L. H. Erbe and H. Wang., *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. 120, 743-748, (1994).
8. D. D. Figueiredo, M. Girardi and M. Matzeu., *Semilinear elliptic equations with dependence on the gradient via mountain–pass techniques*, Differential and Integral Equations. Volume 17. Numbers 1-2, 119-126, (2004).
9. W. Ge and J. Ren., *New existence theorems of positive solutions for Sturm-Liouville boundary value problems*, Appl. Math. Comput. 148, 631-644, (2004).
10. Mawhin, J and Willem, M., *Critical point theory and Hamiltonian systems*, Springer-Verlag, Berlin, 1989.
11. Y. Tian and W. Ge., *Multiple positive solutions for a second order Sturm-Liouville boundary value problem with a p-Laplacian via variational methods*, Rocky Mountain J. Math. 39, 325-342, (2009).
12. Y. Tian and W. Ge., *Second-Order Sturm-Liouville boundary value problem involving the one-dimensional p-Laplacian*, Rocky Mountain J. Math. 38, 309-327, (2008).
13. Zeidler, E., *Nonlinear Functional Analysis and its Applications, Vol. III, Variational Methods and Applications*, Springer, New York, 1985.

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