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Lacunary $\mathcal{I}$-Convergent σ-Asymptotically Equivalent Difference Sequences of Fuzzy Real Numbers

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ABSTRACT: In this paper we have introduced the concept lacunary (Δ, σ) $\mathcal{I}$-asymptotically equivalent sequences of fuzzy real numbers in terms of Orlicz function, which is a natural combination of asymptotic equivalent, σ-convergence, difference sequence, $\mathcal{I}$-statistically limit, $\mathcal{I}$-statistically lacunary sequences and Orlicz function of fuzzy real numbers. Let, θ be a lacunary sequence. Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be lacunary (Δ, σ) $\mathcal{I}$-asymptotically equivalent of multiple L with respect to M, provided that for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left\{ \left| t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) \geq \varepsilon \right| \right\} \geq \delta \right\} \in \mathcal{I}$$

uniformly in $m = 1, 2, 3, ...$ and it is denoted by $X \overset{\mathcal{I}\left(S^{\sigma}_{L,\theta}(\Delta, M)\right)}{\sim} Y$.
We have established some relations between the classes of the sequences related to our study.

Key Words: Orlicz function; lacunary sequence; statistically convergent; asymptotically equivalent; fuzzy real number; Cesáro summable; Ideal convergence; difference sequence.

Contents

1. Introduction

In 1993, Marouf [17] introduced the concept of asymptotically equivalent sequences for real numbers. Later, Patterson [19] extended the concepts by introducing asymptotically statistical equivalent sequences. Nuray and Savaş [18] proposed statistically convergent and statistically Cauchy sequences for fuzzy numbers. Savaş [28] presented the natural combination of the notion of asymptotically equivalent and λ-statistical convergence of fuzzy numbers. Also, Savaş and Gumuş [25] generalized the concept of $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences. Dutta [4] showed some important results of asymptotically equivalent generalized difference sequence of fuzzy real numbers by introducing Orlicz function. Patterson and Savaş [20] showed lacunary sequences in asymptotically statistical equivalent sequences. Moreover, Savaş [22], Savaş and Başarir [24], Savaş and Patterson [27] and many more ([1], [10], [21], [26]) presented some new concept of σ-convergence in various sequence spaces. In recent time, different researchers ([3], [5], [6], [7], [11], [12], [29], [30], [31], [32], [33], [34], [35], [36], [23], [9], [2]) contributed in the field and established some important results. Lindberg [14], Lindenstrauss and Tzafriri [15] studied the idea of Orlicz sequence spaces and established some relations in Banach space theory.

A continuous, non-decreasing and convex mapping $M : [0, \infty) \to [0, \infty)$ with the conditions: $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, for $x \to \infty$ is known as Orlicz function. This Orlicz function M is called the modulus function, if the convexity is replaced by $M(x + y) \geq M(x) + M(y)$.
Lindenstrauss and Tzafriri [15] defined the Orlicz sequence ℓ_M space as follows:
$\ell_M = \left\{ x \in w : \sum_{t=1}^{\infty} M\left(\frac{|x_t|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$, where w denotes the class of all real and complex

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sequences $x=(x_k)$.
Remark 1.1: For any Orlicz function, the inequality $M(\lambda x) \leq \lambda M(x), \forall \lambda$ with $0<\lambda<1$ holds.
A non-empty family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an *ideal* if it satisfies the conditions: (i) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) for each $A \in \mathcal{I}$ and for each $B \subset A$ imply $B \in \mathcal{I}$. A non-empty family of sets $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a *filter* of $\mathbb{N}$ if it satisfies the following conditions: (i) $\phi \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and (iii) for each $A \in \mathcal{F}$, and for each $B \cup A$ imply $B \in \mathcal{F}$. An ideal I is said to be a non-trivial ideal if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I} \neq \phi$. It is clear that I is a non-trivial ideal if and only if $\mathcal{F}(\mathcal{I})=\{\mathbb{N}-B : B \in \mathcal{I}\}$ is a *filter* on $\mathbb{N}$.
Kizmaz [13] introduced the idea of difference sequence space $X(\Delta)$ as follows:
$X(\Delta)=\{x=(x_t) \in w : (\Delta x_t) \in X\}$, for $X=c_0, c$ and ℓ_∞; where $\Delta(x_t)=x_t-x_{t+1}$, for all $t \in \mathbb{N}$. Later, Et and Çolak [8] generalized this as follows:

$$X(\Delta^p)=\{x=(x_t) \in w : (\Delta^p x_t) \in X\}.$$

The generalized difference operator has the following binomial representation:

$$\Delta^p x_t=\sum_{n=0}^{p}(-1)^n \binom{p}{n} x_{n+t}$$

Let, σ be a one-to-one mapping from the set of natural numbers to itself such that $\sigma^t(m)=\sigma\left(\sigma^{t-1}(m)\right)$, $t=1,2,3,\ldots$. A continuous linear functional φ on ℓ_∞ is said to be an σ-mean or an invariant mean if and only if
(i) $\varphi(x) \geq 0$ when $x_t \geq 0$ for all t where $x=(x_t)$
(ii) $\varphi(e)=1$ where $e=(1,1,1,\ldots)$ and
(iii) $\varphi(x_{\sigma(m)})=\varphi(x)$ for all $x \in \ell_\infty$.
Lorentz [15] already showed that the σ-mean is the Banach limit when we consider $\sigma(k)=k+1$ and V_σ is the set of almost convergent sequences.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, more precisely a mapping $X : \mathbb{R} \rightarrow I(=[0,1])$ associating each real number t, with its grade of membership $X(t)$, which satisfy the following properties:
(i) X is *normal* i.e. if there exists $t_0 \in \mathbb{R}$ such that $X(t_0)=1$.
(ii) X is *upper semi-continuous* i.e. if for each $\varepsilon>0$ and for all $a \in I, X^{-1}([0, a+\varepsilon))$, is open in the usual topology of $\mathbb{R}$.
(iii) X is *convex* i.e. if $X(t) \geq X(s) \wedge X(r)=\min(X(s), X(r))$, where $s<t<r$.
(iv) The closure of $\{t \in \mathbb{R} : X(t)>0\}$ is compact.
The class of all *upper semi-continuous, normal, convex* fuzzy real numbers is denoted by $\mathbb{R}(I)$.
The absolute value of $X \in \mathbb{R}(I)$ is defined by

$$|X|(t)=\begin{cases}\max\{X(t), X(-t)\} & \text{for } t \geq 0; \\ 0 & \text{otherwise.}\end{cases}$$

The set $\mathbb{R}$ of all real numbers can be embedded in $\mathbb{R}(I)$. For $r \in \mathbb{R}, \bar{r} \in \mathbb{R}(I)$ is defined by

$$\bar{r}(t)=\begin{cases}1 & \text{for} \quad t=r; \\ 0 & \text{for} \quad t \neq r.\end{cases}$$

We denote the additive identity and multiplicative identity of $\mathbb{R}(I)$ by $\bar{0}$ and $\bar{1}$ respectively.

For any $X, Y, Z \in \mathbb{R}(I)$, the linear structure of $\mathbb{R}(I)$ induces addition $X+Y$ and scalar multiplication $\lambda X, \lambda \in \mathbb{R}$ in terms of α-level set, defined as $[X+Y]^\alpha=[X]^\alpha+[Y]^\alpha$ and $[\lambda X]^\alpha=\lambda[X]^\alpha$, for each $\alpha \in [0,1]$. A subset E of $\mathbb{R}(I)$ is said to be *bounded above* if there exist a fuzzy real number μ such that $X \leq \mu$ for every $X \in E$. We called μ as the upper bound of E and it is called *least upper bound* if $\mu \leq \mu^*$ for all upper bound μ^* of E. A lower bound and greatest lower bound can be defined similarly. The set E is said to be *bounded* if it is both bounded above and bounded below.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$ and $Y = [Y^L, Y^R]$. We define a metric on D by

$$d(X,Y) = \max\left\{|X^L - Y^L|, |X^R - Y^R|\right\}.$$

It is straight forward that (D, d) is a complete metric space.
Define, $\bar{d} : \mathbb{R}(I) \times \mathbb{R}(I) \to \mathbb{R}$ by

$$\bar{d}(X,Y) = \sup_{0\le\alpha\le1} d(X^\alpha, Y^\alpha) \text{ for } X, Y \in \mathbb{R}(I).$$

It is well established that $\left(\mathbb{R}(I), \bar{d}\right)$ is a complete metric space.
A sequence $X = (X_t)$ of fuzzy real number is a function X from the set of natural number into $\mathbb{R}(I)$, where X_t is the t^{th} term of the fuzzy sequence.

Let E^F be the class of sequence of fuzzy real numbers, the linearity of E^F can be understand as follows:
For $(X_t), (Y_t) \in E^F$, $p \in \mathbb{R}$ and for all $t \in \mathbb{N}$,
(i) $(X_t) + (Y_t) = (X_t + Y_t) \in E^F$ and
(ii) $p(X_t) = (pX_t) \in E^F$, where

$$p(X_t)(k) = \begin{cases} X_t(p^{-1}k) & \text{if } p \neq 0; \\ \bar{0} & \text{if } p = 0. \end{cases}$$

2. Definitions and Preliminaries

In this section we mention some definitions related to the topic.

Definition 2.1 *A sequence (X_k) of fuzzy real numbers is said to be convergent to $X_0 \in \mathbb{R}(I)$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \ge n_0$.*

Definition 2.2 *A lacunary sequence is an increasing sequence $\theta = (k_r)$ of positive integer such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$ with $k_0 = 0$. The interval determined by θ is given by $I_r = (k_{r-1}, k_r]$ and the ratio $\dfrac{k_r}{k_{r-1}}$ is denoted by q_r.*

Definition 2.3 *A sequence (X_t) of fuzzy real numbers is said to be statistically convergent to a fuzzy real number X_0 if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{t \le n : \bar{d}(X_t, X_0) \ge \epsilon\}\right| = 0$$

(denoted by $st - \lim X = X_0$).

Definition 2.4 *A sequence (X_t) of fuzzy real numbers is said to be $\mathcal{I}$-convergent to a fuzzy real number X_0 if for every $\varepsilon > 0$,*

$$\{t \in \mathbb{N} : |X_t - X_0| \ge \varepsilon\} \in \mathcal{I}.$$

Definition 2.5 *Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be asymptotically equivalent if*

$$\lim_{t\to\infty} \bar{d}\left(\frac{X_t}{Y_t}, \bar{1}\right) = 0$$

It is denoted by $X \sim Y$.

Now, we give the following definitions in connection to our results.

Definition 2.6 *Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be (Δ, σ)-statistical asymptotically equivalent of multiple L with respect to M if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ t \le n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) \ge \varepsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, ...$(denoted by $X \overset{S_L^\sigma(\Delta)}{\sim} Y$).

Definition 2.7 *Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be (Δ, σ)-strong Cesáro asymptotically equivalent of multiple L with respect to M if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) = 0$$

uniformly in $m = 1, 2, 3, ...$ (denoted by$X \overset{C_L^\sigma(\Delta)}{\sim} Y$).

Definition 2.8 *Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be lacunary (Δ, σ) $\mathcal{I}$-asymptotically equivalent of multiple L with respect to M if for every $\varepsilon > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left\{ \left| t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) \ge \varepsilon \right| \right\} \ge \delta \right\} \in \mathcal{I}$$

uniformly in $m = 1, 2, 3, ...$ (denoted by $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta, M)\right)}{\sim} Y$).

Definition 2.9 *Two sequences $X = (X_t)$ and $Y = (Y_t)$ of fuzzy real numbers are said to be lacunary (Δ, σ) strong $\mathcal{I}$-asymptotically equivalent of multiple L with respect to M if for every $\varepsilon > 0$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left\{ \sum_{t \in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) \right\} \ge \varepsilon \right\} \in \mathcal{I}$$

uniformly in $m = 1, 2, 3, ...$ (denoted by $X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y$).

3. Main Results

Theorem 3.1 *Let, $\theta = (k_r)$ be a lacunary sequence, then*

(a) *If $X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$, then $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y$,*

(b) *If $X, Y \in l_\infty(\Delta, \sigma)$ and $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y$, then $X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$.*

Proof: (a) If $\lim_{r\to\infty} \inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \ge 1 + \delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \ge \dfrac{\delta}{1+\delta}$.

Let, $X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$. Therefore, for any $\alpha > 0$, we have

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{t \in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \ge \alpha \right\} \in \mathcal{I}.$

Now, for a given $\varepsilon > 0$

$$\sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \sum_{t\in I_r \ \& \ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)$$

$$\geq \varepsilon \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right|$$

$$\Rightarrow \frac{1}{\varepsilon h_r} \sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \frac{1}{h_r} \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right|$$

Then for any $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \alpha\right\} \in \mathcal{I}, \text{ where } \alpha = \varepsilon\delta.$$

Therefore, $X \overset{\mathcal{I}\left(S^{\sigma}_{L,\theta}(\Delta)\right)}{\sim} Y$. This complete the proof.

(b) If $\lim_{r\to\infty} \inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1+\delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \geq \dfrac{\delta}{1+\delta}$.

Suppose, $X, Y \in l_\infty(\Delta, \sigma)$ and $X \overset{\mathcal{I}\left(S^{\sigma}_{L,\theta}(\Delta)\right)}{\sim} Y$. Therefore, for any $\gamma, \varepsilon > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}.$$

Then, $\exists$ an integer P such that $\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \leq P$, for all t.

Now, for some $\varepsilon > 0$,

$$\sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)$$

$$= \sum_{t\in I_r \ \& \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) + \sum_{t\in I_r \ \& \ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)<\varepsilon} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)$$

$$\leq P \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| + \varepsilon$$

Then for any $\delta > \varepsilon > 0$ (ε and δ are independent),

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{t \in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I},$$

$\left(\text{where } \gamma = \frac{\alpha}{P} \ \& \ \alpha = (\delta - \varepsilon) > 0\right)$.

Thus, $X \overset{\mathcal{I}(N^{\sigma}_{L}(\Delta))}{\sim} Y$. This complete the proof. □

Theorem 3.2 *Let, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence with $\lim_{r\to\infty} \inf q_r > 1$, then*

(a) $X \overset{\mathcal{I}(S^{\sigma}_{L}(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}\left(S^{\sigma}_{L,\theta}(\Delta,M)\right)}{\sim} Y$,

(b) $X \overset{\mathcal{I}(C^{\sigma}_{L}(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(N^{\sigma}_{L}(\Delta,M))}{\sim} Y$.

Proof: (a) If $\lim_{r\to\infty}\inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1+\delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \geq \dfrac{\delta}{1+\delta}$.

Let, $X \overset{\mathcal{I}(S_L^{\sigma}(\Delta,M))}{\sim} Y$. Then for any $\alpha > 0$ and $\varepsilon > 0$ we can write

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \geq \alpha\right\} \in \mathcal{I}.$$

Now, for any $\varepsilon > 0$ and for $k_{r-1} < n \leq k_r$ we have

$$\frac{1}{n}\left|\left\{t \leq n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{1}{k_r}\left|\left\{t \leq k_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{1}{k_r}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right|$$
$$\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right|$$

Then for any $\alpha > 0$ and for sufficiently large r, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \geq \gamma\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \geq \alpha\right\} \in \mathcal{I}, \text{ where } \alpha = \frac{\gamma\delta}{1+\delta}.$$

This implies, $X \overset{\mathcal{I}\left(S_{L,\theta}^{\sigma}(\Delta,M)\right)}{\sim} Y$. This complete the proof.

(b) If $\lim_{r\to\infty}\inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1+\delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \geq \dfrac{\delta}{1+\delta}$.

Let, $X \overset{\mathcal{I}(C_L^{\sigma}(\Delta,M))}{\sim} Y$, then for a given $\varepsilon > 0$ we can write

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Now, for $k_{r-1} < n \leq k_r$ we get

$$\frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$\geq \frac{1}{k_r}\sum_{t=1}^{k_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$\geq \frac{1}{k_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$= \frac{h_r}{k_r}\frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$

Then for any $\alpha > 0$ and for sufficiently large value of r, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \alpha\right\}$$

$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\} \in \mathcal{I}$, where $\varepsilon = \frac{\alpha\delta}{1+\delta}$.

This implies, $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y$. This complete the proof.

To show the next result, we assume that the *lacunary* sequence satisfies the condition that for any set $A \in \mathcal{F}(\mathcal{I}), \cup\{n : k_{r-1} < n < k_r, r \in A\} \in \mathcal{F}(\mathcal{I})$. □

Theorem 3.3 *Let, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence with $\lim_{r\to\infty} \sup q_r < \infty$, then*

(a) $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta,M)\right)}{\sim} Y \Rightarrow X \overset{\mathcal{I}(S_L^\sigma(\Delta,M))}{\sim} Y$,

(b) $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(C_L^\sigma(\Delta,M))}{\sim} Y$.

Proof: **(a)** If $\lim_{r\to\infty} \sup q_r < \infty$, then $\exists\, Q > 0$ such that $q_r < Q, \forall r \geq 1$.

Let, $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta,M)\right)}{\sim} Y$. For any α, α_1 and $\varepsilon > 0$ we define the following sets,

$A = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| < \alpha\right\}$ and

$B = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| < \alpha_1\right\}$

We observe that $A \in \mathcal{F}(\mathcal{I})$, the filter of $\mathcal{I}$. Also for some $\alpha_2 > 0$ we take,

$$C_p = \frac{1}{h_p}\left|\left\{t \in I_p : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| < \alpha_2, \forall p \in A$$

Now, let n be any integer with $k_{r-1} \leq n \leq k_r$. Then for some $r \in A$ we have,

$$\begin{aligned}
&\frac{1}{n}\left|\left\{t \leq n : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1}}\left|\left\{t \leq k_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\left|\left\{t \in I_1 : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{t \in I_2 : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \\
&+ \frac{1}{k_{r-1}}\left|\left\{t \in I_3 : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| + ... + \frac{1}{k_{r-1}}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_1 k_{r-1}}\left|\left\{t \in I_1 : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| + \frac{k_2 - k_1}{(k_2 - k_1)k_{r-1}}\left|\left\{t \in I_2 : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| + \\
&... + \frac{k_r - k_{r-1}}{(k_r - k_{r-1})k_{r-1}}\left|\left\{t \in I_r : M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_{r-1}}C_1 + \frac{k_2 - k_1}{k_{r-1}}C_2 + ... + \frac{k_r - k_{r-1}}{k_{r-1}}C_r \\
&\leq \{\sup_{p\in A} C_p\}\frac{k_r}{k_{r-1}} \\
&< Q\alpha
\end{aligned}$$

Now, taking $\alpha_1 = Q\alpha$ and as the fact we have $\cup\{n : k_{r-1} < n < k_r, r \in A\} \subset B$, where $A \in \mathcal{F}(\mathcal{I})$ it shows that the set $B \in \mathcal{F}(\mathcal{I})$. This complete the proof.

(b) If $\lim\limits_{r\to\infty}\sup q_r < \infty$, then $\exists Q > 0$ such that $q_r < Q, \forall r \geq 1$.

Let, $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y$. For any $\alpha, \alpha_1, \varepsilon > 0$ we define the following sets,

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) < \alpha \right\} \text{ and}$$
$$B = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) < \alpha_1 \right\}.$$

We observe that $A \in \mathcal{F}(\mathcal{I})$, the filter of $\mathcal{I}$. Also for some $\alpha_2 > 0$, we take
$C_p = \frac{1}{h_p} \sum_{t\in I_p} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) < \alpha_2, \forall p \in A$

Now, let n be any integer with $k_{r-1} \leq n \leq k_r$. Then for some $r \in A$ we have,

$$\frac{1}{n} \sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$\leq \frac{1}{k_{r-1}} \sum_{t=1}^{k_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$= \frac{1}{k_{r-1}} \sum_{t\in I_1} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + \frac{1}{k_{r-1}} \sum_{t\in I_2} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + \frac{1}{k_{r-1}} \sum_{t\in I_3} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + ...$$
$$+ \frac{1}{k_{r-1}} \sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$= \frac{k_1}{k_1 k_{r-1}} \sum_{t\in I_1} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + \frac{k_2 - k_1}{(k_2 - k_1)k_{r-1}} \sum_{t\in I_2} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + ...$$
$$+ \frac{k_r - k_{r-1}}{(k_r - k_{r-1})k_{r-1}} \sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$= \frac{k_1}{k_{r-1}} C_1 + \frac{k_2 - k_1}{k_{r-1}} C_2 + ... + \frac{k_r - k_{r-1}}{k_{r-1}} C_r$$
$$\leq \{\sup_{p\in A} C_p\} \frac{k_r}{k_{r-1}}$$
$$< Q\alpha$$

Now, taking $\alpha_1 = Q\alpha$ and as the fact we have $\cup\{n : k_{r-1} < n < k_r, r \in A\} \subset B$, where $A \in \mathcal{F}(\mathcal{I})$, it shows that the set $B \in \mathcal{F}(\mathcal{I})$. This complete the proof. □

Theorem 3.4 *Let, M be an Orlicz function.*

(a) $X \overset{\mathcal{I}(C_L^\sigma(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(S_L^\sigma(\Delta))}{\sim} Y$.

(b) *If M is bounded, then* $X \overset{\mathcal{I}(S_L^\sigma(\Delta))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(C_L^\sigma(\Delta,M))}{\sim} Y$.

Proof: (a) Suppose, $X \overset{\mathcal{I}(C_L^\sigma(\Delta,M))}{\sim} Y$, then for any $\alpha > 0$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \alpha \right\} \in \mathcal{I}.$$

We consider $\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon$, for a given $\varepsilon > 0$.
Thus we have,

$$\frac{1}{n} \sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$

$$\geq \frac{1}{n} \sum_{t\leq n \;\&\; \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$

$$\geq M(\varepsilon)\frac{1}{n}\left|\left\{t \leq n : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right|$$

Now, for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \alpha\right\} \in \mathcal{I}, \text{ where } \alpha = M(\varepsilon)\delta > 0.$$

Therefore, $X \stackrel{\mathcal{I}(S_L^\sigma(\Delta))}{\sim} Y$. This complete the proof.

(b) Suppose, M is bounded and $X \stackrel{\mathcal{I}(S_L^\sigma(\Delta))}{\sim} Y$. Then for any $\delta_1 > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

We consider $\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon$ for a given $\varepsilon > 0$. Thus we have,

$$\frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$

$$= \frac{1}{n}\sum_{t=1 \;\&\; \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + \frac{1}{n}\sum_{t=1 \;\&\; \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)<\varepsilon}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$

$$\leq \sup M(n)\frac{1}{n}\left|\{t \leq n : \bar{d}(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L) \geq \varepsilon\}\right| + M(\varepsilon)$$

Now, for any $\delta > 0$ we have,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{t=1}^{n} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{t \leq n : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\}\right| \geq \delta_1\right\} \in \mathcal{I}, \left(\text{taking } \delta_1 = \frac{\delta - M(\varepsilon)}{\sup M(\varepsilon)}\right).$$

Therefore, $X \stackrel{\mathcal{I}(C_L^\sigma(\Delta,M))}{\sim} Y$. This complete the proof. □

Lemma 3.1 *Let, M be an Orlicz function and we consider $0 < \delta < 1$. Then for $y \neq 0$ and each $\left(\frac{x}{y}\right) > \delta$, we have $M\left(\frac{x}{y}\right) \leq 2M(1)\delta^{-1}\left(\frac{x}{y}\right)$.*

Theorem 3.5 *Let, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence. Then $X \stackrel{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y \Rightarrow X \stackrel{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y$.*

Proof: If $\lim_{r\to\infty} \inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1 + \delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

Let, $X \stackrel{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$, where $X = (X_t), Y = (Y_t) \in w_F$. Then for any $\varepsilon > 0$ we have,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

For any $\alpha, \beta, \gamma > 0$, we define the sets,

$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{t \in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) < \alpha \right\}$ and

$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{t \in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) < \beta \right\}$.

Clearly, $A \in \mathcal{F}(\mathcal{I})$, the filter of $\mathcal{I}$.

Now, for a given $\varepsilon_1 > 0$, we choose $0 < \delta < 1$ such that $M(p) < \varepsilon_1$, for $0 \leq p \leq \delta$.

Let, r be any integer such that $r \in A$. By using the Lemma 3.8 we have,

$$\frac{1}{h_r} \sum_{t \in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right)$$
$$= \frac{1}{h_r} \sum_{t \in I_r \ \&\ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \leq \delta} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) + \frac{1}{h_r} \sum_{t \in I_r \ \&\ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) > \delta} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right)$$
$$\leq \frac{1}{h_r}(h_r \varepsilon_1) + \frac{1}{h_r} 2M(1)\delta^{-1} \sum_{t \in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right)$$
$$\leq \varepsilon_1 + \frac{1}{h_r} 2M(1)\delta^{-1} h_r \varepsilon$$
$$\leq \varepsilon_1 + 2M(1)\delta^{-1}\varepsilon$$

Taking $\beta > \left(\varepsilon_1 + \frac{1}{h_r} 2M(1)\delta^{-1} h_r \varepsilon \right) > 0$ and as the fact that $\cup\{r : r \in A\} \subset B$, it shows that the set $B \in \mathcal{F}(\mathcal{I})$.

Therefore, $X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y$. This complete the proof. □

Theorem 3.6 *Let, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence. Then,*

$X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y \Leftrightarrow X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$, *provided* $\lim_{t \to \infty} \frac{M(t)}{t} = \gamma > 0$.

Proof: If $\lim_{r \to \infty} \inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1 + \delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}$.
We already proved that $X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y$.
Now we show that,

$$X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y \text{ if } \lim_{t \to \infty} \frac{M(t)}{t} = \gamma > 0.$$

Let, $X \overset{\mathcal{I}(N_L^\sigma(\Delta, M))}{\sim} Y$, where $X = (X_t)$ and $Y = (Y_t)$.

Let $\gamma > 0$ such that $M(t) \geq \gamma t, \forall t \geq 0$. Therefore, for any $\delta > 0$ we have,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{t \in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L \right) \right) > \delta \right\} \in \mathcal{I}.$$

Now, we have

$$\frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right) \geq \frac{1}{h_r}\sum_{t\in I_r} \gamma\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right)$$
$$= \frac{\gamma}{h_r}\sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)$$

Then for any $\varepsilon > 0$,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right) > \varepsilon\right\}$$
$$\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right) > \gamma\varepsilon\right\} \in \mathcal{I}, \text{ where } \delta = \gamma\varepsilon.$$

Therefore, $X \overset{\mathcal{I}(N_L^\sigma(\Delta))}{\sim} Y$. This complete the proof. □

Theorem 3.7 *Let, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence.*

(a) $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y.$

(b) *If M is bounded, then* $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y \Rightarrow X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y.$

Proof: (a) If $\lim_{r\to\infty}\inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1+\delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \geq \dfrac{\delta}{1+\delta}$.

Let, $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y$. Therefore, for any $\gamma > 0$ we have,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right) > \gamma\right\} \in \mathcal{I}.$$

Now for some $\varepsilon > 0$,

$$\frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right) \geq \frac{1}{h_r}\sum_{t\in I_r\ \&\ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\geq\varepsilon} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right)$$
$$\geq M(\varepsilon)\frac{1}{h_r}\left|\left\{t\in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right) \geq \varepsilon\right\}\right|$$

Then for any $\delta > 0$,

$$\left\{r\in\mathbb{N} : \left|\left\{t\in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right) \geq \varepsilon\right\}\right| > \delta\right\}$$
$$\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right)\right) > \gamma\right\} \in \mathcal{I}, \text{ where } \gamma = M(\varepsilon)\delta.$$

Therefore, $X \overset{\mathcal{I}(N_L^\sigma(\Delta,M))}{\sim} Y \Rightarrow X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y$. This complete the proof.

(b) If $\lim_{r\to\infty}\inf q_r > 1$, then $\exists\ \delta > 0$ such that $q_r \geq 1+\delta$, for sufficiently large r. We have, $h_r = (k_r - k_{r-1})$ which implies, $\dfrac{h_r}{k_r} \geq \dfrac{\delta}{1+\delta}$.

Let, $X \overset{\mathcal{I}\left(S_{L,\theta}^\sigma(\Delta)\right)}{\sim} Y$. This implies for any $\delta_1 > 0$ and $\varepsilon > 0$

$$\left\{r\in\mathbb{N} : \left|\left\{t\in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}},L\right) \geq \varepsilon\right\}\right| > \delta_1\right\} \in \mathcal{I}.$$

Suppose, M is bounded. Then we have

$$\frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$= \frac{1}{h_r}\sum_{t\in I_r \ \& \ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) + \frac{1}{h_r}\sum_{t\in I_r \ \& \ \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)<\varepsilon} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right)$$
$$\leq \sup M(I_r)\frac{1}{h_r}\left|\left\{t\in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon\right\}\right| + M(\varepsilon)$$

Then for any $\delta > M(\varepsilon) > 0$, we have

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{t\in I_r} M\left(\bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\right) > \delta\right\}$$
$$\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{t\in I_r : \bar{d}\left(\frac{\Delta X_{\sigma^t(m)}}{\Delta Y_{\sigma^t(m)}}, L\right)\geq\varepsilon\right\}\right| > \delta_1\right\} \in \mathcal{I}, \text{ where } \delta_1 = \frac{\delta - M(\varepsilon)}{\sup M(I_r)}.$$

Therefore, $X \overset{\mathcal{I}\left(S^{\sigma}_{L,\theta}(\Delta)\right)}{\sim} Y \Rightarrow X \overset{\mathcal{I}(N^{\sigma}_{L}(\Delta,M))}{\sim} Y$. This complete the proof. □

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