Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.67413

A note on new unified fractional derivative *

Pinakadhar Baliarsingh

ABSTRACT: Recently, a new and unified definition of fractional derivative of the Caputo type is introduced Zheng et al.(Int. J. Non-Linear Mechanics, 2019). Indeed, the proposed formula is very interesting and unified one as it generalizes the notion of fractional derivative of Caputo type in global sense adopting full memory effect and also is capable to capture the short memory effect through local fractional derivatives. Some basic properties of the proposed fractional differential operator such as linearity, backward compatibility, identity, consistency, semigroup, etc. are discussed in detail. Integral transforms such as Fourier transform and Laplace transform of the proposed derivatives are also determined. Upon further investigation, it is found that some of the properties of this operator lack validity and consistency. The prime objective of this note is to point out these properties and study deeply by comparing the results of fractional derivatives of the Caputo type.

Key Words: Fractional derivatives, Caputo fractional derivatives, fractional dynamics.

Contents

1. Introduction

Nowadays, fractional derivatives (FD) and their applications are frequently arisen in different dynamic models which are directly associated most of physical world problems. In this context, varieties of definitions of fractional derivatives and integrals have been developed. amongst all, definitions given by Riemann-Liouville, Caputo, Grunwald and Letnikov etc, are very ancient and popular. Then their subsequent modifications and extensions have been appeared in order to tackle specific dynamic problems in modeling many diffusion, physical, electrochemical, biological processes, etc. For basic formulas on fractional derivatives and properties the reader may refer to the articles [1,2,3,4,5,6,8,9].

Now, we present the basic definitions of FD of order β [7]. Let β be a positive real number such that $n-1 \le \beta < n, n \in \mathbb{N}$ (the set of all natural numbers).

(i) Riemann-Liouville FD

$$({}_a^{RL}D_t^{\beta} f)(t) = \frac{1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-x)^{n-\beta-1} f(x)\mathrm{d}x \ \ (t>a).$$

(ii) Caputo fractional derivative FD

$$({}_a^{C}D_t^{\beta})(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-x)^{n-\beta-1} f^{(n)}(x)\mathrm{d}x \ \ (t>a).$$

(iii) Grunwald-Letnikov FD

$$(D_t^{\beta} f)(t) = \lim_{h\to 0} \frac{1}{h^{\beta}} \sum_{j=0}^{\left[\frac{t-\beta}{h}\right]} (-1)^j \binom{\beta}{j} f(t-jh).$$

* This work is partially supported by NBHM, DAE, Mumbai, under the Grant No. 02011/7/2020/NBHM.
Submitted March 07, 2023. Published March 18, 2025
2010 *Mathematics Subject Classification:* 26A33; 33C60.

where $\Gamma(.)$ is the gamma function, $f^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)$, and $[\beta]$ is the integral part of β. Further, a development of fractional derivative of Caputo and Riemann-Liouville, Zheng et al. [10] introduced a new unified definition of fractional derivative. Suppose $n-1 \leq \beta < n$, $f(t)$ is continuous and its $n-$order successive derivatives are continuous. Then, due to Zheng et al. [10], we have

$$D_{l,t}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_{t-l(t,\beta)}^{t} (t-x)^{n-\beta-1} f^{(n)}(x) \mathrm{d}x, \tag{1.1}$$

and

$$D_{l,t}^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_{t-l(t,\beta)}^{t} (t-x)^{\beta-1} f(x) \mathrm{d}x, \tag{1.2}$$

where $l(t,\beta)$, is the changeable interval of variable time t and the order β. In fact, this definition is being used for local memory effect of the real physical process, whereas Riemann-Liouville, Caputo and Grunwald-Letnikov fractional derivatives are used for capturing the global for global memory effect.

Now refereing [10], we mention some general properties of the fractional differential operators defined in (1.1) and (1.2).

- **Linearity:** Let $f(t), g(t)$ be two differentiable functions, $\lambda_1, \lambda_2 \in \mathbb{R}$ and β is a fraction. Then

$$D_{l,t}^{\beta}\left[\lambda_1 f(t) + \lambda_2 g(t)\right] = \lambda_1 D_{l,t}^{\beta} f(t) + \lambda_2 D_{l,t}^{\beta} g(t),$$

- **Backward compatibility:** If β is an integer, then above-defined fractional derivative provides same results as the ordinary derivatives, i.e.,

$$D_{l,t}^{\beta} f(t) = \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} f(t).$$

- **Identity:** For $\beta = 0$,

$$D_{l,t}^{\beta} f(t) = f(t).$$

- **Consistency:** If $\beta \to n$, then

$$\lim_{\beta \to n} D_{l,t}^{\beta} f(t) = f^{(n)}(t).$$

  Note that the fractional derivative defined in (1.1) is of Caputo type, hence it vanishes for a constant function.

- **Semigroup property:**
  (i) $D^m D_{l,t}^{\beta} f(t) = D_{l,t}^{\beta} D^m f(t) = D_{l,t}^{m+\beta} f(t)$, for all $m \in \mathbb{N}$;
  (ii) $D_{l,t}^{-\alpha} D_{l,t}^{-\beta} f(t) = D_{l,t}^{-(\alpha+\beta)} f(t)$, for $\alpha > 0, \beta > 0$;
  (iii) $D_{l,t}^{\alpha} D_{l,t}^{\beta} f(t) = D_{l,t}^{\alpha+\beta} f(t)$, for $\lfloor \alpha + \beta \rfloor = \lfloor \alpha \rfloor + \lfloor \beta \rfloor$, where $\lfloor \alpha \rfloor$ is the largest integer less than equal to α.

- **Fourier transform:** Suppose $f_l(t) = \begin{cases} f(t), & t \in [t-l(t,\beta), t] \\ 0, & \text{otherwise} \end{cases}$ and $\mathscr{F}_l(w)$ denotes the Fourier transform of $f_l(t)$. Then the Fourier transform of $D_{l,t}^{\beta} f(t)$ is defined by

$$\mathscr{F}\{D_{l,t}^{\beta} f(t)\} = (iw)^{\beta} \mathscr{F}_l(w).$$

7 **Laplace transform:** Suppose $\mathscr{L}_l(s)$ denotes the Laplace transform of $f_l(t)$. Then the Laplace transform of $D_{l,t}^{\beta} f(t)$ is defined by

$$\mathscr{L}\{D_{l,t}^{\beta} f(t)\} = s^{\beta} \mathscr{L}_l(s) - \sum_{k=0}^{n-1} f^{(k)}(0).$$

8 **Relation:**

$$D^{\beta}_{l,t}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t-l(t,\beta)}^{t}(t-x)^{n-\beta-1}f(x)\mathrm{d}x - \sum_{k=0}^{n-1}\frac{l^{k-\beta}}{\Gamma(k-\beta+1)}f^{(k)}(t-l).$$

2. Remarks and examples

In the current section, we investigate the validity of some of the properties of new FD given by [10]. Now, we quote the following result due to [7] which are useful for our main work.

Lemma 2.1 If $\beta \in \mathbb{R}^+$(set of all positive reals), $f(t) = (t-a)^\eta$ for $\eta \in \mathbb{R}\setminus Z^-$(set of all negative integers) and $t > a$, then

$${}^{RL}_{a}D^{\beta}_{t}f(t) = {}^{C}_{a}D^{\beta}_{t}f(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\beta)}(t-a)^{\eta-\beta}.$$

In particular, if $f(t) = \lambda$ for all t, and $0 < \beta < 1$, then

$${}^{C}_{0}D^{\beta}_{t}f(t) = D^{\beta}_{l,t}f(t) = 0,$$

and

$${}^{RL}_{0}D^{\beta}_{t}f(t) = \frac{\lambda t^{-\beta}}{\Gamma(1-\beta)}.$$

Remark 2.1 Let $0 < \beta < 1$ and $f(t) = t^\eta$ for $\eta \in \mathbb{R}\setminus Z^-$. Then the formula

$$D^{\beta}_{l,t}f(t) = t^{\eta-\beta}\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\beta)},$$

holds only if $[t - l(t,\beta)] = 0$.

Proof: Suppose $f(t) = t^\eta$ for $\eta \in \mathbb{R}\setminus Z^-$. Since $0 < \beta < 1$, the choice of n could be taken as $n = 1$. Using the new definition of fractional derivative indicated in (1.1), we have

$$\begin{aligned} D^{\beta}_{l,t}f(t) &= \frac{1}{\Gamma(1-\beta)}\int_{t-l(t,\beta)}^{t}(t-x)^{1-\beta-1}\eta x^{\eta-1}\mathrm{d}x \\ &= \frac{\eta}{\Gamma(1-\beta)}\int_{t-l(t,\beta)}^{t}(t-x)^{-\beta}x^{\eta-1}\mathrm{d}x \\ &= \frac{\eta t^{\eta-\beta-1}}{\Gamma(1-\beta)}\int_{t-l(t,\beta)}^{t}\left(1-\frac{x}{t}\right)^{-\beta}\left(\frac{x}{t}\right)^{\eta-1}\mathrm{d}x \\ &= \frac{\eta t^{\eta-\beta}}{\Gamma(1-\beta)}\int_{\frac{t-l(t,\beta)}{t}}^{1}(1-z)^{-\beta}z^{\eta-1}\mathrm{d}z, \quad \left(\text{set } z = \frac{x}{t}\right). \end{aligned}$$

From the above calculation it is concluded that, if $\frac{t-l(t,\beta)}{t} = 0$, then only the following formula holds.

$$D^{\beta}_{l,t}(t^\eta) = t^{\eta-\beta}\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\beta)}.$$

This completes the proof. □

Now, we have the following counter example.

Example 2.1 *Consider the function $f(t) = t$ and $0 < \beta < 1$. Then*

$$\begin{aligned} D^{\beta}_{l,t} f(t) &= \frac{1}{\Gamma(1-\beta)} \int_{t-l(t,\beta)}^{t} (t-x)^{-\beta} \mathrm{d}x \\ &= \frac{1}{\Gamma(1-\beta)} \left[\frac{-(t-x)^{-\beta+1}}{(1-\beta)} \right] \Bigg|_{t-l(t,\beta)}^{t} . \\ &= \frac{l(t,\beta)^{-\beta+1}}{\Gamma(2-\beta)}, \end{aligned}$$

which is not same as that derived from Lemma 2.1 using existing well-known fractional derivatives formulas.

Remark 2.2 *The proposed definition of fractional differential operator does not satisfy the semigroup properties, i.e., if $\alpha > 0$, $\beta > 0$ and $m \in \mathbb{N}$, then following formulas do not hold true.*

(i) $D^{-\alpha}_{l,t} D^{-\beta}_{l,t} f(t) = D^{-(\alpha+\beta)}_{l,t} f(t)$;

(ii) $D^{m} D^{\alpha}_{l,t} f(t) = D^{\alpha}_{l,t)} D^{m} f(t) = D^{m+\alpha}_{l,t} f(t)$;

(iii) $D^{\alpha}_{l,t} D^{\beta}_{l,t} f(t) = D^{\alpha+\beta}_{l,t} f(t)$, *for* $\lfloor \alpha + \beta \rfloor = \lfloor \alpha \rfloor + \lfloor \beta \rfloor$.

Proof: We prove only the part (i) of the theorem and other parts also follow the similar arguments. Following to the definition in (1.2), one can have

$$\begin{aligned} D^{-\alpha}_{l,t} D^{-\beta}_{l,t} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t-l(t,\alpha)}^{t} (t-x)^{\alpha-1} (D^{-\beta}_{l,t} f(x)) \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)} \int_{t-l(t,\alpha)}^{t} (t-x)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_{x-l(t,\beta)}^{x} (x-s)^{\beta-1} f(s) \mathrm{d}s \right) \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t-l(t,\alpha)}^{t} (t-x)^{\alpha-1} \left(\int_{x-l(t,\beta)}^{x} (x-s)^{\beta-1} f(s) \mathrm{d}s \right) \mathrm{d}x. \end{aligned}$$

Now, setting $l(t,\alpha) = l(t,\beta) = l$ and interchanging the orders of above integration, we have

$$\begin{aligned} D^{-\alpha}_{l,t} D^{-\beta}_{l,t} f(t) =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t-2l}^{t-l} (t-x)^{\alpha-1} \left(\int_{t-l}^{s+l} (x-s)^{\beta-1} f(s) \mathrm{d}x \right) \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t-l}^{t} (t-x)^{\alpha-1} \left(\int_{s}^{t} (x-s)^{\beta-1} f(s) \mathrm{d}x \right) \mathrm{d}s. \end{aligned}$$

Using the region formed by the above double integral in $xs-$plane, it is formulated that

$$\begin{aligned}
D_{l,t}^{-\alpha}D_{l,t}^{-\beta}f(t) &= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}\left(\int_{s}^{t}(t-x)^{\alpha-1}(x-s)^{\beta-1}\mathrm{d}x\right)f(s)\mathrm{d}s \\
&= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}\left(\int_{s}^{t}(t-x+s-s)^{\alpha-1}(x-s)^{\beta-1}\mathrm{d}x\right)f(s)\mathrm{d}s \\
&= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}\left(\int_{0}^{t-s}(t-s-z)^{\alpha-1}z^{\beta-1}\mathrm{d}z\right)f(s)\mathrm{d}s, \quad (\text{set } z = x-s) \\
&= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}(t-s)^{\alpha+\beta-2}\left(\int_{0}^{t-s}\left(1-\frac{z}{t-s}\right)^{\alpha-1}\left(\frac{z}{t-s}\right)^{\beta-1}\mathrm{d}z\right)f(s)\mathrm{d}s \\
&= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}(t-s)^{\alpha+\beta-1}\left(\int_{0}^{1}(1-p)^{\alpha-1}p^{\beta-1}\mathrm{d}p\right)f(s)\mathrm{d}s, \ \left(\text{set } p = \frac{z}{t-s}\right) \\
&= \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_{t-l}^{t}(t-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}f(s)\mathrm{d}s, \ (\text{by Beta function}) \\
&= \frac{2}{\Gamma(\alpha+\beta)}\int_{t-l}^{t}(t-s)^{\alpha+\beta-1}f(s)\mathrm{d}s \\
&= 2D_{l,t}^{-\alpha-\beta}f(t).
\end{aligned}$$

Therefore, one may conclude that

$$D_{l,t}^{-\alpha}D_{l,t}^{-\beta}f(t) \neq D_{l,t}^{-\alpha-\beta}f(t).$$

This completes the proof. □

Now in this connection, we give a counter example below.

Example 2.2 Consider the function $f(t) = 1$ and $\alpha = \beta = 1/2$. Then

$$\begin{aligned}
D_{l,t}^{-\alpha}f(t) = D_{l,t}^{-1/2}f(t) &= \frac{1}{\Gamma(1/2)}\int_{t-l}^{t}(t-x)^{1/2-1}1\mathrm{d}x \\
&= \frac{1}{\sqrt{\pi}}\left[\frac{-(t-x)^{-1/2+1}}{(-1/2+1)}\right]\Bigg|_{t-l}^{t} \\
&= \frac{2l(t,\alpha)^{1/2}}{\sqrt{\pi}}.
\end{aligned}$$

Applying $D_{l,t}^{-\beta}$ operation in above equation, i.e.,

$$\begin{aligned}
D_{l,t}^{-\beta}(D_{l,t}^{-\alpha}f(t) = D_{l,t}^{-1/2}(D_{l,t}^{-1/2}f(t)) &= \frac{1}{\Gamma(1/2)}\int_{t-l}^{t}(t-x)^{1/2-1}.\frac{2l(t,\alpha)^{1/2}}{\sqrt{\pi}}\mathrm{d}x \\
&\neq D_{l,t}^{-1/2-1/2}f(t) \\
&= D_{l,t}^{-1}f(t) \\
&= t.
\end{aligned}$$

Remark 2.3 The proposed definition of fractional derivatives does not satisfy the following relation in general

$$D_{l,t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t-l(t,\alpha)}^{t}(t-x)^{n-\alpha-1}f(x)\mathrm{d}x - \sum_{k=0}^{n-1}\frac{l^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(t-l).$$

We prove the theorem by using given counter example. For this, we consider the function $f(t) = t$ and $0 < \alpha < 1$. Hence, the left hand side of the equation in Remark 2.3 becomes

$$D^{\alpha}_{l,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t-l(t,\alpha)}^{t} (t-x)^{-\alpha} \mathrm{d}x = \frac{l^{-\alpha+1}}{\Gamma(2-\alpha)}. \tag{2.1}$$

But from the right hand side of the result given in Remark 2.3, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t-l(t,\alpha)}^{t} (t-x)^{1-\alpha-1} x \mathrm{d}x - \sum_{k=0}^{0} \frac{l^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(t-l) \\
=&\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t-l(t,\alpha)}^{t} (t-x)^{-\alpha} x \mathrm{d}x - \frac{l^{-\alpha}}{\Gamma(-\alpha+1)}(t-l) \\
=&\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{(t-l)l^{-\alpha+1}}{-\alpha+1} + \frac{l^{-\alpha+2}}{(-\alpha+1)(-\alpha+2)} \right] - \frac{l^{-\alpha}}{\Gamma(-\alpha+1)}(t-l) \\
=&\frac{(t-l)(-\alpha+1)l^{-\alpha}l' + l^{-\alpha+1}}{\Gamma(2-\alpha)} + \frac{l^{-\alpha+1}l'}{\Gamma(2-\alpha)} - \frac{l^{-\alpha}}{\Gamma(-\alpha+1)}(t-l),
\end{aligned} \tag{2.2}$$

where $l' = \frac{\mathrm{d}l(t,\alpha)}{\mathrm{d}t}$. By combining (2.1) and (2.2), the proof is completed.

Conclusion : In this note, we have shown that the new fractional derivative formula defined by [10] satisfies most of the known properties such as linearity, identity, backward compatibility, etc., but does not satisfy consistency (see, [10, p.4]) and semigroup properties (see, [10, p.5]). This suggests some dynamic natures of the fractional derivative operator defined in [10].

Acknowledgment: I am very much grateful to the referees for their valuable suggestions and comments, which have improved the presentation of this article.

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P. Baliarsingh,
Department of Mathematics,
Institute of Mathematics and Applications,
Bhubaneswar-751029, India.
E-mail address: pb.math10@gmail.com