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New approach to solve Volterra q-integral equations by Differential transform method

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ABSTRACT: The q-differential transformation approach is used in this study to solve Volterra q-integral equations. Investigations have been conducted to find exact solutions of linear and nonlinear q-integral equations. To illustrate the method, several problems are discussed for the effectiveness and performance of the proposed method.

Key Words: q-differential transform method, Volterra q-integral equations.

Contents

1. Introduction

In q-calculus we are looking for q-analogues of mathematical functions that have the original function as limits when q tends to unity. The history of q-calculus (and q-hypergeometric functions) dates back to Euler and Jacobi [1,2,3], who first introduced the q in his introduction [4] in the tracks of Newton's infinite series. In recent years the interest in the subject has exploded [5]. This is of course due to the fact that q-analysis has proved itself extremely fruitful in quantum theory, mechanics, number theory and theory of relativity [6] and today has wide-ranging applications in areas like computer science and particle physics and also acts as an important research tool for researchers working with analytic number theory or in theoretical physics.

The differential transform is a numerical method for solving system of equations involving algebraic, differential, integral and integro-differential differential equations. The concept of differential transform was first proposed by Zhou [7]. The main applications of this method is to solve both linear and non-linear initial value problems in electric circuit analysis and gives exact values of the nth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner [8].

The main motive to solve the integral equations are there use in mathematical models for many physical situations. Integral equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations [9]. In this study, q-differential transform is introduced to solve Volterra q-integral equations. The concept of one-dimensional q-differential was first proposed and applied to solve linear and nonlinear initial value problems by H. Jafari et al [10].

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2. Preliminaries and q-notations

Some basic concepts of q-calculus are given in [2,3,11]. The q-derivative of a real continuous function $\zeta(x)$ is defined as

$$(D_q\zeta)(x) = \frac{d_\zeta(x)}{d_q(x)} = \frac{\zeta(qx)-\zeta(x)}{(q-1)x}, \quad x \neq 0, \quad 0<q<1.$$
$$(D_q\zeta)(0) = q^{-1}\zeta'(0),$$

where $\zeta'(0)$ exists.
The q-integral of real valued contineous function $\zeta(x)$ and $\psi(x)$ is defined in [12] as

$$\int_b^a \zeta(x)\psi(x)d_q(x) = \zeta(a)\int_a^c \zeta(x)\psi(x)d_q(x)$$

which is q-analogue of Bonnet's theorem and by putting $\zeta(x)-\zeta(a)$ for $\zeta(x)$ in Bonnet's analogue, we find that

$$\int_a^b \zeta(x)\psi(x)d_q(x) = \zeta(a)\int_a^c \psi(x)d_q(x) + \zeta(b)\int_c^b \psi(x)d_q(x) \tag{2.1}$$

where $qa \leq c \leq qb$, this result is an analogue of du Bois-Reymond's theorem.
Definition 1. The series representation of the q-exponential function is defined as [13]

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}$$

further,

$$\frac{\delta_q}{\delta_q x} e_q^x = e_q^x.$$

Also note that

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+...+q^{n-1}, \qquad [n]_q! = [1]_q[2]_q...[n]_q$$
$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}, \qquad m,n \in \mathbb{N}_0.$$

For further details in q-calculs go through [2,6,11,14].
If all the q-differentials of a function $\zeta(x)$ exists in some neighborhood of a. Then q-Taylor formula in [15] is defined as

$$\zeta(x) = \sum_{k=0}^{\infty} \frac{d_q^k \zeta(x,a)}{[k]_q!}.$$

3. Analysis of q-differential transform method

For the function $f(y)$ the differential transform of kth derivative is defined as

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(y)}{dy^k}\right]_{y=y_0} \tag{3.1}$$

The differential inverse transform of $F(k)$ is defined as

$$f(y) = \sum_{k=0}^{\infty} F(k)(y-y_0)^k \tag{3.2}$$

from equations (3.1) and (3.2), we get

$$f(y) = \sum_{k=0}^{\infty} \frac{(y-y_0)^k}{k!} \frac{d^k f(y)}{dy^k}|_{y=y_0}.$$

However, the basic definitions and results of the differential transform method are given in [7,16,17,18,19,20,21,22,23,24].
H. Jafari et al. has defined the q-analogue of differential transform in [10].
Definition 2. The q-differential transform of the function $f(z,t)$ is defined as

$$F(k) = \frac{1}{[k]_q!} \left[\frac{d_q^k}{d_q t^k} f(z,t) \right]_{t=a}, \tag{3.3}$$

where all the q-differentials of $f(z,t)$ exist in some neighbourhood of $t = a$, and F(k) is the transformed function.
Definition 3. The inverse q-differential transform of the function $F(k)$ is defined as

$$f(z,t) = \sum_{k=0}^{\infty} F(k)(t-a)^k \tag{3.4}$$

from the definitions (3.3) and (3.4), we get

$$f(z,t) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left[\frac{d_q^k}{d_q t^k} f(z,t) \right]_{t=a} (t-a)^k$$

The fundamental operations of q-differential transform can readily be obtained and are very useful on our apporach for solving q-integral equations listed below in table 1.

4. Results

Theorem 1. Suppose that $F(k)$ and $G(k)$ are the q-differential transformations of the function $f(y)$ and $g(y)$ respectively, then we have with the heip of (1)
(a) If $h(y) = \int_{y_0}^{y} g(t)f(t)d_q(t)$, then

$$H(k) = \sum_{l=0}^{k-1} G(l)\frac{F(k-l-1)}{k} + \sum_{m=0}^{n-1} G(m)\frac{F(n-m-1)}{n}, \qquad H(0) = 0.$$

(b) If $h(y) = g(y)\int_{y_0}^{y} u(t)d_q(t)$, then

$$H(k) = \sum_{l=0}^{k-1} G(l)\frac{F(k-l-1)}{k-1}, \qquad H(0) = 0.$$

Proof:
From the results (4) and (5) and from the properties of q-differential transform as given in table below the proof follows from those.

5. Applications and numerical results

In this section, we apply the method to solve some Volterra q-integral equations. These results shows that the method is simple and effective and leads to exact solution.

Table 1: Operations of q-differential transformations

Original function	Transformed function
$f(\eta)=u(\eta)\pm v(\eta)$	$F(k)=U(k)\pm V(k)$
$f(\eta)=\alpha u(\eta)$	$F(k)=\alpha U(k)$
$f(\eta)=\dfrac{d_q u(\eta)}{d_q\eta}$	$F(k)=(k+1)U(k+l)$
$f(\eta)=\dfrac{d_q^r u(\eta)}{d_q^r\eta}$	$F(k)=[k+1]_q[K+2]_q...[k+r]_qU(k+r)$
$f(\eta)=\int\limits_{\eta_0}^{\eta} u(t)d_qt$	$F(k)=\dfrac{U(k-1)}{k},\ k\geq 1,\ F(0)=0$
$f(\eta)=exp_q(\lambda\eta)$	$F(k)=\dfrac{\lambda^k}{[k]_q!}$
$f(\eta)=\eta^m$	$F(k)=\delta(k-m)$
$f(\eta)=sin_q(w\eta+\alpha)$	$F(k)=\dfrac{w^k}{[k]_q!}sin_q(\frac{\pi k}{2}+\alpha)$
$f(\eta)=cos_q(w\eta+\alpha)$	$F(k)=\dfrac{w^k}{[k]_q!}cos_q(\frac{\pi k}{2}+\alpha)$
$f(\eta)=u(\eta)v(\eta)$	$F(k)=\sum\limits_{n=0}^{k}U(k-n)V(n)$

Example 1. We consider the following Volterra q-integral equation

$$v(y)=1-y-\frac{y^2}{2}+\int_0^y (y-t)u(t)d_q(t),\quad 0<y<1$$

Sol:
By using operations from table and according to theorem as stated above, we have the following relation

$$V(k)=\delta(k)-\delta(k-1)-\frac{\delta(k-2)}{2}+\sum_{l=0}^{k-1}\delta(l-1)\frac{V(k-l-1)}{k-l}-\sum_{l=0}^{k-1}\delta(l-l)\frac{V(k-l-1)}{k},\quad k\geq 1,\quad V(0)=1.$$

consequently, we find
$V(1)=-1,\quad V(2)=0$
$V(3)=\dfrac{-1}{[3]_q!},\quad V(4)=0$
$V(5)=\dfrac{-1}{[5]_q!}$ ...
therefore from (5), the solution of above integral equation is given by

$$\begin{aligned}v(y)&=1-y-\frac{1}{[3]_q!}y^3-\frac{1}{[5]_q!}y^5-\cdots\\&=1-sin_qh(y)\end{aligned}$$

which is the solution of above q-integral equation.
Example 2. Consider the linear Volterra q-integral equation

$$\psi(x)=x+\int_0^x (t-x)u(t)d_q(t),\quad 0<x<1,\quad |q|<1.$$

Sol: The recurrence relation is given by

$$\Psi(k) = \delta(k-1) + \sum_{l=0}^{k-1} \delta(l-l)\frac{\Psi(k-l-1)}{k} - \sum_{l=0}^{k-1} \delta(l-1)\frac{\Psi(k-l-1)}{k-1}, k \geq 1, \Psi(0) = 0.$$

consequently, we have
$\Psi(1) = 1, \quad \Psi(2) = 0$
$\Psi(3) = \frac{-1}{[3]_q!}, \quad \Psi(4) = 0$
$\Psi(5) = \frac{-1}{[5]_q!}$ ...
therefore from (5) the solution of above integral equation is given by

$$\psi(x) = \frac{1}{[1]_q!}x - \frac{1}{[3]_q!}x^3 - \frac{1}{[5]_q!}x^5 - \cdots = sin_q(x)$$

which is an exact solution.
Example 3. Consider the nonlinear Volterra q-integral equation

$$\psi(y) + \int_0^y \psi^2(t) + \psi(t)d_q(t) = \frac{3}{2} - \frac{1}{2}exp_q(-2y)., \quad 0 < y < 1.$$

Sol: consider the following recurrence relation

$$\Psi(k) + \sum_{l=0}^{k-1} \Psi(l)\frac{\Psi(k-l-1)}{k}\frac{\Psi(k-1)}{k} = \frac{3}{2}\delta(k) - \frac{(-2)^k}{[2k]_q!}, \quad k \geq 1, \quad \Psi(0) = 1.$$

consequently, we find
$\Psi(1) = -1, \quad \Psi(2) = \frac{1}{[2]_q!}$
$\Psi(3) = \frac{-1}{[3]_q!}, \quad \Psi(4) = \frac{1}{[4]_q!}$
$\Psi(5) = \frac{-1}{[5]_q!}$ ...
therefore from (5), we have
$\psi(y) = 1 - y + \frac{1}{[2]_q!}y^2 - \frac{1}{[3]_q!}y^3 - \cdots$
$= exp_q(-y)$
which is an exact solution.
Example 4. Consider the nonlinear Volterra q-integral equation

$$v(y) = cos_q(y) + \frac{1}{2}sin_q(2y) + 3x - 2\int_0^y (1 + v^2(t))d_q(t)$$

Sol: consider the following relation

$$V(k) = \frac{-1}{[k]_q!}cos_q(\frac{\pi k}{2}) + \frac{2^k - 1}{[k]_q!}sin_q(\frac{\pi k}{2}) + \delta(k-1) - \sum_{l=0}^{k-1} V(l)\frac{V(k-l-1)}{k}, \quad k \geq 1, \quad V(0) = 1.$$

consequently, we find
$V(1) = 0, \quad V(2) = \frac{-1}{[2]_q!}$
$V(3) = 0, \quad V(4) = \frac{1}{[4]_q!}$
$V(5) = 0$ ...

therefore from (5) the solution is given as

$$v(y) = 1 - \frac{1}{[2]_q!}y^2 + \frac{1}{[4]_q!}y^4 - \cdots$$
$$= cos_q(y)$$

which is an exact solution of above integral equation.

6. Conclusion

In this work, the q-differential transform method has been used to solve Volterra q-integral equations. Integral equations are used as mathematical models for many physical situations such as ordinary and partial differential equations. Our results are highly compatible and involves less computation. Some results have been described and are tested with the help of examples.

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