Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.67418

Diameter of a direct power of alternating groups

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ABSTRACT: In this paper we estimate the diameter of a direct power of alternating groups A_k for $k \geq 4$. We show that there exists a generating set of minimum size for A_4^n, for which the diameter of A_4^n is $O(n)$. For $k \geq 5$, we show that there exists a generating set of minimum size for A_k^2, for which the diameter of A_k^2 is at most $O(ke^{(c+1)(\log k)^4 \log\log k})$, for an absolute constant $c > 0$. Finally for $1 \leq n \leq 8$, we provide generating sets of size two for A_5^n and we show that the diameter of A_5^n with respect to those generating sets is $O(n)$. These results leads us to the sense that the best upper bound known for the diameter of the direct power of non-abelian simple groups (specially alternating groups), i.e. $O(n^3)$ [5], may be improved to $O(n)$. Furthermore, these results are more pieces of evidence for a conjecture which has been presented in [9] in 2015.

Key Words: Diameter of a group, rank of a group, non-abelian simple groups

Contents

1. Introduction

Let G be a finite group with a generating set A. By diameter of G with respect to A we mean the maximum over $g \in G$ of the length of the shortest word in A expressing g. We define the diameter of G to be the maximum over the generating set A of the diameter of G with respect to A. Finding a bound for the diameter of a finite group is an important area of research in finite group theory. We mention the most important conjecture in this area, known as the Babai's conjecture [2]: every non-abelian finite simple group G has diameter less than or equal to $log^k|G|$, where k is an absolute constant. The conjecture is still open, despite great progress towards a solution both for alternating groups and for groups of Lie type.

Producing a bound for the diameter of a direct product of simple groups, depending on the diameter of their factors, have been used more than once for proving Babai's conjecture. In [3], it is shown that for $G = T_1 \times T_2 \times \cdots T_n$, in which T_i's are non-abelian simple groups, $\text{diam}(G) \leq 20n^3h^2$, such that h is the maximum diameter of the T_i's. In [8], this bound improved to be a bound linear on h instead of quadratic, when the factors are alternating groups; and then in [5], it is generalized for all non-abelian simple groups. So far, all the upper bounds presented, are cubic on n.

Furthermore, it has been proven that if G is an abelian group , then the diameter of G^n with respect to any generating set is $O(n)$; and if G is nilpotent, symmetric or dihedral, then there exists a generating set of minimum size, for which the diameter of G^n is $O(n)$ [10].

This paper is organized as follows:

In Section 3, we find generating sets of minimum size for A_4^n for $n \geq 1$, and we show that the diameter of A_4^n, with respect to those generating sets is $O(n)$ for $n \geq 2$.

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Submitted March 07, 2023. Published June 02, 2023
2010 *Mathematics Subject Classification:* 20B05; 20B30; 20D60

In Section 4, for $k \geq 5$, we find generating sets of size two for A_k^2 , for which the diameter of A_k^2 is $O(ke^{(c+1)(\log k)^4 \log\log k})$, for an absolute constant $c > 0$.

In Section 5, we show that there exist generating sets of size two for A_5^n $(1 \leq n \leq 8)$ for which, the diameter of A_5^n is at most $n\,(|A_5| - \mathrm{rank}(A_5)) = 58\,n$.

2. Preliminaries

Throughout the paper all groups are considered to be finite . The subset $A \subseteq G$ is a generating set of G, if every element of G can be expressed as a sequence of elements in A.[1] By the rank of G, denoted by $\mathrm{rank}(G)$, we mean the cardinality of any of the smallest generating sets of G. By the length of a non identity element $g \in G$, with respect to A, we mean the minimum length of a sequence expressing g in terms of elements in A. Denote this parameter by $l_A(g)$.

Remark 2.1 *We consider the length of the identity to be zero, i.e.* $l_A(1) = 0$ *for every generating set* A*.*

Definition 2.1 *Let G be a finite group with generating set A. By the* diameter *of G with respect to A we mean*

$$\mathrm{diam}(G, A) := \max\{l_A(g) : g \in G\},$$

and by the diameter of G, denoted by $D(G)$, we mean

$$D(G) := \max\{\mathrm{diam}(G, A) : G =< A >\}.$$

The next definition introduces a generating set (let us call it canonical) for any direct power G^n with respect to a generating set of G.

Definition 2.2 *Let G be a finite group with a generating set A. By the* canonical generating set *of G^n with respect to A, we mean the set*

$$C^n(A) := \{(1, \ldots, \overbrace{a}^{i\,\text{th}}, \ldots, 1) : i \in \{1, 2, \ldots, n\}, a \in A\}.$$

Remark 2.2 If G is a group with the property that $\mathrm{rank}(G^n) = n\,\mathrm{rank}(G)$ then the canonical generating set of G^n is a generating set of minimum size and the diameter of G^n with respect to $C^n(A)$ is at most $O(n)$ [10]. Note that the alternating groups A_k for $k \geq 4$, do not have the property that $\mathrm{rank}(G^n) = n\,\mathrm{rank}(G)$.

We explain the following easy fact as a remark.

Remark 2.3 Let $(g_1, g_2, \ldots, g_n) \in G^n = \langle A \rangle$. Since $(g_1, g_2, \ldots, g_n)$ is a product of n elements of the form $(1, \ldots, g_i, \ldots, 1)$, then we have

$$l_A(g_1, g_2, \ldots, g_n) \leq \sum_{i=1}^{n} l_A(1, \ldots, g_i, \ldots, 1). \tag{2.1}$$

Definition 2.3 *By an n-*basis *of a group G we mean any ordered set of n elements $x_1, x_2, \ldots, x_n$ of G which generates G. Furthermore, two n-bases $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ of G will be called* equivalent *if there exists an automorphism θ of G which transforms one into the other:*

$$x_i\theta = y_i,$$

for each $i = 1, 2, \ldots, n$. Otherwise the two bases will be called non-equivalent*.*

In general, we have the following lemma for the rank of a direct power of a finite group G:

[1] Usually $A \subseteq G$ is considered to be a generating set, if every element of G can be expressed as a sequence of elements in $A \cup A^{-1}$. When G is finite the definitions coincide.

Lemma 2.1 [12] *Let G be a finite group and k be a positive integer. The following inequalities hold:*

$$k\,\mathrm{rank}(G/G') \leq \mathrm{rank}(G^k) \leq k\,\mathrm{rank}(G), \tag{2.2}$$

where G' is the commutator subgroup of G.

Definition 2.4 *A group is said to be* perfect *if it equals its own commutator subgroup; otherwise it is called* imperfect*.*

Remark 2.4 By Lemma 2.1, if G is a perfect group, then the lower bound in the inequality (2.2) is zero, hence the first inequality in lemma 2.1 is trivial. If G is imperfect, then the first inequality in lemma 2.1 gives a lower bound, depending on k, for the rank of G^k.

Since the alternating group A_4 is imperfect; and for $n \geq 5$ alternating groups A_n are perfect, then by Remark 2.4, we need to verify them in the separate sections.

3. The diameter of a direct power of alternating group A_4

We use the following lemma for finding a generating set of minimum size for a direct power of the alternating group A_4.

Lemma 3.1 *Let G be a finite imperfect group. If G is generated by k elements of mutually coprime orders, then* $\mathrm{rank}(G^n) = n$*, for $n \geq k$.*

Proof: Because G is not perfect, it follows from Lemma 2.1 that $\mathrm{rank}(G^n) \geq n$. Suppose $A = \{a_1, a_2, \dots, a_k\}$ is a generating set of G such that the a_i's are of mutually coprime orders. Let $n \geq k$. We construct a generating set of size n for G^n. For $1 \leq i \leq n$, define the elements $g_i \in G^n$ as follows:

$$g_i = (1, \dots, \overbrace{a_1}^{i\,\text{th}}, a_2, \dots, a_k, \dots, 1) \qquad \text{for } 1 \leq i \leq n-k+1,$$

$$g_i = (a_{n-i+2}, a_{n-i+3}, \dots, a_k, 1, \dots, 1, \overbrace{a_1}^{i\,\text{th}}, \dots, a_{n-i+1}) \qquad \text{for } n-k+2 \leq i \leq n.$$

We prove that $C = \{g_1, g_2, \dots, g_n\}$ is a generating set of G^n. If we show that C generates $C^n(A)$, then we are done. Choose an arbitrary element $(1, \dots, a_i, \dots, 1) \in C^n(A)$. Since the a_i's are of mutually coprime orders, there exists a positive integer ℓ such that

$$(1, \dots, a_i, \dots, 1) = (1, \dots, a_1, \dots, a_i, \dots, a_k, \dots, 1)^{\ell}, \text{ for } 1 \leq i \leq n-k+1,$$

$$(1, \dots, a_i, \dots, 1) = (a_{n-i+2}, \dots, a_k, 1, \dots, 1, \overbrace{a_1}^{i\,\text{th}}, \dots, a_{n-i+1})^{\ell}, \text{ for } n-k+2 \leq i \leq n.$$

This yields the desired conclusion. □

Now we have the following Lemma for the rank of A_4^n.

Lemma 3.2 *The rank of A_4^n is equal to n, for $n \geq 2$.*

Proof: The alternating group A_4 is generated by the following two elements

$$\alpha = (1\ 2)(3\ 4),\ \ \beta = (1\ 2\ 3).$$

(see [4]). Since A_4 is not perfect and α, β have coprime orders by Lemma 3.1, the rank of A_4^n is equal to n, for $n \geq 2$. □

Theorem 3.1 *There exists a generating set of minimum size for A_4^n, for which the diameter of A_4^n is at most $10n$.*

Proof: As we mentioned before in Example 3.2, the generating set C constructed in the proof of Lemma 3.1 is a generating set of minimum size for A_4^n for $n \geq 2$. We show that $\mathrm{diam}(A_4^n, C) \leq 10n$. Let $(g_1, g_2, \ldots, g_n) \in A_4^n$. By Remark 2.3, it is enough to show that $l_C(1, \ldots, 1, g_i, 1, \ldots, 1) \leq 10$, for $1 \leq i \leq n$. Because of the following equalities

$$(1, \ldots, \overbrace{\alpha}^{i\,\mathrm{th}}, \beta, \ldots, 1)^3 = (1, \ldots, \overbrace{\alpha}^{i\,\mathrm{th}}, 1, \ldots, 1),$$
$$(1, \ldots, \alpha, \overbrace{\beta}^{i\,\mathrm{th}}, \ldots, 1)^4 = (1, \ldots, 1, \overbrace{\beta}^{i\,\mathrm{th}}, \ldots, 1),$$
$$(1, \ldots, \alpha, \overbrace{\beta}^{i\,\mathrm{th}}, \ldots, 1)^2 = (1, \ldots, 1, \overbrace{\beta^2}^{i\,\mathrm{th}}, \ldots, 1),$$

we have

$$l_C(1, \ldots, \overbrace{\alpha}^{i\,\mathrm{th}}, \ldots, 1) \leq 3,$$
$$l_C(1, \ldots, \overbrace{\beta}^{i\,\mathrm{th}}, \ldots, 1) \leq 4,$$
$$l_C(1, \ldots, \overbrace{\beta^2}^{i\,\mathrm{th}}, \ldots, 1) \leq 2.$$

On the other hand, the elements of A_4 can be represented over the generating set $\{\alpha, \beta\}$ as follows:

$$A_4 = \{\alpha, \beta, \alpha^2, \alpha\beta, \beta\alpha, \beta^2, \alpha\beta\alpha, \alpha\beta^2, \beta\alpha\beta = \alpha\beta^2\alpha, \beta^2\alpha, \beta^2\alpha\beta, \beta\alpha\beta^2\}.$$

Now it is easy to see that the length of $(1, \ldots, \overbrace{g}^{i\,\mathrm{th}}, \ldots, 1)$ in the generating set C is at most 10 for every element $g \in A_4$, which completes the proof. □

4. The diameter of A_n^2, for $n \geq 5$

Note that alternating groups A_n for $n \geq 5$ are perfect. There is a different approach to compute the rank of the direct power of perfect groups using the Eulerian function of a group (see [6,12]). The following lemma is a consequence of the results in [6].

Lemma 4.1 *Let G be a non-abelian simple group. If G is generated by n elements, then the set $\{(a_{i1}, a_{i2} \ldots, a_{ik}) : i = 1, \ldots, n\}$ will generate G^k if and only if the following conditions are satisfied:*

1. *the set $\{a_{1i}, a_{2i}, \ldots, a_{ni}\}$ is a generating set of G for $i = 1, \ldots, k$;*
2. *there is no automorphism $f : G \to G$ which maps $(a_{1i}, a_{2i}, \ldots, a_{ni})$ to $(a_{1j}, a_{2j}, \ldots, a_{nj})$ for any $i \neq j$.*

Furthermore, in [6] Hall shows that the alternating group A_5 satisfies Lemma 4.1 with $n = 2$ for $1 \leq k \leq 19$ and not for $k \geq 20$.

Therefore, the following is an immediate consequence of Lemma 4.1.

Corollary 4.1 *A pair $(s_1, \ldots, s_k), (t_1, \ldots, t_k)$ will generate A_5^k if and only if the following conditions are satisfied:*

1. *the set $\{s_i, t_i\}$ is a generating set of A_5 for $i = 1, ..., k$;*
2. *there is no automorphism $f : A_5 \to A_5$ which maps (s_i, t_i) to (s_j, t_j) for any $i \neq j$.*

Furthermore, $k = 19$ is the largest number for which these conditions can be satisfied. That is, the rank of A_5^k is equal to 2 if and only if $1 \leq k \leq 19$.

Now we are ready to prove the following theorem.

Theorem 4.1 *Let $k \geq 5$. There exists a generating set of size two for A_k^2, for which the diameter of A_k^2 is at most $O(ke^{(c+1)(\log\ k)^4 \log\log k})$, for an absolute constant $c > 0$.*

Proof: For $k \geq 5$, let $a = (1\,2\,3\,\cdots\,k)$, $b = (1\,2)(3\,4)$, $a' = (1\,2\,3\,\cdots\,k{-}1)$ and $b' = (k{-}3\,k{-}2)(k{-}1\,k)$. It is easy to see that $A = \{a, b\}$ and $A' = \{a', b'\}$ are generating sets of A_k for k odd and k even, respectively. Furthermore, if k is odd, then $(a, b), (b, a)$ are two non-equivalent 2-bases of A_k and if k is even, then $(a', b'), (b', a')$ are two non-equivalent 2-bases of A_k. By Lemma 4.1, if k is odd, then $A_k^2 =< (a, b), (b, a) >$ and if k is even, then $A_k^2 =< (a', b'), (b', a') >$. Let $S = \{(a, b), (b, a)\}$ and $S' = \{(a', b'), (b', a')\}$. Suppose for the moment that k is odd. For $(x, y) \in A_k^2$ we have $l_A(x, y) \leq l_A(x, 1) + l_A(1, y)$, for every generating set A. Combining this with the following equalities:

$$(a, b)^2 = (a^2, 1), (b, a)^k = (b, 1), (b, a)^2 = (1, a^2), \text{and } (a, b)^k = (1, b). \tag{4.1}$$

we obtain

$$\operatorname{diam}(A_k^2, S) \leq 2\,k \operatorname{diam}(A_k, \{b, a^2\}). \tag{4.2}$$

Replacing $\operatorname{diam}(A_k, \{b, a^2\})$ with $O(e^{(c+1)(\log k)^4 \log\log k})$, for an absolute constant $c > 0$, (see Theorem 6.6 in [7]) in 4.2 we get the desired conclusion. Similar arguments apply for the case that k is even. □

5. The diameter of a direct power of A_5

We know that 19 is the largest number for which the group A_5^k is generated by two elements for $1 \leq k \leq 19$ (see [6]). Let $a = (12)(34), b = (12345), c = (123), d = (135), e = (245), f = (12354), g = (12543), h = (12534), i = (13254)$. We have checked with the Groups, Algorithms, Programming (GAP) - a System for Computational Discrete Algebra- that the pairs

$$(a, b), (b, a), (a, b^2), (b^2, a), (c, b), (b, c), (c, b^2), (b^2, c), (b, c^2), (c^2, b), (b^2, c^2), (c^2, b^2),$$
$$(d, a), (a, d), (d, e), (b, f), (b, g), (b, h), (b, i)$$

are 19 non-equivalent 2-basis of A_5.

By Corollary 4.1 we can build generating sets of size two for A_5^k, $1 \leq k \leq 19$; but for proving theorem 5.1 we just need 8 of them. Let

$$\begin{aligned}
C_1 &= \{a, b\},\\
C_2 &= \{(a, b), (b, a)\},\\
C_3 &= \{(a, b, a), (b, a, b^2)\},\\
C_4 &= \{(a, b, a, b^2), (b, a, b^2, a)\},\\
C_5 &= \{(a, b, a, b^2, c), (b, a, b^2, a, b)\},\\
C_6 &= \{(a, b, a, b^2, c, b), (b, a, b^2, a, b, c)\},\\
C_7 &= \{(a, b, a, b^2, c, b, c), (b, a, b^2, a, b, c, b^2)\},\\
C_8 &= \{(a, b, a, b^2, c, b, c, b^2), (b, a, b^2, a, b, c, b^2, c)\}.
\end{aligned}$$

Then for $1 \leq n \leq 8$, the sets C_n are generating sets of minimum size for groups A_5^n.

Theorem 5.1 *The diameter of A_5^n, for $1 \leq n \leq 8$, is at most $58\,n$.*

Proof: Using GAP [11] we check that $\operatorname{diam}(A_5, C_1) = 10$ and $\operatorname{diam}(A_5^2, C_2) = 18$. Let (x, y, z) be an arbitrary element in A_5^3. Then we have

$$l_{C_3}(x, y, z) = l_{C_3}(x, 1, z) + l_{C_3}(1, y, 1).$$

On the other hand, $(x,z) \in A_5^2 = \langle (a,a),(b^2,b^4)\rangle$ and $\mathrm{diam}(A_5^2,\{(a,a),(b^2,b^4)\}) = 20$ and $y \in A_5 = \langle a, b^2\rangle$ and $\mathrm{diam}(A_5,\{a,b^2\}) = 9$. These facts together with the following equalities

$$\begin{aligned}(a,b,a)^2 &= (1,b^2,1),\\ (b,a,b^2,)^5 &= (1,a,1),\\ (a,b,a)^5 &= (a,1,a),\\ (b,a,b^2,)^2 &= (b^2,1,b^4)\end{aligned}$$

lead to

$$\mathrm{diam}(A_5^3, C_3) \leq 5\times 9 + 5\times 20 = 5\times(9+20) = 145.$$

Let (x,y,z,w,k,h,l) be an arbitrary element in A_5^7. Then we have

$$l_{C_7}(x,y,z,w,k,h,l) = l_{C_7}(x,1,z,1,k,1,l) + l_{C_7}(1,y,1,w,1,h,1).$$

On the other hand, $(x,z,k,l) \in A_5^4 = \langle (a,a,c^2,c^2),(b,b^2,b,b^2)\rangle$ and

$$\mathrm{diam}(A_5^4,\{(a,a,c^2,c^2),(b,b^2,b,b^2)\} = 30.$$

$(y,w,h) \in A_5^3 = \langle (a,a,c^2),(b,b^2,b)\rangle$ and

$$\mathrm{diam}(A_5^3,\{(a,a,c^2),(b,b^2,b)\} = 25.$$

These facts together with the following equalities

$$\begin{aligned}(a,b,a,b^2,c,b,c)^6 &= (1,b,1,b^2,1,b,1),\\ (b,a,b^2,a,b,c,b^2)^6 &= (b,1,b^2,1,b,1,b^2),\\ (a,b,a,b^2,c,b,c)^5 &= (a,1,a,1,c^2,1,c^2),\\ (b,a,b^2,a,b,c,b^2)^5 &= (1,a,1,a,1,c^2,1)\end{aligned}$$

lead to

$$diam(A_5^7, C_7) \leq 6\times(25+30) = 330.$$

In the same manner we can see that

$$\begin{aligned}\mathrm{diam}(A_5^4, C_4) &\leq 5\times(20+20) = 200,\\ \mathrm{diam}(A_5^5, C_5) &\leq 6\times(20+25) = 270,\\ \mathrm{diam}(A_5^6, C_6) &\leq 6\times(25+25) = 300,\\ \mathrm{diam}(A_5^8, C_8) &\leq 6\times(30+30) = 360.\end{aligned}$$

□

Forasmuch as by increasing n the Cayley Graph of A_5^n is growing exponentially, we could not calculate the diameter of A_5^n for $n \geq 4$ with GAP, hence we could not estimate the diameter of A_5^n for $n \geq 9$ with the technique which is used in the proof of theorem 5.1.

In 2015, the second author conjectured that the diameter of G^n is growing polynomially with respect to n. More precisely, she conjectured that if G is a finite group, then the diameter of G^n is at most $n(|G| - \mathrm{rank}(G))$, which is called the strong conjecture. The strong conjecture has been proved for abelian groups in [10]. Another version of the strong conjecture called the weak conjecture states that if G is a finite group, then there exists a generating set of minimum size for G^n, for which the diameter of G^n is at most $n(|G| - \mathrm{rank}(G))$. The weak conjecture is proved for nilpotent groups, dihedral groups and some power of imperfect groups in [10]. Recently, it was shown that for a solvable group G, the diameter of G^n grows polynomially with respect to n [1]. In this paper, theorems 3.1, 4.1 and 5.1 are more pieces of evidence for the weak conjecture.

Acknowledgments

The second author wishes to thank Arak University, for the invitation and hospitality, and the International Science and Technology Interactions (ISTI) for financial support.

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