Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm
(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.67430

Frequently supercyclic operators and frequently supercyclic C_0-semigroups

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ABSTRACT: In this paper, the concept of frequent supercyclicity for operators and for C_0-semigroups is defined. It is proved that if an operator T is frequently supercyclic, then T^n and λT are frequently supercyclic for any natural number n and any non-zero scalar λ. Also, it is established that frequent supercyclicity of a C_0-semigroup implies frequent supercyclicity of any of its operators. Moreover, by using discretization and autonomous discretization of a C_0-semigroup, some equivalent conditions for frequent supercyclicity are stated.

Key Words: Frequent Supercyclicity, Supercyclicity, Operators, C_0-semigroups.

Contents

1. Introduction

Assume X is a Banach space. Suppose $T : X \to X$ is a linear and continuous operator or briefly an operator. We show the collection of operators on X by $B(X)$ in this paper. If $orb(T, x) = \{T^n x : n = 0, 1, 2, ...\}$ is dense in X for some $x \in X$, we say T is hypercyclic. If $\mathbb{C}orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n = 0, 1, 2, ...\}$ is dense in X for some $x \in X$, then we say that T is supercyclic. Hypercyclic operators can be constructed only on infinite-dimensional spaces, while supercyclic operators can be made on both finite-dimensional spaces and infinite-dimensional spaces [11]. One can see [2], [11], and [13] for a history of these operators.

An operator T on X is called frequently hypercyclic if there is some $x \in X$ such that $\underline{dens}\{n \in \mathbb{N}_0 : T^n x \in U\} > 0$ for any non-empty open set U, where $\mathbb{N}_0$ is the set of non-negative integers. By $\underline{dens}(B)$ we mean the lower density of B. Remind for a subset B of natural numbers is defined by $\liminf_{N\to+\infty} \frac{\text{card}\{m\leq N : m\in B\}}{N}$.

In fact, in frequent hypercyclicity, the number of times an element's orbit meets with an open set is of interest.

It is shown that there are hypercyclic operators that are not frequently hypercyclic [3, Example 2.9]. Also, frequent hypercyclicity of an operator implies that any of its powers are frequently hypercyclic [3, Theorem 4.7]. Moreover, some criteria for frequent hypercycliciy are stated in [7]. Also, in [14] frequent hypercyclicity of Toeplitz operators are investigated. One can also see [12], [20], and [22] for more information.

Another structure that considers in dynamical systems is a C_0-semigroup. By a C_0-semigroup $(T_t)_{t\geq 0}$ on X, we mean a family of operators on X with these properties:

(i) $T_0 = I$,

(ii) $T_{s+t} = T_s T_t$ for any $s \geq 0$ and $t \geq 0$,

(iii) $\lim_{s\to t} T_s x = T_t x$, for any $x \in X$ and $t \geq 0$.

Submitted March 08, 2023. Published July 11, 2023
2010 *Mathematics Subject Classification:* 47A16, 37B99, 47D03.

The concept of hypercyclicity for C_0-semigroups is defined by Desh, Schappacher and Webb in [10]. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called hypercyclic if there is $x \in X$ such that $orb((T_t), x) = \{T_t x : t \geq 0\}$ be dense in X.

Like hypercyclic operators, hypercyclic C_0-semigroups exist only on infinite-dimensional spaces [11, Proposition 7.12]. Any infinite-dimensional Banach space support a hypercyclic C_0-semigroup [5]. In [6], there are some theorems about the existence and non-existence of this type of C_0-semigroups. Moreover, we have the following theorem about the behavior of operators of a C_0-semigroup.

Theorem 1.1 *([9]) Suppose $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup. Then T_t is hypercyclic for any $t > 0$.*

One can also see [19] for some extensions.
We say $(T_{t_n})_n$ is a discretization of $(T_t)_{t\geq 0}$, when $(t_n) \subseteq \mathbb{R}^{\geq 0}$ is an increasing sequence tending to infinity. If $t_n = ns$ for some $s > 0$, we say $(T_{t_n})_n = (T_{ns})_n$ is an autonomous discretization of $(T_t)_{t\geq 0}$.

Hypercyclicity of an autonomous discretization of a C_0-semigroup is equivalent to hypercyclicity of the C_0-semigroup and it is equivalent to hypercyclicity of any discretization of the C_0-semigroup [11, Theorem 7.26].

The concept of frequent hyperciclicity is defined for C_0-semigroups, either. The C_0-semigroup $(T_t)_{t\geq 0}$ on X is named frequently hypercyclic if $x \in X$ exists such that for any non-empty open set U of X, $\underline{dens}\{t \in \mathbb{R}^{\geq 0} : T_t x \in U\} > 0$ [11, Definition 9.29]. It is proved in [9, Theorem 3.2] that frequent hypercyclicity of $(T_t)_{t\geq 0}$ implies that for any $s > 0$, T_s is frequently hypercyclic. Some criteria for frequent supercyclicity and frequent hypercyclicity of tranlation C_0-semigroups are studied in [4]. Also, one can see f-frequent hypercyclic operators in [8].

A C_0-semigroup is called supercyclic if $\{\lambda T_t x : t \in \mathbb{R}^{\geq 0}, \lambda \in \mathbb{C}\}$ for some $x \in X$. It is proved that a Banach space X can support supercyclic C_0-semigroups if $\dim(X) = 1$ or $\dim(X) = \infty$ [21, Lemma 2.3]. Also, the supercyclicity of a C_0-semigroup implies supercyclicity of its operators as follows.

Theorem 1.2 *([21]) Suppose $(T_t)_{t\geq 0}$ is a supercyclic C_0-semigroup. Then T_t is supercyclic for any $t > 0$.*

In [16] some sufficient conditions for supercyclicity of C_0-semigroups are stated. One can see some results about this type of C_0-semigroup in [15].

In this paper, we extend the concept of supercyclicity and we define frequently supercyclic operators and frequently supercyclic C_0-semigroups.

In Section 2, we introduce frequently supercyclic operators. We show that the collection of frequently hypercyclic operators is a proper subset of the collection of the frequently supercyclic operators. We prove that frequent supercyclicity preserves under quasiconjugacy. Moreover, we establish that if T is frequently supercyclic, then T^n and λT are frequently supercyclic for any $n \in \mathbb{N}$ and any $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.

In Section 3, we define frequently supercyclic C_0-semigroups. We show that there are frequently supercyclic C_0-semigroups that are not frequently hypercyclic. We establish that if a C_0-semigroup contains a frequently supercyclic operator, then it is frequently supercyclic. Moreover, frequent supercyclicity of a C_0-semigroup implies that any of its operators is frequently supercyclic. We establish a C_0-semigroup is frequently supercyclic if and only if some autonomous discretization of it is frequently supercyclic. Also, we state some other equivalent conditions for frequent supercyclicity.

2. Frequently supercyclic operators

By the idea of frequently hypercyclic operators and supercyclic operators we introduce the concept of frequent supercyclicity as follows.

Definition 2.1 Let $T \in B(X)$. We say that T is frequently supercyclic if, for some $x \in X$,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n x \in U; \lambda \in \mathbb{C}\}$$

is positive for any non-empty open set U of X.

The vector x is called a frequently supercyclic vector for T. By $FSH(T)$, we show the set of all frequently supercyclic vectors of T .

by definition, it is clear that frequently hypercyclic operators are frequently supercyclic.

Frequently supercyclic operators can be constructed on both infinite-dimensional and finite-dimensional spaces as it is shown in the next examples.

Example 2.1 By [11, Example 9.12], the differentiation operator D on the space of holomorphic operators on $\mathbb{C}$, $H(\mathbb{C})$, is frequently hypercyclic. So, it is frequently supercyclic.

Also, it is shown in [11, Example 9.15] that the Rolewicz 's operators $T = \lambda B$, where $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and B is the backward shift on l^p , $1 \leq p < \infty$, is frequently hypercyclic. So, it is frequently supercyclic, either.

Example 2.2 Consider $T : \mathbb{C} \to \mathbb{C}$ for any $x \in \mathbb{C}$ is defined with $Tx = \lambda x$, where $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. We state 1 is frequently supercyclic vector for T. For this, consider U is a non-empty open set of X. Let $y \in U$. Then $\frac{y}{\lambda} \in \mathbb{C}$ and

$$\frac{y}{\lambda} T1 = \frac{y}{\lambda}\lambda = y \in U.$$

Also, $\frac{y}{\lambda^2} \in \mathbb{C}$ and

$$\frac{y}{\lambda^2} T^2 1 = \frac{y}{\lambda^2}\lambda^2 = y \in U.$$

Similarly, for any $n \in \mathbb{N}$, $\frac{y}{\lambda^n} T^n 1 \in U$. Hence,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n 1 \in U; \lambda \in \mathbb{C}\} > 0.$$

T is not frequently hypercyclic. Since, frequently hypercyclic operators are hypercyclic and hypercyclic operators do not exist on finite-dimensional spaces [11, Proposition 7.12].

Lemma 2.1 *If T is a frequently supercyclic operator on X, then it is supercyclic.*

Proof: Let U be a non-empty open subset of X. Suppose $x \in X$ is a frequently supercyclic vector for T. Hence,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n x \in U; \lambda \in \mathbb{C}\} > 0. \tag{2.1}$$

Therefore, there is $n_0 \in \mathbb{N}_0$ and $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 T^{n_0} x \in U$. So, we can conclude that $\overline{\mathbb{C}orb(T, x)} = \overline{\{\lambda T^n x; n \in \mathbb{N}_0, \lambda \in \mathbb{C}\}} = X$. Hence, x is a supercyclic vector for T.

□

In the following lemmas we show that while an operator has a frequently supercyclic vector, it has many of them.

Lemma 2.2 *Let $T \in B(X)$. If x is a frequently supercyclic vector for T, then $T^m x$ is a frequently supercyclic vector for T for any $m \in \mathbb{N}$.*

Proof: Let U be a non-empty open subset of X. Hence, $T^{-m}(U)$ is an open set in X. By frequent supercyclicity of T,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n x \in T^{-m}(U); \lambda \in \mathbb{C}\} > 0.$$

Hence,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^{m+n} x \in U; \lambda \in \mathbb{C}\} > 0. \tag{2.2}$$

So by (2.2),

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n (T^m x) \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, $T^m x$ is a frequently supercyclic vector for T.

□

Lemma 2.3 *Let $T \in B(X)$. If x is a frequently supercyclic vector for T, then βx is a frequently supercyclic vector for T for any $\beta \in \mathbb{C}$ with $\beta \neq 0$.*

Proof: Let U be a non-empty open subset of X. Hence, $\beta^{-1}U$ is an open set in X. By frequent supercyclicity of T,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n x \in \beta^{-1}U; \lambda \in \mathbb{C}\} > 0.$$

Hence,

$$\underline{dens}\{n \in \mathbb{N}_0 : \beta\lambda T^n x \in U; \lambda \in \mathbb{C}\} > 0. \tag{2.3}$$

So by (2.3),

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n(\beta x) \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, βx is a frequently supercyclic vector for T.

□

By Lemma 2.2 and Lemma 2.3, we can conclude that when $x \in X$ is a frequently supercyclic vector for $T \in B(X)$, then $\beta T^m x$ is a frequently supercyclic vector for T, for any non-zero $\beta \in \mathbb{C}$ and any $m \in \mathbb{N}_0$.

We call an operator $S : Y \to Y$ is conjugate to $T : X \to X$ if a continuous map $\Phi : Y \to X$ with dense range exists such that $T \circ \Phi = \Phi \circ S$. In the next theorem, we investigate that frequent supercyclicity preserves under quasiconjugacy or not.

Theorem 2.1 *Frequent supercyclicity of operators preserves under quasiconjugacy.*

Proof: Let $S \in B(Y)$ be a frequently supercyclic operator. Suppose $T \in B(X)$ is quasiconjugate to S. So, a continuous map $\Phi : Y \to X$ with dense range exists such that

$$T \circ \Phi = \Phi \circ S. \tag{2.4}$$

Let x be a frequent supercyclic vector for S. Suppose U is a non-empty open set of X. $\Phi^{-1}(U)$ is open and non-empty. Also, by frequent supercyclicity of S,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda S^n x \in \Phi^{-1}(U); \lambda \in \mathbb{C}\} > 0.$$

So,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda \Phi \circ S^n x \in U\} > 0.$$

Hence by (2.4),

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n \circ \Phi x \in U\} > 0. \tag{2.5}$$

Therefore (2.5) implies that,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n \circ (\Phi x) \in U\} > 0.$$

Hence, Φx is a frequently supercyclic vector for T.

□

If T is frequently supercyclic, then any power of T is frequently supercyclic as we prove in the following theorem.

Theorem 2.2 *If $T \in B(X)$ is frequently supercyclic, then T^n is frequently supercyclic for any $n > 1$.*

Proof: Let x be a frequently supercyclic vector for T. Hence, x is a supercyclic vector for T. By [1, Theorem 2], x and Tx are supercyclic vectors for T^2.

Let $w \in X$ be an arbitrary element. Hence, there is $p, q \in \mathbb{N}$ and $\gamma, \mu \in \mathbb{C}$ such that

$$\|\gamma T^{2p}x - w\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\mu T^{2q+1}x - w\| < \frac{\varepsilon}{2}. \tag{2.6}$$

Let

$$A = \{n \in \mathbb{N}_0 : \|\lambda T^n x - x\| < \frac{\varepsilon}{2\max\{|\gamma|, |\mu|\}\max\{\|T^{2p}\|, \|T^{2q+1}\|\}}; \lambda \in \mathbb{C}\}. \tag{2.7}$$

By frequent supercyclicity of T, $\underline{dens}A > 0$. Consider $n \in A$. Hence by (2.6), there exists $\lambda \in \mathbb{C}$ such that

$$\|\lambda T^n x - x\| < \frac{\varepsilon}{2\max\{|\gamma|, |\mu|\}\max\{\|T^{2p}\|, \|T^{2q+1}\|\}}. \tag{2.8}$$

If $n = 2k$, then by (2.6) and (2.8),

$$\begin{aligned}\|\gamma\lambda T^{2(k+p)}x - w\| &\leq \|\gamma\lambda T^{2(k+p)}x - \gamma T^{2p}x\| + \|\gamma T^{2p}x - w\| \\ &\leq \|\gamma T^{2p}\|\|\lambda T^{2k}x - x\| + \|\gamma T^{2p}x - w\| \\ &< \varepsilon.\end{aligned} \tag{2.9}$$

If $n = 2k+1$, then by (2.6) and (2.8),

$$\begin{aligned}\|\lambda\mu T^{2(k+q+1)}x - w\| &\leq \|\lambda\mu T^{2(k+q+1)}x - \mu T^{2q+1}x + \mu T^{2q+1}x - w\| \\ &\leq \|\mu T^{2q+1}\|\|\lambda T^{2k+1}x - x\| + \|\mu T^{2q+1}x - w\| \\ &< \varepsilon.\end{aligned} \tag{2.10}$$

Consider $B = \{k + p; 2k \in A\} \cup \{k + p + 1 : 2k + 1 \in A\}$. If for some $\delta > 0$ and a natural number M, $card(A \cap \{0, 1, ..., N\}) \geq \delta M$ then

$$card(A \cap \{1, 3, ..., 2[\frac{M-1}{2}] + 1\}) \geq \frac{\delta M}{2} \tag{2.11}$$

or

$$card(A \cap \{0, 2, 4, ..., 2[\frac{M}{2}] + 1\}) \geq \frac{\delta M}{2} \tag{2.12}$$

If (2.11) occurs, then

$$card(B \cap \{1, 3, ..., 2[\frac{M-1}{2}] + 1\}) \geq \frac{\delta M}{2}$$

and if (2.12) occurs, then

$$card(B \cap \{0, 2, 4, ..., 2[\frac{M}{2}] + 1\}) \geq \frac{\delta M}{2}.$$

Hence, B has positive lower density too. So, we establish that T^2 is frequently supercyclic. Other powers are similarly proved.

□

Corollary 2.1 *Let $T \in B(X)$. Then T is frequently supercyclic if and only if T^n is frequently supercyclic for some $n \in \mathbb{N}$ with $n > 1$.*

Proof: Let T^p is frequently supercyclic for some $p \in \mathbb{N}$ with $p > 1$. Let $x \in X$ be a frequently supercyclic vector for T^p. If U is a non-empty open subset of X, then

$$\{n \in \mathbb{N}_0 : \lambda(T^p)^n x \in U; \lambda \in \mathbb{C}\} \subseteq \{n \in \mathbb{N}_0 : \lambda T^n x \in U; \lambda \in \mathbb{C}\} \tag{2.13}$$

The left set of (2.13) has positive lower density. So, the right set has positive lower density too.

□

It is stated in [11, Theorem 9.35] that hypercyclicity of T implies hypercyclicity of λT for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. In the next theorem, we prove this for frequently supercyclic operators for any λ with $\lambda \neq 0$. That means, if T is frequently supercyclic, then any non-zero multiple of T is frequently supercyclic as we prove in the following theorem.

Theorem 2.3 *If $T \in B(X)$ is frequently supercyclic operator, then αT is frequently supercyclic for any $\alpha \in \mathbb{C}$ with $\alpha \neq 0$.*
Moreover, $FSH(T) = FSH(\alpha T)$.

Proof: Let x be a frequently supercyclic vector for T. Fix $0 \neq \alpha \in \mathbb{C}$. Suppose U is a non-empty open set of X. By frequent supercyclicity of T,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda T^n x \in U, \lambda \in \mathbb{C}\} > 0.$$

But

$$\{n \in \mathbb{N}_0 : \lambda T^n x \in U, \lambda \in \mathbb{C}\} = \{n \in \mathbb{N}_0 : \lambda\alpha T^n x \in U, \lambda \in \mathbb{C}\}.$$

Hence,

$$\underline{dens}\{n \in \mathbb{N}_0 : \lambda\alpha T^n x \in U, \lambda \in \mathbb{C}\} > 0.$$

Hence, αT is frequently supercyclic.

□

Corollary 2.2 *If $T \in B(X)$ is frequently supercyclic operator, then αT^n is frequently supercyclic for any $\alpha \in \mathbb{C}$ with $\lambda \neq 0$ and any $n \in \mathbb{N}$.*

3. Frequently supercyclic C_0-semigroups

In this section, we extend the concept of frequent supercyclicity for C_0-semigrous. We begin with defininition of frequently supercyclic C_0-semigrous.

Definition 3.1 Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. We call $x \in X$ a frequently supercyclic vector for $(T_t)_{t\geq 0}$ if for any non-empty open subset U of X, the set $\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in U; \lambda \in \mathbb{C}\}$ has positive lower density.
Mind if $F \subseteq \mathbb{R}^{\geq 0}$, then the lower density of F is defined as follows

$$\underline{dens}(F) = \liminf_{p\to+\infty} \frac{card\{t \leq p : t \in F\}}{p}.$$

It is clear that frequently hypercyclic C_0-semigroups are frequently supercyclic.
Frequent supercyclicity implies supercyclicity, as it is proved in the next lemma.

Lemma 3.1 *Frequently supercyclicity of the C_0-semigroup $(T_t)_{t\geq 0}$ on X implies its supercyclicity.*

Proof: Let U be a non-empty open subset of X. Suppose $x \in X$ is a frequently supercyclic vector for $(T_t)_{t\geq 0}$. Hence,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, there is $t_0 \geq 0$ and $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 T_{t_0} x \in U$. So, we can conclude that $\mathbb{C}orb((T_t), x) = \{\lambda T_t x; t \geq 0, \lambda \in \mathbb{C}\}$ is dense in X. Hence, x is a supercyclic C_0-semigroup.

□

In the next lemmas we show that similar to Lemma 2.2 for operators, while a C_0-semigroup has a frequently supercyclic vector, it has many of them.

Lemma 3.2 *Consider $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Suppose x is a frequently supercyclic vector for $(T_t)_{t\geq 0}$. Then $T_s x$ for any $s > 0$, is a frequently supercyclic vector for T.*

Proof: Consider $s > 0$. Let U be an open subset of X. Hence, $T_s^{-1}(U)$ is an open set in X. By frequent supercyclicity of T,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in T_s^{-1}(U); \lambda \in \mathbb{C}\} > 0.$$

Hence,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_{t+s} x \in U; \lambda \in \mathbb{C}\} > 0.$$

So by definition of a C_0-semigroup,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t(T_s x) \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, $T_s x$ is a frequently supercyclic vector for $(T_t)_{t\geq 0}$.

□

Lemma 3.3 *Consider $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Suppose x is a frequently supercyclic vector for $(T_t)_{t\geq 0}$. Then βx for any $\beta \in \mathbb{C}$ with $\beta \neq 0$, is a frequently supercyclic vector for $(T_t)_{t\geq 0}$.*

Proof: Let U be an open subset of X. Hence, $\beta^{-1}U$ is an open set in X. By frequent supercyclicity of $(T_t)_{t\geq 0}$,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in \beta^{-1}U; \lambda \in \mathbb{C}\} > 0.$$

Hence,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \beta\lambda T_t x \in U; \lambda \in \mathbb{C}\} > 0. \tag{3.1}$$

So, we can deduce from (3.1) that

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t(\beta x) \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, βx is a frequently supercyclic vector for $(T_t)_{t\geq 0}$.

□

By Lemma 3.2 and Lemma 3.3, we can conclude that when $x \in X$ is a frequently supercyclic vector for $(T_t)_{t\geq 0}$, then $\beta T_s x$ is a frequently supercyclic vector for T, for any $\beta \in \mathbb{C}$ with $\beta \neq 0$ and any $m \geq 0$.

If a C_0-semigroup contains a frequently supercyclic operator, then the C_0-semigroup is frequently supercyclic, either.

Theorem 3.1 *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If T_s is a frequently supercyclic operator for some $s > 0$, then $(T_t)_{t\geq 0}$ is a frequently supercyclic C_0-semigroup on X.*

Proof: Let T_s be a frequently supercyclic operator for some $s > 0$. Let U be a non-empty open subset of X. So, by definition of the frequent supercyclicity,

$$\underline{dens}\{n \in \mathbb{N} : \lambda T_s^n x \in U; \lambda \in \mathbb{C}\} > 0. \tag{3.2}$$

But for any $p \in \mathbb{R}$ with $p > 0$,

$$\begin{aligned} & card\{n \in \mathbb{N} : n \leq p \text{ and } \lambda T_s^n x \in U; \lambda \in \mathbb{C}\} \\ \leq & \, card\{t \in \mathbb{R}^{\geq 0} : t \leq ps \text{ and } \lambda T_s x \in U; \lambda \in \mathbb{C}\}. \end{aligned} \tag{3.3}$$

By (3.3),

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in U; \lambda \in \mathbb{C}\} \geq \underline{dens}\{n \in \mathbb{N} : n \leq p \text{ and } \lambda T_s^n x \in U; \lambda \in \mathbb{C}\}.$$

Hence, by(3.2),

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t x \in U; \lambda \in \mathbb{C}\} > 0.$$

Therefore, $(T_t)_{t\geq 0}$ is a frequently supercyclic C_0-semigroup on X. □

The following example shows that frequently supercyclic operators can be made on finite-dimensional spaces.

Example 3.1 Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on $\mathbb{C}$ that is for any $x \in \mathbb{C}$ and any $t \geq 0$ is defined by

$$T_t x = \lambda_t x,$$

where $(\lambda_t) \subseteq \mathbb{C}$ such that $\lambda_0 = 1$, $\lambda_t \neq 0$ and $\lambda_s \lambda_t = \lambda_{s+t}$ for any $s \geq 0$ and $t \geq 0$.

It is shown in Example 2.2, for any $s > 0$, T_s is a frequently supercyclic operator. So, $(T_t)_{t\geq 0}$ is a frequently supercyclic C_0-semigroup by Theorem 3.1. But $(T_t)_{t\geq 0}$ is not frequently hypercyclic. Since frequently hypercyclic C_0-semigroups are hypercyclic and hypercyclic C_0-semigroups do not exist on finite-dimensional spaces [11, Theorem 7.15].

Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. suppose $(S_t)_{t\geq 0}$ is a C_0-semigroup on Y. Then $(T_t)_{t\geq 0}$ is called quasiconjugate to $(S_t)_{t\geq 0}$ if a continuous map $\Phi : Y \to X$ with dense range exists such that for any $t > 0$, $T_t \circ \Phi = \Phi \circ S_t$.

Theorem 3.2 *Frequent supercyclicity of C_0-semigroups preserves under quasiconjugacy.*

Proof: Let $(S_t)_{t\geq 0}$ be a frequently supercyclic C_0-semigroup on a Banach space Y. Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X quasiconjugate to $(S_t)_{t\geq 0}$. So, a continuous map $\Phi : Y \to X$ with dense range is exists such that for any $t > 0$,

$$T_t \circ \Phi = \Phi \circ S_t. \tag{3.4}$$

Let x be a frequent supercyclic vector for $(S_t)_{t\geq 0}$. Suppose U is an open and non-emptyset of X. $\Phi^{-1}(U)$ is open and non-empty. Also, by frequent supercyclicity of $(S_t)_{t\geq 0}$,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda S_t x \in \Phi^{-1}(U); \lambda \in \mathbb{C}\} > 0.$$

So,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda \Phi \circ S_t x \in U\} > 0. \tag{3.5}$$

Hence by (3.4) and (3.5),

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t \circ \Phi x \in U\} > 0.$$

Therefore,

$$\underline{dens}\{t \in \mathbb{R}^{\geq 0} : \lambda T_t \circ (\Phi x) \in U\} > 0.$$

Hence, Φx is a frequently supercyclic vector for $(T_t)_{t\geq 0}$.

□

It is proved in [11, Theorem 9.36] that frequently hypercyclicity of $(T_t)_{t\geq 0}$ implies that any of its operators is frequently hypercyclic and the set of their frequently hypercyclic vectors are the same. Now, we want to prove similar theorem for frequent supercyclic C_0-semigroups. First, we need the next lemma. The proof of the following lemma is similar to Lemma 3.1 in [9]. By $FSC((T_t)_{t\geq 0})$, we mean the set of frequently supercyclic vectors of $(T_t)_{t\geq 0}$. Also, $\mathcal{U}_0$ denotes a base of open and balanced neighbourhoods for zero.

Lemma 3.4 *Let $(T_t)_{t\geq 0}$ be a frequently supercyclic C_0-semigroup on X. Suppose x is a frequently supercyclic vector for $(T_t)_{t\geq 0}$. Then for any $y \in X$, $m \in \mathbb{N}$, and $U \in \mathcal{U}_0$,*

$$\underline{dens}\{t \in \cup_{n\in\mathbb{N}}[\frac{n-1}{m}, n) : \lambda T_t x - y \in U\} > 0.$$

Theorem 3.3 *If $(T_t)_{t\geq 0}$ is a frequently supercyclic C_0-semigroup on X, then T_s is frequently supercyclic operator for any $s > 0$.*

Moreover, for any $s > 0$, $FSC((T_t)_{t\geq 0}) = FSC(T_s)$.

Proof: Suppose $s > 0$. If x is a frequent supercyclic vector for T_s, then it is easy to see that x is a frequently supercyclic vector for $(T_t)_{t\geq 0}$.

Suppose that x is a frequently supercyclic vector for $(T_t)_{t\geq 0}$. Suppose $y \in X$, W is an open and balanced neighbourhood of zero. Without loss of generality, we can assume that $s = 1$. Let $p \in \mathbb{N}$, W' is an open and balanced neighbourhood of zero and $\lambda \in \mathbb{C}$ be such that $W' + W' \subseteq W$ and $\lambda T_t y - y \in W'$ for any $0 \leq t \leq \frac{1}{p}$. Also, by properties of a C_0-semigroup, there exists $V \in \mathcal{U}_0$ such that $T_t(V) \subseteq W'$ for any $0 \leq t \leq \frac{1}{p}$. By Lemma 3.4,

$$\underline{dens}\{t \in \cup_{n\in\mathbb{N}}[\frac{n-1}{m}, n) : \lambda T_t x - y \in U\} > 0. \tag{3.6}$$

Suppose for some $n \in \mathbb{N}$, $t \in [\frac{n-1}{p}, n)$ and suppose $\lambda T_t x - y \in V$. consider $\alpha_t := [t] + 1 - t$. Hence, $0 < \alpha_t \leq \frac{1}{k}$ and $t + \alpha_t \in \mathbb{N}$ for any α_t. Therefore,

$$\lambda T_{t+\alpha_t} x - y = T_{\alpha_t}(\lambda T_t x - y) + (T_{\alpha_t} y - y) \in T_{\alpha_t}(V) + W' \subseteq W.$$

So,

$$\begin{aligned} &\underline{dens}(\{n \in \mathbb{N}_0 : \lambda T_n x - y \in U; \lambda \in \mathbb{C}\}) \qquad (3.7)\\ &\geq \underline{dens}\{t \in \cup_{n\in\mathbb{N}}[\frac{n-1}{m}, n) : \lambda T_t x - y \in U\}. \end{aligned}$$

By (3.6) and (3.7), $\underline{dens}(\{n \in \mathbb{N}_0 : \lambda T_n x - y \in U; \lambda \in \mathbb{C}\}) > 0$. Hence, x is a frequently supercyclic vector for T_s. □

By Theorem 3.3 we state the following equivalent condition.

Corollary 3.1 *Consider $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. The following statements are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is frequently supercyclic.*

(ii) *For any $t \geq 0$, T_t is a frequently supercyclic operator.*

(iii) For some $t \geq 0$ T_t is a frequently supercyclic operator.

By Theorem 3.3 and Corollary 3.1, we can state some equivalent conditions between a frequent supercyclic C_0-semigroup and its discretization and its autonomous discretization as follows.

Theorem 3.4 *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then the following statements are equivalent:*

(i) $(T_t)_{t\geq 0}$ is a frequently supercyclic C_0-semigroup.

(ii) Any discretization of $(T_t)_{t\geq 0}$ is frequently supercyclic.

(iii) Any autonomous discretization of $(T_t)_{t\geq 0}$ is frequently supercyclic.

(iv) Some autonomous discretization of $(T_t)_{t\geq 0}$ is frequently supercyclic.

Proof: $(ii) \to (iii)$ is clear. Also, $(iii) \to (iv)$ is obvious.

For $(i) \to (ii)$, let (t_n) be an increasing sequence such that $t_n \to \infty$. By (i), $(T_t)_{t\geq 0}$ is frequently supercyclic. So, by Theorem 3.3, T_t is frequently supercyclic for any $t > 0$. Especially, T_{t_1} is frequently supercyclic. Now, by Theorem 3.1, $(T_{t_n})_n$ is frequently supercyclic.

For $(iv) \to (i)$, assume there is $s > 0$ such that $(T_{t_n})_n = (T_{ns})_n$ is frequently supercyclic. Assume x is a frequently supercyclic vector for $(T_{ns})_n$. Let U be an open subset of X. Hence,

$$\underline{dens}(\{n \in \mathbb{N}_0 : \lambda T_{ns} x \in U; \lambda \in \mathbb{C}\}) > 0.$$

So by properties of a C_0-semigroup,

$$\underline{dens}(\{n \in \mathbb{N}_0 : \lambda T_s^n x \in U; \lambda \in \mathbb{C}\}) > 0.$$

Therefore, T_s is frequently supercyclic and by Theorem 3.1, $(T_t)_{t\geq 0}$ is frequently supercyclic.

□

Using the proof of Theorem 3.4, we can conclude the next corollary about frequently supercyclic vectors of a C_0-semigroup.

Corollary 3.2 *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then the following are equivalent:*

(i) x is a frequently supercyclic vector for the C_0-semigroup $(T_t)_{t\geq 0}$.

(ii) x is a frequently supercyclic vector for any discretization of $(T_t)_{t\geq 0}$ is frequently supercyclic.

(iii) x is a frequently supercyclic vector for any autonomous discretization of $(T_t)_{t\geq 0}$.

(iv) x is a frequently supercyclic vector for some autonomous discretization of $(T_t)_{t\geq 0}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Funding Statement

This study did not receive funding from any sources.

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