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On solutions of a two-dimensional $(m+1)$ −order system of difference equations via Pell numbers

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ABSTRACT: In this paper, we are interested in the closed-form solution of the following two-dimensional system of difference equations of $(m+1)$ −order,

$$x_{n+1}^{(i)} = \frac{1}{14 + x_{n-m}^{(i+1) \bmod 2}}, i \in \{1,2\}, n, m \in \mathbb{N}_0,$$

and the initial values $x_{-j}^{(i)}$, $i \in \{1,2\}$, $j \in \{0,1,...,m\}$ are real numbers do not equal $-1/14$. We show that the solutions of this system are associated with Pell numbers and some other numbers. It is shown that the global stability of positive solutions of this system holds. Our results are illustrated via numerical examples.

Key Words: Stability, Pell numbers, balancing numbers, Pell-Lucas numbers, Binet formula, system of difference equations.

Contents

1. Introduction

Despite the tremendous development of difference equations, it remains to construct solutions to them explicitly useful (see., [1] − [24], [26] − [34]). In particular, great importance is given to the difference equations related to Fibonacci, Bell, Jacobthal, Padovan, Lucas and their generalizations. This led to the emergence of a new problem, which is how to turn nonlinear difference equations or systems into linear difference equations or systems, while homogeneous linear difference equations of the 2nd-order are well established and very common equations due to their many applications, which have the following form

$$x_{n+1} = \alpha x_n + \beta x_{n-1}, n \geq 1,$$

where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$ such that $\beta \neq 0$, in particular, we give information about the Pell sequence that establishes a significant part of our study, defined as follows

$$P_{n+1} = 2P_n + P_{n-1},\ n \geq 1,$$

with initial conditions $P_0 = 0$ and $P_1 = 1$. The following Binet formula of the Pell numbers gives, $P_n = (a^n - b^n) / (a - b)$, where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$, and the closed-form expression for the Pell-Lucas numbers are $Q_n = a^n + b^n$. Now, in this paper, we seek to provide a class of system of nonlinear difference equations which can be solved in explicit form, but the solutions are expressed by Pell numbers, is the following two-dimensional system of difference equations,

$$x_{n+1}^{(i)} = \frac{1}{14 + x_{n-m}^{(i+1) \bmod 2}}, i \in \{1,2\}, n, m \in \mathbb{N}_0, \tag{1.1}$$

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and the initial values $x_{-j}^{(i)}$, $i \in \{1,2\}$, $j \in \{0,1,...,m\}$ are real numbers do not equal $-1/14$.

2. Main results

To solve system (2.5) we require to utilize the following lemmas.

Lemma 2.1 *Consider the homogeneous linear difference equation with constant coefficients*

$$y_{n+1} - 14y_n - y_{n-1} = 0, n \geq 0, \tag{2.1}$$

with initial conditions y_0, $y_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ y_n = \frac{y_0}{5} P_{3(n+1)} + \frac{y_{-1}}{5} P_{3n},$$

where $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. Difference equation (2.1) is ordinarily solved by using the following characteristic polynomial,

$$\lambda^2 - 14\lambda - 1 = 0,$$

roots of this equation are

$$\lambda_1 = 7 + 5\sqrt{2} = a^3, \lambda_2 = 7 - 5\sqrt{2} = b^3.$$

These roots are linked to the roots of the Pell number sequence. Then the closed form of general solution of the equation (2.1) is

$$\forall n \geq -1,\ y_n = c_1 a^{3n} + c_2 b^{3n},$$

where y_0, y_{-1} are initial values such that

$$\begin{cases} y_0 = c_1 + c_2 \\ y_{-1} = \dfrac{c_1}{a^3} + \dfrac{c_2}{b^3} \end{cases},$$

and we have

$$c_1 = \frac{a^3 y_0 + y_{-1}}{10\sqrt{2}}, c_2 = -\frac{b^3 y_0 + y_{-1}}{10\sqrt{2}},$$

after some calculations, we get

$$y_n = \frac{y_0}{5}\left(\frac{a^{3(n+1)} - b^{3(n+1)}}{a-b}\right) + \frac{y_{-1}}{5}\left(\frac{a^{3n} - b^{3n}}{a-b}\right).$$

The lemma is proved.□

Lemma 2.2 *Consider the homogeneous linear difference equation with constant coefficients*

$$z_{n+1} + 14z_n - z_{n-1} = 0, n \geq 0, \tag{2.2}$$

with initial conditions z_0, $z_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ z_n = (-1)^n \left(\frac{z_0}{5} P_{3(n+1)} - \frac{z_{-1}}{5} P_{3n}\right),$$

where $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. Difference equation (2.2) is ordinarily solved by using the following characteristic polynomial, $\lambda^2 + 14\lambda - 1 = 0$, roots of this equation are

$$\lambda_1 = -a^3, \lambda_2 = -b^3.$$

These roots are linked to the roots of the Pell number sequence. Then the closed form of general solution of the equation (2.2) is

$$\forall n \geq -1,\ z_n = (-1)^n \left(\widetilde{c}_1 a^{3n} + \widetilde{c}_2 b^{3n}\right),$$

where z_0, z_{-1} are initial values such that

$$\begin{cases} z_0 = \widetilde{c}_1 + \widetilde{c}_2 \\ z_{-1} = -\dfrac{\widetilde{c}_1}{a^3} - \dfrac{\widetilde{c}_2}{b^3} \end{cases},$$

and we have

$$\widetilde{c}_1 = \frac{a^3 z_0 - z_{-1}}{10\sqrt{2}}, \widetilde{c}_2 = \frac{-b^3 z_0 + z_{-1}}{10\sqrt{2}},$$

after some calculations, we get

$$\begin{aligned} z_n &= (-1)^n \left(a\left(\delta_0 + \delta_{-1}\right) a^n + \left(-b\delta_0 - a\delta_{-1}\right) b^{3n}\right) \\ &= (-1)^n \left(\frac{z_0}{5}\left(\frac{a^{3(n+1)} - b^{3(n+1)}}{a-b}\right) - \frac{z_{-1}}{5}\left(\frac{a^{3n} - b^{3n}}{a-b}\right)\right). \end{aligned}$$

The lemma is proved.□

Lemma 2.3 *Consider the following system of difference equations*

$$t_{n+1}^{(i)} = 14 t_n^{(i+1) \bmod 2} + t_{n-1}^{(i)},\ i \in \{1,2\},\ n \geq 0, \tag{2.3}$$

with initial conditions $t_{-j}^{(i)} \in \mathbb{R},\ i \in \{1,2\},\ j \in \{0,1\}$. *Then,*

$$t_{2n}^{(i)} = \tfrac{1}{5} t_0^{(i)} P_{3(2n+1)} + \tfrac{1}{5} t_{-1}^{(i+1) \bmod 2} P_{6n}, \quad t_{2n+1}^{(i)} = \tfrac{1}{5} t_0^{(i+1) \bmod 2} P_{6(n+1)} + \tfrac{1}{5} t_{-1}^{(i)} P_{3(2n+1)},$$

for $i \in \{1,2\}$, $n \geq 0$.

Proof. From system (2.3), we get the following system

$$\begin{cases} t_{n+1}^{(1)} + t_{n+1}^{(2)} = 14\left(t_n^{(1)} + t_n^{(2)}\right) + \left(t_{n-1}^{(1)} + t_{n-1}^{(2)}\right) \\ t_{n+1}^{(1)} - t_{n+1}^{(2)} = -14\left(t_n^{(1)} - t_n^{(2)}\right) + \left(t_{n-1}^{(1)} - t_{n-1}^{(2)}\right) \end{cases}, n \geq 0, \tag{2.4}$$

Using the change of variables $y_n = t_n^{(1)} + t_n^{(2)}$ and $z_n = t_n^{(1)} - t_n^{(2)}$, we can write (2.4) as

$$\begin{cases} y_{n+1} = 14 y_n + y_{n-1} \\ z_{n+1} = -14 z_n + 2 z_{n-1} \end{cases}, n \geq 0,$$

by Lemmas 2.1 − 2.2, we have

$$\begin{aligned} &\forall n \geq 0,\ y_n = \frac{y_0}{5} P_{3(n+1)} + \frac{y_{-1}}{5} P_{3n}, \\ &\forall n \geq 0,\ z_n = (-1)^n \left(\frac{z_0}{5} P_{3(n+1)} - \frac{z_{-1}}{5} P_{3n}\right), \end{aligned}$$

hence, the closed form of general solution of the system (2.3) is $\left(t_n^{(1)}, t_n^{(2)}\right) = \left(\left(y_n + z_n\right)/2\,, \left(y_n - z_n\right)/2\right)$, $n \geq 0$. The lemma is proved.□

2.1. On the system (2.5)

In this subsection, we consider the following system of difference equations of 1st-order,

$$x_{n+1}^{(i)} = \frac{1}{14 + x_n^{(i+1) \bmod 2}}, i \in \{1, 2\}, n \in \mathbb{N}_0. \tag{2.5}$$

To find the closed form of the solutions of the system (2.5) we consider the following change variables

$$x_n^{(i)} = \frac{t_{n-1}^{(i+1) \bmod 2}}{t_n^{(i)}}, \ i \in \{1, 2\}, n \geq 0,$$

then the system (2.5) becomes

$$t_{n+1}^{(i)} = 14 t_n^{(i+1) \bmod 2} + t_{n-1}^{(i)}, \ i \in \{1, 2\}, n \geq 0.$$

By Lemma 2.3, the closed form of general solution of the equation (2.5) is easily obtained, in the following Theorem

Theorem 2.1 *Let $\left\{x_n^{(1)}, x_n^{(2)}, n \geq 0\right\}$ be a solution of equation (2.5). Then,*

$$x_{2n}^{(i)} = \frac{P_{6n} + x_0^{(i)} P_{3(2n-1)}}{P_{3(2n+1)} + x_0^{(i)} P_{6n}}, \ i \in \{1, 2\},$$
$$x_{2n+1}^{(i)} = \frac{P_{3(2n+1)} + x_0^{(i+1) \bmod 2} P_{6n}}{P_{6(n+1)} + x_0^{(i+1) \bmod 2} P_{3(2n+1)}}, \ i \in \{1, 2\},$$

where $(P_n, n \geq 0)$ is the Pell sequence.

Proof. Straightforward and hence omitted.□

2.2. On the system (1.1)

In this paper, we study the System (1.1), which is an extension of System (2.5). Therefore, the System (1.1) can be written as follows

$$x_{(m+1)(n+1)-t}^{(i)} = \frac{1}{14 + x_{(m+1)n-t}^{(i+1) \bmod 2}}, i \in \{1, 2\},$$

for $t \in \{0, 1, ..., m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$x_{n,t}^{(i)} = x_{(m+1)n-t}^{(i)}, i \in \{1, 2\}, t \in \{0, 1, ..., m\},$$

we can get $(m+1)$ −systems similar to System (2.5),

$$x_{n+1,t}^{(i)} = \frac{1}{14 + x_{n,t}^{(i+1) \bmod 2}}, i \in \{1, 2\}, n \in \mathbb{N}_0,$$

for $t \in \{0, 1, ..., m\}$. Through the above discussion, we can introduce the following Theorem

Theorem 2.2 *Let $\left\{x_n^{(1)}, x_n^{(2)}, n \geq -m\right\}$ be a solution of equation (1.1). Then, for $t \in \{0, 1, ..., m\}$,*

$$x_{2(m+1)n-t}^{(i)} = \frac{P_{6n} + x_{-t}^{(i)} P_{3(2n-1)}}{P_{3(2n+1)} + x_{-t}^{(i)} P_{6n}}, i \in \{1, 2\},$$
$$x_{(m+1)(2n+1)-t}^{(i)} = \frac{P_{3(2n+1)} + x_{-t}^{(i+1) \bmod 2} P_{6n}}{P_{6(n+1)} + x_{-t}^{(i+1) \bmod 2} P_{3(2n+1)}}, i \in \{1, 2\},$$

where $(P_n, n \geq 0)$ is the Pell sequence.

Proof. The proof of Theorem 2.2 is based on Theorem 2.1 for $(m+1)$ −systems (1.1).□

Corollary 2.1 *Let* $\left\{x_n^{(1)}, x_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1)*. Then, for* $t \in \{0,1,...,m\}$,

$$x_{2(m+1)n-t}^{(i)} = \frac{Q_{6n+1}+Q_{6n-1}+x_{-t}^{(i)}\left(Q_{3(2n-1)+1}+Q_{3(2n-1)-1}\right)}{Q_{3(2n+1)+1}+Q_{3(2n+1)-1}+x_{-t}^{(i)}\left(Q_{6n+1}+Q_{6n-1}\right)},\ i \in \{1,2\},$$
$$x_{(m+1)(2n+1)-t}^{(i)} = \frac{Q_{3(2n+1)+1}+Q_{3(2n+1)-1}+x_{-t}^{(i+1) \bmod 2}\left(Q_{6n+1}+Q_{6n-1}\right)}{Q_{6(n+1)+1}+Q_{6(n+1)-1}+x_{-t}^{(i+1) \bmod 2}\left(Q_{3(2n+1)+1}+Q_{3(2n+1)-1}\right)},\ i \in \{1,2\},$$

where $(Q_n, n \geq 0)$ *is the Pell-Lucas sequence.*

Proof. We see that it suffices to remark

$$8P_n = Q_{n+1}+Q_{n-1},\ \text{(see., [25])}.\square$$

Corollary 2.2 *Let* $\left\{x_n^{(1)}, x_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1)*. Then, for* $t \in \{0,1,...,m\}$,

$$x_{2(m+1)n-t}^{(i)} = \frac{2P_{3n}Q_{3n}+x_{-t}^{(i)}\left(2P_{3n-1}Q_{3n-1}-Q_{3(2n-1)}\right)}{2P_{3n+2}Q_{3n+2}-Q_{3(2n+1)}+2x_{-t}^{(i)}P_{3n}Q_{3n}},\ i \in \{1,2\},$$
$$x_{(m+1)(2n+1)-t}^{(i)} = \frac{2P_{3n+2}Q_{3n+2}-Q_{3(2n+1)}+2x_{-t}^{(i+1) \bmod 2}P_{3n}Q_{3n}}{2P_{3(n+1)}Q_{3(n+1)}+x_{-t}^{(i+1) \bmod 2}\left(2P_{3n+2}Q_{3n+2}-Q_{3(2n+1)}\right)},\ i \in \{1,2\},$$

where $(P_n, n \geq 0)$ *is the Pell sequence and* $(Q_n, n \geq 0)$ *is the Pell-Lucas sequence.*

Proof. We see that it suffices to remark

$$P_{2n} = P_nQ_n \text{ and } 2P_{2n+1} = 2P_{n+1}Q_{n+1}-Q_{2n+1} \text{ (see., [25])}.\square$$

Corollary 2.3 *Let* $\left\{x_n^{(1)}, x_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1)*. Then, for* $t \in \{0,1,...,m\}$,

$$x_{2(m+1)n-t}^{(i)} = \frac{B_{12n}+x_{-t}^{(i)}B_{6(2n-1)}}{B_{6(2n+1)}+x_{-t}^{(i)}B_{12n}},\ i \in \{1,2\},$$
$$x_{(m+1)(2n+1)-t}^{(i)} = \frac{B_{6(2n+1)}+x_{-t}^{(i+1) \bmod 2}B_{12n}}{B_{12(n+1)}+x_{-t}^{(i+1) \bmod 2}B_{6(2n+1)}},\ i \in \{1,2\},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence.*

Proof. We see that it suffices to remark

$$P_n = 2B_{2n},\ \text{(see., [25])}.\square$$

Remark 2.1 *There are many systems whose solutions can be expressed by Pell, Balancing and Pell-Lucas numbers, which are*

$$x_{n+1}^{(i)} = \frac{1}{\theta_l + x_{n-m}^{(i+1) \bmod 2}}, i \in \{1,2\}, n, m \in \mathbb{N}_0, l \geq 1,$$

where $\theta_l = a^{3l}+b^{3l} \in \{14; ...\}, l \geq 1$*. Using the results of Theorem* 2.2*, we get*

$$x_{2(m+1)n-t}^{(i)} = \frac{P_{6ln}+x_{-t}^{(i)}P_{3l(2n-1)}}{P_{3l(2n+1)}+x_{-t}^{(i)}P_{6ln}}, i \in \{1,2\},$$
$$x_{(m+1)(2n+1)-t}^{(i)} = \frac{P_{3l(2n+1)}+x_{-t}^{(i+1) \bmod 2}P_{6ln}}{P_{6l(n+1)}+x_{-t}^{(i+1) \bmod 2}P_{3l(2n+1)}}, i \in \{1,2\},$$

for $t \in \{0,1,...,m\}$.

3. Global stability of positive solutions of (1.1)

In the following, we will study the global stability character of the solutions of system (1.1). Obviously, the positive equilibriums of system (1.1) are

$$E=\left(\overline{x}^{(1)},\overline{x}^{(2)}\right)=-b^{3}\left(1,1\right),\overline{E}=-a^{3}\left(1,1\right)\ .$$

Let the functions $h_1,h_2:(0,+\infty)^{2(m+1)}\rightarrow(0,+\infty)$ defined by

$$h_i\left(\left(\underline{x}_{0:m}^{(1)}\right)',\left(\underline{x}_{0:m}^{(2)}\right)'\right)=\frac{1}{14+x_{n-m}^{(i+1)\bmod 2}},i\in\{1,2\},$$

where $\underline{u}_{0:m}=(u_0,u_1,...,u_m)'$. Now, it is usually useful to linearize the system (1.1) around the equilibrium point E in order to facilitate its study. For this purpose, introducing the vectors $\underline{X}_n':=\left(\left(\underline{X}_n^{(1)}\right)',\left(\underline{X}_n^{(2)}\right)'\right)$ where $\underline{X}_n^{(i)}=\left(x_n^{(i)},x_{n-1}^{(i)},...,x_{n-m}^{(i)}\right)'$ $,i\in\{1,2\}$. With these notations, we obtain the following representation

$$\underline{X}_{n+1}=F_m\underline{X}_n, \tag{3.1}$$

where

$$F_m=\left(\begin{array}{llll}\underline{O}'_{(m-1)} & 0 & \underline{O}'_{(m-1)} & -a^{-6}\\ I_{(m-1)} & \underline{O}_{(m-1)} & O_{(m-1)} & \underline{O}_{(m-1)}\\ \underline{O}'_{(m-1)} & -a^{-6} & O_{(m-1)} & 0\\ O_{(m-1)} & \underline{O}_{(m-1)} & I_{(m-1)} & \underline{O}_{(m-1)}\end{array}\right),$$

with $O_{(k,l)}$ denotes the matrix of order $k\times l$ whose entries are zeros, for simplicity, we set $O_{(k)}:=O_{(k,k)}$ and $\underline{O}_{(k)}:=O_{(k,1)}$ and $I_{(m)}$ is the $m\times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 3.1 *The positive equilibrium point E is locally asymptotically stable.*

Proof. After some preliminary calculations, the characteristic polynomial of F_m is

$$P_{F_m}(\lambda)=\det\left(F_m-\lambda I_{(2(m+1))}\right)=\Lambda_1(\lambda)-\Lambda_2(\lambda),$$

where $\Lambda_1(\lambda)=\lambda^{2(m+1)}$ and $\Lambda_2(\lambda)=a^{-12}$, then $|\Lambda_2(\lambda)|<|\Lambda_1(\lambda)|,\forall\lambda:|\lambda|=1$. So, according to Rouche's Theorem, all zeros of $\Lambda_1(\lambda)-\Lambda_2(\lambda)=0$ lie in the unit disc $|\lambda|<1$. Thus, the positive equilibrium point E is locally asymptotically stable.□

Corollary 3.1 *For every well defined solution of system (1.1), we have $\lim x_n^{(1)}=\lim x_n^{(2)}=-b^3$.*

Proof. From Theorem 2.2, we have

$$\begin{aligned}\lim x_{2(m+1)n-t}^{(i)}&=\lim\frac{P_{6n}+x_{-t}^{(i)}P_{3(2n-1)}}{P_{3(2n+1)}+x_{-t}^{(i)}P_{6n}}\\&=\lim\frac{1+x_{-t}^{(i)}\frac{P_{3(2n-1)}}{P_{6n}}}{\frac{P_{3(2n+1)}}{P_{6n}}+x_{-t}^{(i)}}\\&=\frac{1+x_{-t}^{(i)}a^{-3}}{a^3+x_{-t}^{(i)}}\\&=a^{-3}=-b^3,\end{aligned}$$

$$\begin{aligned}\lim x^{(i)}_{(m+1)(2n+1)-t} &= \lim \frac{P_{3(2n+1)} + x_{-t}^{(i+1) \bmod 2} P_{6n}}{P_{6(n+1)} + x_{-t}^{(i+1) \bmod 2} P_{3(2n+1)}} \\ &= \lim \frac{1 + x_{-t}^{(i+1) \bmod 2} \frac{P_{6n}}{P_{3(2n+1)}}}{\frac{P_{6(n+1)}}{P_{3(2n+1)}} + x_{-t}^{(i+1) \bmod 2}} \\ &= \frac{1 + x_{-t}^{(i+1) \bmod 2} a^{-3}}{a^3 + x_{-t}^{(i+1) \bmod 2}} \\ &= a^{-3} = -b^3,\end{aligned}$$

which completes the proof of Corollary 3.1.□

The following result is an immediate consequence of Theorem 3.1 and Corollary 3.1.

Corollary 3.2 *The positive equilibrium point E is globally asymptotically stable.*

4. Numerical Examples

In order to clarify and shore up the theoretical results of the previous section, we consider some interesting numerical examples in this section.

Example 4.1 *We consider interesting numerical example for the difference equations system* (1.1) *when $m = 1$ with the initial conditions $x^{(1)}_{-1} = 0.3$, $x^{(1)}_0 = 2.2$, $x^{(2)}_{-1} = 1.2$ and $x^{(2)}_0 = -2.1$. The plot of the solutions is shown in Figure 1.*

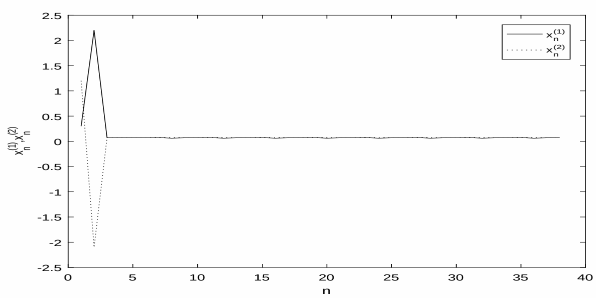

Figure 1. The plot of the solutions of system (1.1); when $m = 1$ and we put the initial conditions $x^{(1)}_{-1} = 0.3, x^{(1)}_0 = 2.2, x^{(2)}_{-1} = 1.2$ and $x^{(2)}_0 = -2.1$.

Example 4.2 *We consider interesting numerical example for the difference equations system* (1.1) *when $m = 2$ with the initial conditions*

i	0	1	2
$x^{(1)}_{-i}$	-3	2	-1
$x^{(2)}_{-i}$	2	-1	-0.1

Table 1. The initial conditions.

The plot of the solutions is shown in Figure 2.

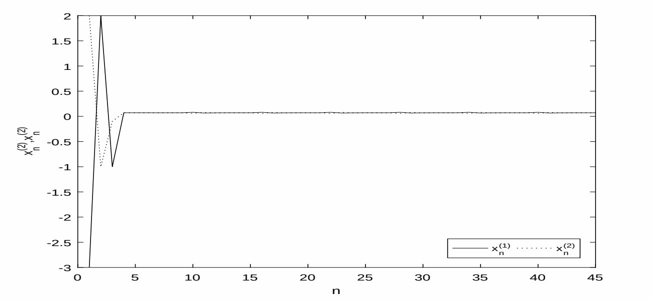

Figure 2. The plot of the solutions of system (1.1); when we put the initial conditions in Table 1.

Example 4.3 *We consider interesting numerical example for the difference equations system* (1.1) *when* $m = 3$ *with the initial conditions*

i	0	1	2	3
$x_{-i}^{(1)}$	0.3	0.1	0.2	0.2
$x_{-i}^{(2)}$	-0.4	0.3	-0.1	0.2

Table 2. The initial conditions.

The plot of the solutions is shown in Figure 3.

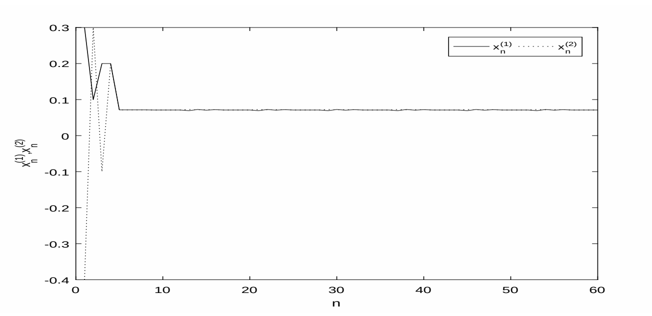

Figure 3. The plot of the solutions of system (1.1); when we put the initial conditions in Table 2.

In these examples, we show that the solutions of the system (1.1) for some cases are globally asymptotically stable.

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Conflicts of Interest

The corresponding author declares no conflict of interest.

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