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Korovkin-Type Approximation Theorems for Positive Linear Operators via Statistical Martingale Sequences

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ABSTRACT: In this paper, we introduce and study the notions of statistical product convergence and statistical product summability via deferred Cesàro and deferred Nörlund product means for martingale sequences of random variables. After that, we establish an inclusion theorem concerning the relation between these two lovely and potentially useful notions. We also describe and prove a set of new Korovkin-type approximation theorems for a martingale sequence over a Banach space based on the principles we have put forward. Additionally, we demonstrate that most (if not all) of the prior findings both in statistical and classical forms can be improved upon by our approximation theorems. Finally, we offer an example of martingale sequence by using the generalized Bernstein polynomials to show that our established theorems are significantly stronger than the traditional and statistical versions of several theorems already exist in the literature.

Key Words: Stochastic sequences, martingale sequences, statistical convergence of the product mean, Korovkin-type approximation theorems, Bernstein polynomials, positive linear operators.

Contents

1. Introduction and Motivation

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability measurable space, and let (Y_k) is a random variable defined over this space. Suppose that $\mathbb{F}_k \subseteq \mathbb{F}$ $(k \in \mathbb{N})$ be a monotonically increasing sequence of σ-fields of measurable sets. Now, considering the random variable (Y_k) with respect to measurable functions $(\mathbb{F}_k)$, we adopt a stochastic sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$.

A given stochastic sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ is said to be a martingale sequence, if

(i) $\mathbb{E}|Y_k| < \infty$,

(ii) $\mathbb{E}(Y_{k+1}|\mathbb{F}_k) = Y_k$ almost surely (a.s.) and

(iii) $(\mathbb{F}_k)$ is a measurable sequence of functions,

where $\mathbb{E}$ is the mathematical expectation.

We now recall the notion of convergence of martingale sequences of random variables.

A martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ with $\mathbb{E}|Y_k|$ is bounded and $\text{Prob}(Y_k) = 1$ (that is, with probability 1) is said to be convergent to a martingale $(Y_0, \mathbb{F}_0)$, if

$$\lim_{k\to\infty} (Y_k, \mathbb{F}_k) \longrightarrow (Y_0, \mathbb{F}_0) \qquad (\mathbb{E}|Y_0| < \infty).$$

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The concept of statistical convergence has recently emerged as one of the most important ideas in both the theory of sequence spaces and the theory of summability. The investigation and study of statistical convergence are presently very useful in sequence space because it is quite more powerful than the classical convergence. Two renowned mathematicians, Fast [6] and Steinhaus [29], separately offered such a lovely idea for the first time. As a result, various researchers developed many interesting and practical results in a variety of mathematical areas, such as Fourier series, Approximation theory, Probability theory, Machine Learning, Signal Processing, Measure theory, and so on. Moreover, the introduction of statistical probability convergence has enhanced the splendour of this development. For some recent research works in this direction, see [2], [3], [4], [7], [9], [10], [12], [13], [19], [20], [23], [24], [26] and [31].

Let $\mathfrak{G} \subseteq \mathbb{N}$ and, let

$$\mathfrak{G}_k = \{\lambda : \lambda \leqq k \quad \text{and} \quad \lambda \in \mathfrak{G}\} \qquad (k \in \mathbb{N}).$$

Then $\delta(\mathfrak{G})$ is the natural density of $\mathfrak{G}$, defined by

$$\delta(\mathfrak{G}) = \lim_{k\to\infty} \frac{|\mathfrak{G}_k|}{n} = a,$$

where a is a real finite number and $|\mathfrak{G}_k|$ is the cardinality of $\mathfrak{G}_k$.

A given sequence (x_k) is statistically convergent to μ if, for each $\epsilon > 0$

$$\mathfrak{G}_\epsilon = \{\lambda : \lambda \leqq k \quad \text{and} \quad |x_\lambda - b| \geqq \epsilon\} \qquad (k \in \mathbb{N})$$

has zero natural density (see [6] and [29]). Thus, for each $\epsilon > 0$, we have

$$\delta(\mathfrak{G}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{G}_\epsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} x_k = b.$$

The definition of statistical convergence of the martingale sequence is now given below.

Definition 1.1 *A bounded martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ having its probability 1 is said to be statistically convergent to a martingale $(Y_0, \mathbb{F}_0)$ with $\mathbb{E}|Y_0| < \infty$ if, for all $\epsilon > 0$, we have*

$$\mathfrak{G}_\epsilon = \{\lambda : \lambda \leqq k \quad \textit{and} \quad |(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, we have

$$\delta(\mathfrak{G}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{G}_\epsilon|}{k} = 0.$$

We write

$$stat_{\text{mart}} \lim_{k\to\infty} (Y_k, \mathbb{F}_k) = (Y_0, \mathbb{F}_0).$$

Example 1.1 *Let $(\mathbb{F}_k, k \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables over σ-fields. Also, let $(Y_k) \in \mathbb{F}_k$ be such that*

$$Y_k = \begin{cases} 1 & (k = 2^m;\ m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that the martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ is statistically convergent to zero, but not simply martingale convergent.

Motivated by the aforementioned developments, we investigate the concepts of statistical product convergence and statistical product summability via deferred Cesàro and deferred Nörlund product means for martingale sequences of random variables. We establish an inclusion theorem concerning the relationship between these two important and useful notions. Additionally, we present new Korovkin-type approximation theorems for a martingale sequence over a Banach space based on our proposed concepts. Our theorems extend and enhance the majority, if not all, earlier findings on both statistical and classical versions. Furthermore, we demonstrate that our theorems significantly improve upon several previously existing results. Finally, we provide an example of a martingale sequence based on generalized Bernstein polynomials to show that our established theorems are considerably stronger than the traditional and statistical versions of many existing results in the literature.

2. Product Means for Martingale Sequence

Let (a_k) and (b_k) be sequences of non-negative integers such that $a_k < b_k$ and

$$\lim_{k\to\infty} b_k = +\infty.$$

Then the deferred Cesàro mean for the martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ is defined by

$$\begin{aligned}\mathcal{C}_k(Y_k, \mathbb{F}_k) &= \frac{(Y_{a_k+1}, \mathbb{F}_{a_k+1}) + (Y_{a_k+2}, \mathbb{F}_{a_k+2}) + \cdots + (Y_{b_k}, \mathbb{F}_{b_k})}{b_k - a_k} \\ &= \frac{1}{b_k - a_k} \sum_{i=a_k+1}^{b_k} (Y_i, \mathbb{F}_i).\end{aligned}$$

Similarly, let (q_j) be a sequence of non-negative numbers such that

$$Q_k = \sum_{j=a_k+1}^{b_k} q_j.$$

Then the deferred weighted mean for the martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ of random variables is defined by

$$\mathcal{N}_k(Y_k, \mathbb{F}_k) = \frac{1}{Q_k} \sum_{j=a_k+1}^{b_k} q_j (Y_j, \mathbb{F}_j).$$

For the martingale sequence, we now define the product of deferred Cesàro and deferred Nörlund means as follows:

$$\begin{aligned}\Lambda_k(Y_k, \mathbb{F}_k) = (\mathcal{C}\mathcal{N})_k &= \frac{1}{b_k - a_k} \sum_{i=a_k+1}^{b_k} (\mathcal{N}_i) \\ &= \frac{1}{b_k - a_k} \sum_{i=a_k+1}^{b_k} \frac{1}{Q_i} \sum_{j=a_k+1}^{b_k} q_j (Y_j, \mathbb{F}_j).\end{aligned}$$

We now present the definitions of statistical deferred Cesàro and deferred Nörlund product mean convergence (that is, the DCN-mean convergence) as well as statistically deferred Cesàro and deferred Nörlund product mean summability (that is, the DCN-mean summability) for martingale sequences as follows.

Definition 2.1 *Let (a_k) and (b_k) be sequences of non-negative integers. Also, let (q_λ) be a sequence of non-negative numbers. A bounded martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ having probability* 1 *is statistically* *deferred Cesàro and deferred weighted product mean convergent (DCN-mean convergent) to a martingale* $(Y_0, \mathbb{F}_0)$ *with* $\mathbb{E}|Y_0| < \infty$ *if, for all* $\epsilon > 0$

$$\mathfrak{G}_\epsilon = \{\lambda : \lambda \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_\lambda|(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, *we have*

$$\lim_{k\to\infty} \frac{|\{\lambda : \lambda \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_\lambda|(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}|}{(b_k - a_k)Q_k} = 0.$$

We write

$$\Lambda_{k\text{stat}} \lim_{k\to\infty} (Y_k, \mathbb{F}_k) = (Y_0, \mathbb{F}_0).$$

Definition 2.2 *Let* (a_k) *and* (b_k) *be sequences of non-negative integers. A bounded martingale sequence* $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ *having probability* 1 *is statistically deferred Cesàro and deferred weighted product summable (DCN-mean summable) to a martingale* $(Y_0, \mathbb{F}_0)$ *with* $\mathbb{E}|Y_0| < \infty$ *if, for all* $\epsilon > 0$

$$\mathfrak{G}_\epsilon = \{\lambda : a_k < \lambda \leqq b_k \quad \textit{and} \quad |\Lambda_\lambda(Y_k, \mathbb{F}_k) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}$$

has zero natural density. This means that, for every $\epsilon > 0$, *we have*

$$\lim_{k\to\infty} \frac{|\{\lambda : a_k < \lambda \leqq b_k \quad \textit{and} \quad |\Lambda_\lambda(Y_k, \mathbb{F}_k) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}|}{b_k - a_k} = 0.$$

We write

$$stat_{\Lambda_k} \lim_{k\to\infty} \Lambda_k(Y_k, \mathbb{F}_k) = (Y_0, \mathbb{F}_0).$$

We now develop an inclusion theorem relating to the two aforementioned new and intriguing ideas that, every statistical DCN-product mean convergent martingale sequence is statistically DCN-product mean summable, but the converse is not generally true.

Theorem 2.1 *If a given martingale sequence* $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ *is statistical DCN- mean convergent to a martingale* $(Y_0, \mathbb{F}_0)$ *with* $\mathbb{E}|Y_0| < \infty$, *then it is statistically DCN- mean summable to the same martingale, but not conversely.*

Proposition 2.1 *Suppose the given martingale sequence* $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ *is statistically DCN- mean convergent to a martingale* $(Y_0, \mathbb{F}_0)$ *with* $\mathbb{E}|Y_0| < \infty$. *Then, by Definition 2.1, we have*

$$\lim_{k\to\infty} \frac{|\{\lambda : \lambda \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_\lambda|(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}|}{(b_k - a_k)Q_k} = 0.$$

Now, for the following two sets:

$$\mathcal{R}_\epsilon = \{\lambda : \lambda \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_\lambda|(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| \geqq \epsilon\}$$

and

$$\mathcal{R}_\epsilon^c = \{\lambda : \lambda \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_\lambda|(Y_\lambda, \mathbb{F}_\lambda) - (Y_0, \mathbb{F}_0)| < \epsilon\},$$

we find that

$$
\begin{aligned}
|\Lambda_k(Y_k,\mathbb{F}_k)-(Y_0,\mathbb{F}_0)| &= \left|\frac{1}{b_k-a_k}\sum_{i=a_k+1}^{b_k}\frac{1}{Q_i}\sum_{j=a_k+1}^{b_k}q_j(Y_j,\mathbb{F}_j)-(Y_0,\mathbb{F}_0)\right| \\
&\leqq \left|\frac{1}{b_k-a_k}\sum_{i=a_k+1}^{b_k}\left[\frac{1}{Q_i}\sum_{j=a_k+1}^{b_k}q_j(Y_j,\mathbb{F}_j)-(Y_0,\mathbb{F}_0)\right]\right| \\
&\quad + \left|\frac{1}{b_k-a_k}\sum_{i=a_k+1}^{b_k}(Y_0,\mathbb{F}_0)-(Y_0,\mathbb{F}_0)\right| \\
&\leqq \frac{1}{(b_k-a_k)Q_k}\sum_{\substack{i=a_k+1\\(\lambda\in\mathcal{R}_\epsilon)}}^{b_k}|(Y_k,\mathbb{F}_k)-(Y_0,\mathbb{F}_0)| \\
&\quad + \frac{1}{(b_k-a_k)Q_k}\sum_{\substack{i=a_k+1\\(\lambda\in\mathcal{R}_\epsilon^c)}}^{b_k}|(Y_k,\mathbb{F}_k)-(Y_0,\mathbb{F}_0)| \\
&\quad + |(Y_0,\mathbb{F}_0)|\left|\frac{1}{b_k-a_k}\sum_{\mu=a_k+1}^{b_k}-1\right| \\
&\leqq \frac{1}{(b_k-a_k)Q_k}|\mathcal{R}_\epsilon|+\frac{1}{(b_k-a_k)Q_k}|\mathcal{R}_\epsilon^c| = 0.
\end{aligned}
$$

Thus, we clearly obtain

$$|\Lambda_k(Y_k,\mathbb{F}_k)-(Y_0,\mathbb{F}_0)| < \epsilon.$$

Therefore, the martingale sequence $(Y_k,\mathbb{F}_k;k\in\mathbb{N})$ *is statistically DCN-mean summable to the martingale* $(Y_0,\mathbb{F}_0)$ *with* $\mathbb{E}|Y_0|<\infty$.

We next provide an example to show that a statistically DCN- mean summable martingale sequence is not always statistically DCN- mean convergent, which evidently supports the non-validity of the converse proposition.

Example 2.1 *Let us set*

$$a_k = 2k, \quad b_k = 4k \quad \textit{and} \quad q_k = k \quad (k\in\mathbb{N}).$$

Also, let $(\mathbb{F}_k, k\in\mathbb{N})$ *be a monotonically increasing sequence of* 0*-mean independent random variables of* σ*-fields with* $(Y_k)\in\mathbb{F}_k$ *such that for* k *is even*

$$Y_k = \begin{cases} 1 & (k=m^2;\ m\in\mathbb{N}) \\ \\ 0 & (\text{otherwise}) \end{cases} \tag{2.1}$$

and for k *is odd*

$$Y_k = \begin{cases} -1 & (k=m^2;\ m\in\mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases} \tag{2.2}$$

It is easy to see that, the martingale sequence $(Y_k,\mathbb{F}_k;k\in\mathbb{N})$ *is neither ordinarily DCN- mean convergent nor statistically DCN- mean convergent; however, it is statistically DCN- mean summable to* 0.

3. Korovkin-Type Theorems for Positive Linear Operator

The approximation features of the Korovkin-type theorems have lately been extended (or generalized) in a variety of different areas of mathematics such as (for example) sequence spaces, Banach spaces, Probability spaces, Measurable spaces, and others. This concept is extremely valuable in Real Analysis, Functional Analysis, Harmonic Analysis, and other related areas. Here, in this connection, we choose to refer the interested readers to the recent works [5], [15], [21], [22], [23] and [25]

We establish here the statistical versions of new approximation of Korovin-type theorems for martingale sequences of positive linear operators via our proposed DCN-product summability mean.

Let $C([0,1])$ be the space of all real-valued continuous functions defined on $[0,1]$ under the norm $\|\cdot\|_\infty$. Also, let $C[0,1]$ be a complete norm linear space. Then, for $g \in C[0,1]$, the norm of g denoted by $\|g\|$ is given by

$$\|g\|_\infty = \sup\{|g(t)| : t \in [0,1]\}.$$

We say that the operator $\mathcal{L}$ is a martingale sequence of positive linear operators, provided that

$$\mathcal{L}(g;t) \geqq 0 \quad \text{whenever} \quad t \geqq 0 \text{ with } \mathcal{L}(g;t) < \infty \text{ and } \mathrm{Prob}\big(\mathcal{L}(g;t)\big) = 1.$$

Theorem 3.1 *Let*

$$\mathcal{L}_i : C[0,1] \to C[0,1]$$

be a martingale sequence of positive linear operators. Then, for all $g \in C[0,1]$,

$$\Lambda_{\mathrm{kstat}} \lim_{i\to\infty} \|\mathcal{L}_i(g;t) - g(t)\|_\infty = 0 \tag{3.1}$$

if and only if

$$\Lambda_{\mathrm{kstat}} \lim_{i\to\infty} \|\mathcal{L}_i(1;t) - 1\|_\infty = 0, \tag{3.2}$$

$$\Lambda_{\mathrm{kstat}} \lim_{i\to\infty} \|\mathcal{L}_i(2t;t) - 2t\|_\infty = 0 \tag{3.3}$$

and

$$\Lambda_{\mathrm{kstat}} \lim_{i\to\infty} \|\mathcal{L}_i(3t^2;t) - 3t^2\|_\infty = 0. \tag{3.4}$$

Proposition 3.1 *Since each of the following functions:*

$$g_0(t) = 1, \quad g_1(t) = 2t \quad \text{and} \quad g_2(t) = 3t^2$$

belong to $C[0,1]$ *and are continuous, the implication given by* (3.1) *implies that the conditions* (3.2) *to* (3.4) *is obvious.*

In order to complete the proof of the Theorem 3.1, we first assume that the conditions (3.2) to (3.4) hold true. If $g \in C[0,1]$, *then there exists a constant* $K > 0$ *such that*

$$|g(t)| \leqq K \quad (\forall\ t \in [0,1]).$$

We thus find that

$$|g(y) - g(t)| \leqq 2K \qquad (y, t \in [0,1]). \tag{3.5}$$

Clearly, for a given $\epsilon > 0$, *there exists* $\delta^* > 0$ *such that*

$$|g(y) - g(t)| < \epsilon \tag{3.6}$$

whenever

$$|y - t| < \delta^* \quad \text{for all} \quad y, t \in [0,1].$$

Let us choose

$$\varphi_1 = \varphi_1(y,t) = 4(y-t)^2.$$

If $|y-t| \geqq \delta^*$, *then we find that*

$$|g(y)-g(t)| < \frac{2K}{\delta^{2*}}\varphi_1(y,t). \tag{3.7}$$

Thus, from the equations (3.6) and (3.7), we get

$$|g(y)-g(t)| < \epsilon + \frac{2K}{\delta^{2*}}\varphi_1(y,t),$$

which implies that

$$-\epsilon - \frac{2K}{\delta^{2*}}\varphi_1(y,t) \leqq g(y)-g(t) \leqq \epsilon + \frac{2K}{\delta^{2*}}\varphi_1(y,t). \tag{3.8}$$

Now, since $\mathcal{L}_i(1;t)$ *is monotone and linear, by applying the operator* $\mathcal{L}_i(1;t)$ *to this inequality, we have*

$$\begin{aligned}\mathcal{L}_i(1;t)\left(-\epsilon - \frac{2K}{\delta^{2*}}\varphi_1(y,t)\right) &\leqq \mathcal{L}_i(1;t)(g(y)-g(t)) \\ &\leqq \mathcal{L}_i(1;t)\left(\epsilon + \frac{2K}{\delta^{2*}}\varphi_1(y,t)\right).\end{aligned}$$

We note that t *is fixed and so* $g(t)$ *is a constant number. Therefore, we have*

$$\begin{aligned}-\epsilon\mathcal{L}_i(1;t) - \frac{2K}{\delta^{2*}}\mathcal{L}_i(\varphi_1;t) &\leqq \mathcal{L}_i(g;t) - g(t)\mathcal{L}_i(1;t) \\ &\leqq \epsilon\mathcal{L}_i(1;t) + \frac{2K}{\delta^{2*}}\mathcal{L}_i(\varphi_1;t).\end{aligned} \tag{3.9}$$

We also know that

$$\mathcal{L}_i(g;t) - g(t) = [\mathcal{L}_i(g;t) - g(t)\mathcal{L}_i(1;t)] + g(t)[\mathcal{L}_i(1;t)-1]. \tag{3.10}$$

So, by using (3.9) and (3.10), we have

$$\mathcal{L}_i(g;t) - g(t) < \epsilon\mathcal{L}_i(1;t) + \frac{2K}{\delta^{2*}}\mathcal{L}_i(\varphi_1;t) + g(t)[\mathcal{L}_i(1;t)-1]. \tag{3.11}$$

We now estimate $\mathcal{L}_i(\varphi_1;t)$ *as follows:*

$$\begin{aligned}\mathcal{L}_i(\varphi_1;t) &= \mathcal{L}_i((2y-2t)^2;t) = \mathcal{L}_i(4y^2 - 8ty + 4t^2;t) \\ &= \mathcal{L}_i(4y^2;t) - 8t\mathcal{L}_i(y;t) + 4t^2\mathcal{L}_i(1;t) \\ &= 4[\mathcal{L}_i(y^2;t) - t^2] - 8t[\mathcal{L}_i(y;t) - t] \\ &\quad + 4t^2[\mathcal{L}_i(1;t)-1].\end{aligned}$$

Thus, by using (3.11), we obtain

$$\begin{aligned}\mathcal{L}_i(g;t) - g(t) &< \epsilon\mathcal{L}_i(1;t) + \frac{2K}{\delta^{2*}}\{4[\mathcal{L}_i(y^2;t) - t^2] \\ &\quad - 8t[\mathcal{L}_i(y;t) - t] + 4t^2[\mathcal{L}_i(1;t)-1]\} \\ &\quad + g(t)[\mathcal{L}_i(1;t)-1]. \\ &= \epsilon[\mathcal{L}_i(1;t)-1] + \epsilon + \frac{2K}{\delta^{2*}}\{4[\mathcal{L}_i(y^2;t) - t^2] \\ &\quad - 8t[\mathcal{L}_i(y;t) - t] + 4t^2[\mathcal{L}_i(1;t)-1]\} \\ &\quad + g(t)[\mathcal{L}_i(1;t)-1].\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathcal{L}_i(g;t) - g(t)| &\leqq \epsilon + \left(\epsilon + \frac{8K}{\delta^{2*}} + K\right)|\mathcal{L}_i(1;t) - 1| \\ &\quad + \frac{16K}{\delta^{2*}}|\mathcal{L}_i(y;t) - t| + \frac{8K}{\delta^{2*}}|\mathcal{L}_i(y^2;t) - t^2| \\ &\leqq \mathcal{E}(|\mathcal{L}_i(1;t) - 1| + |\mathcal{L}_i(y;t) - t| \\ &\quad + |\mathcal{L}_i(y^2;t) - t^2|),\end{aligned} \tag{3.12}$$

where

$$\mathcal{E} = \max\left(\epsilon + \frac{8K}{\delta^{2*}} + K, \frac{16K}{\delta^{2*}}, \frac{8K}{\delta^{2*}}\right).$$

Now, for a given $\kappa > 0$, there exists $\epsilon > 0$ $(\epsilon < \kappa)$, we get

$$\mathfrak{T}_i(t;\kappa) = \{i : i \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_i\,|\mathcal{L}_i(g;t) - g(t)| \geqq \kappa\}.$$

Furthermore, for $j = 0, 1, 2$, we have

$$\mathfrak{T}_{i,j}(t;\kappa) = \left\{i : i \leqq (b_k - a_k)Q_k \quad \textit{and} \quad q_i\,|\mathfrak{T}_m(g;t) - g_j(t)| \geqq \frac{\kappa - \epsilon}{3\mathcal{E}}\right\},$$

so that

$$\mathfrak{T}_i(t;\kappa) \leqq \sum_{j=0}^{2} \mathfrak{T}_{i,j}(t;\kappa).$$

Clearly, we obtain

$$\frac{\|\mathfrak{T}_i(t;\kappa)\|_{C[0,1]}}{(b_k - a_k)Q_k} \leqq \sum_{j=0}^{2} \frac{\|\mathfrak{T}_{i,j}(t;\kappa)\|_{C[0,1]}}{(b_k - a_k)Q_k}. \tag{3.13}$$

Now, using the above assumption about the implications in (3.2) to (3.4), and by Definition 2.1, the right-hand side of (3.13) tends to 0 as $n \to \infty$. Consequently, we get

$$\lim_{k\to\infty} \frac{\|\mathfrak{T}_i(t;\kappa)\|_{\mathcal{C}[0,1]}}{(\beta_k - \alpha_k)P_k} = 0 \ (\delta,\ \kappa > 0).$$

Therefore, the implication (3.1) holds true. This completes the proof of Theorem 3.1.

Next, by using Definition 2.2, we present the following theorem.

Theorem 3.2 *Let $\mathfrak{T}_i : C[0,1] \to C[0,1]$ be a martingale sequence of positive linear operators. Also, let $g \in C[0,1]$. Then*

$$\mathrm{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathfrak{T}_i(g;x) - g(t)\|_\infty = 0 \tag{3.14}$$

if and only if

$$\mathrm{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathfrak{T}_i(1;t) - 1\|_\infty = 0, \tag{3.15}$$

$$\mathrm{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathfrak{T}_i(2t;t) - 2t\|_\infty = 0 \tag{3.16}$$

and

$$\mathrm{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathfrak{T}_i(3t^2;t) - 3t^2\|_\infty = 0. \tag{3.17}$$

Proposition 3.2 *The proof of Theorem 3.2 is similar to the proof of Theorem 3.1. We therefore, choose to skip the details involved.*

We provide below an illustrative example for the martingale sequence of positive linear operators that does not meet the conditions of the DCN product mean of statistical convergence versions of Korovkin-type approximation Theorem 3.1, and also the results of Jena and Paikray *et al.* ([7], [8]), and Srivastava *et al.* [18], but it meets the conditions of statistical DCN product mean summability versions of our Korovkin-type approximation Theorem 3.2. Thus, our non-trivial Theorem 3.2 is quite powerful than the results asserted by Theorem 3.1, and so also the results of Jena and Paikray *et al.* (see [7] and [8]), and Srivastava *et al.* [18].

We now recall the operator

$$\nu(1+\nu\mathcal{D}) \qquad \left(\mathcal{D}=\frac{d}{d\nu}\right),$$

which was used by Al-Salam [1], and more recently, by Viskov and Srivastava [30] (see [17] and [27]). Here, in our Example 3.1 below, we use this operator in conjunction with the Bernstein polynomials.

Example 3.1 *Let us consider the Bernstein polynomials $\mathcal{B}_k(g;\nu)$ on $C[0,1]$ given by (see also [28])*

$$\mathcal{B}_k(g;\nu)=\sum_{i=0}^{k} g\left(\frac{i}{k}\right)\binom{k}{i}\nu^i(1-\nu)^{k-i} \quad \big(\nu\in[0,1]\big). \tag{3.18}$$

Next, we present the martingale sequences of positive linear operators on $C[0,1]$ defined as follows:

$$\mathcal{L}_i(g;\nu)=[1+(Y_k,\mathbb{F}_k)]\nu(1+\nu D)\mathcal{B}_i(g;\nu) \quad (\forall\ g\in C[0,1]) \tag{3.19}$$

with $(Y_k,\mathbb{F}_k)$ as already mentioned in the above Example 2.1.

Now, by using our proposed operators (3.19), we calculate the values of the functions 1, 2ν and $3\nu^2$ as follows:

$$\mathcal{L}_i(1;\nu)=[1+(Y_i,\mathbb{F}_i)]\nu(1+\nu D)1=[1+(Y_i,\mathbb{F}_i)]\nu,$$

$$\mathcal{L}_i(2\nu;\nu)=[1+(Y_i,\mathbb{F}_i)]\nu(1+\nu D)2\nu=[1+(Y_i,\mathbb{F}_i)]\nu(1+2\nu),$$

and

$$\begin{aligned}\mathcal{L}_i(3\nu^2;\nu)&=[1+(Y_i,\mathbb{F}_i)]\nu(1+\nu D)3\left\{\nu^2+\frac{\nu(1-\nu)}{i}\right\}\\&=[1+(Y_i,\mathbb{F}_i)]\left\{\nu^2\left(6-\frac{9\nu}{i}\right)\right\},\end{aligned}$$

so that, we have

$$\begin{aligned}&\mathrm{stat}_{\Lambda_k}\lim_{i\to\infty}\|\mathcal{L}_i(1;\nu)-1\|_\infty=0,\\&\mathrm{stat}_{\Lambda_k}\lim_{i\to\infty}\|\mathcal{L}_i(2\nu;\nu)-2\nu\|_\infty=0\end{aligned}$$

and

$$\mathrm{stat}_{\Lambda_k}\lim_{i\to\infty}\|\mathcal{L}_i(3\nu^2;\nu)-3\nu^2\|_\infty=0.$$

Consequently, the sequence $(\mathcal{L}_i(g;\nu))$ satisfies the conditions (3.15) to (3.17). Therefore, by Theorem 3.2, we have

$$\mathrm{stat}_{\Lambda_k}\lim_{i\to\infty}\|\mathcal{L}_i(g;\nu)-g\|_\infty=0.$$

Here clearly, the given martingale sequence $(Y_i,\mathbb{F}_i)$ of functions in Example 2.1 is statistically DCN product mean summable; however, not DCN product mean statistically convergent. Thus, the martingale operators defined by (3.19) satisfy Theorem 3.2. However, these operators do not satisfy Theorem 3.1.

4. Remarkable Conclusion and Observations

This final section of our investigation presents a number of additional remarks and observations about the numerous findings that we have proved in this article.

Remark 4.1 *Let $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ be a martingale sequence given in Example 2.1. Then, since*

$$\text{stat}_{\Lambda_k} \lim_{i\to\infty} Y_i = \frac{1}{2} \text{ on } [0,1],$$

we have

$$\text{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathcal{L}_i(g_j;t) - g_j(t)\|_\infty = 0 \qquad (j = 0,1,2). \tag{4.1}$$

Thus, by Theorem 3.2, we can write

$$\text{stat}_{\Lambda_k} \lim_{i\to\infty} \|\mathcal{L}_i(g;t) - g(t)\|_\infty = 0, \tag{4.2}$$

where

$$g_0(t) = 1, \quad g_1(t) = 2t \quad \textit{and} \quad g_2(t) = 3t^2.$$

Here, the martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ is neither statistically convergent nor it converges uniformly in the traditional sense. Thus, the ordinary and statistical versions of Korovkin-type theorems certainly do not work here under the operators defined by (3.19). However, our Theorem 3.2 still works. As a result, this application shows that our Theorem 3.2 is a non-trivial generalization of the classical as well as statistical versions of Korovkin-type theorems (see [6] and [11]).

Remark 4.2 *Let $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ be a martingale sequence as given in Example 2.1. Then, since*

$$\text{stat}_{\Theta_k} \lim_{i\to\infty} Y_i = \frac{1}{2} \textit{ on } [0,1],$$

so (4.1) holds true. Now, by applying (4.1) and Theorem 3.2, the condition (4.2) also holds true. However, since the martingale sequence $(Y_k, \mathbb{F}_k; k \in \mathbb{N})$ is not statistically DCN product mean convergent, but it is statistically DCN product mean summable. Thus, Theorem 3.2 is certainly a non-trivial extension of Theorem 3.1. Therefore, Theorem 3.2 is powerful than Theorem 3.1.

Remark 4.3 *Inspired by a recently published article by Srivastava et al. [18] and [19], we draw the attention of interested readers to the potential of establishing Korovkin-type approximation theorems over Banach spaces using martingale sequences. Furthermore, in light of recent results by Paikray et al. [14] and Saini et al. [16], we encourage curious readers to explore further research in the area of fuzzy approximation theorems.*

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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