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Automorphism groups of groups of order p^2q^2

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ABSTRACT: In this paper, we have computed the automorphism groups of all groups of order p^2q^2 up to isomorphism, where p and q are distinct primes.

Key Words: Automorphism group, finite groups, semidirect product.

Contents

1. Introduction

A. S. Hadi, M. Ghorbani and F. N. Larki [5] classified the groups of order p^2q^2 upto isomorphism, where p and q are distinct primes. They showed that any group of order p^2q^2 is either a direct product of groups or a semidirect product of groups. Our aim in this paper is to find the automorphism group of all groups of order p^2q^2, where p and q are distinct primes.

Bidwell et. al. [1] proved that if H and K are finite groups with no common direct factors and $G = H \times K$, then the automorphism group $Aut(G)$ of G can be expressed as the 2×2 matrices of maps satisfying some conditions (see [1, Theorem 3.6, p. 487]). Also, Bidwell and Curran [2] proved that if H and K are finite groups with a group homomorphism $\phi : K \longrightarrow Aut(H)$ and $G = H \rtimes_\phi K$ is the semidirect product of H and K, then the automorphism group $Aut(G)$ of G is in one to one correspondence with the group of 2×2 matrices of maps satisfying some conditions (see [2, Lemma 2.1, p. 489]). Recently, the automorphism groups of all groups of order p^2q are obtained by Campedel et. al. [3]. So, it motivates us to study the automorphism groups of all groups of order p^2q^2.

We will use the results mentioned in the previous paragraph in order to achieve our goal. In the second section, we have recalled some results which will be useful for us in achieving our goal of this paper. In the third section, we have listed all groups of order p^2q^2 upto isomorphism, where p and q are distinct primes. In the last section, we have found the structures of the automorphism groups of all groups of order p^2q^2 upto isomorphism and have listed them in the Tables 1 and 2.

Throughout the paper, we will identify the internal direct product $G = HK$ (semidirect product $G = HK$) with the external direct product $G = H \times K$ (semidirect product $G = H \rtimes_\phi K$, where the group homomorphism $\phi : K \longrightarrow Aut(H)$ is defined by $\phi(k)(h) = h^k = k^{-1}hk$, for all $h \in H$ and $k \in K$). The commutator of two elements $x, y \in G$ is defined as $[x, y] = x^{-1}y^{-1}xy$ and the conjugate of an element x is defined as $x^y = y^{-1}xy$. The cyclic group of order n, symmetric group on n symbols, alternating group on n symbols and dihedral group of order $2n$ are denoted by $\mathbb{Z}_n, S_n, A_n$ and D_n respectively. The quaternion group of order 8 is denoted by Q_8. The greatest integer function is denoted by $[\cdot]$.

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2. Preliminaries

In this section, we will recall some results about the automorphism group of direct product of groups and semidirect product of groups.

Theorem 2.1 *[1, Theorem 3.6, p. 487] Let $G = H \times K$, where H and K have no common direct factor, and let*

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in Aut(H) \right\}, \qquad B = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in Hom(K, Z(H)) \right\},$$
$$C = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mid \gamma \in Hom(H, Z(K)) \right\}, \quad and \quad D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in Aut(K) \right\}.$$

Then A, B, C, D are subgroups of $Aut(G)$ and $Aut(G) = ABCD$, where $AD = A \times D$ normalizes B and C.

Note that in [2], for the group $G = H \rtimes_\phi K$, the group homomorphism $\phi : K \longrightarrow Aut(H)$ is given by $\phi(k)(h) = khk^{-1}$ for all $k \in K$ and $h \in H$. In this paper, we define the group homomorphism as $\phi(k)(h) = k^{-1}hk$ for all $k \in K$ and $h \in H$. Thus the following theorem is immediate from the Lemma [2, Lemma 2.1, p. 489].

Theorem 2.2 *[2, Lemma 2.1, p. 489] Let $G = H \rtimes K$ be the semidirect product of groups H and K and let*

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \begin{matrix} \alpha \in Map(H, H), & \beta \in Map(K, H), \\ \gamma \in Hom(H, K), & \delta \in Hom(K, K) \end{matrix} \right\},$$

where the maps $\alpha, \beta, \gamma, \delta$ satisfy the following properties for all $h \in H$ and $k \in K$,

(i) $\alpha(hh') = \alpha(h)\alpha(h')^{\gamma(h)^{-1}}$,

(ii) $\beta(kk') = \beta(k)\beta(k')^{\delta(k)^{-1}}$,

(iii) $\gamma(h^k) = \gamma(h)^{\delta(k)}$,

(iv) $\alpha\left(h^{k^{-1}}\right)\beta(k)^{\gamma\left(h^{k^{-1}}\right)^{-1}} = \beta(k)\alpha(h)^{\delta(k^{-1})}$,

(v) for any $h'k' \in G$, where $h' \in H$ and $k' \in K$, there is a unique $hk \in G$ such that $\alpha(h)\beta(k)^{\gamma(h)^{-1}} = h'$ and $\gamma(h)\delta(k) = k'$.

Then there is a one to one correspondence between the automorphism group of G, $Aut(G)$ and $\mathcal{A}$ given by $\theta \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\theta(h) = \alpha(h)\gamma(h)$ and $\theta(k) = \beta(k)\delta(k)$. Also, if $\theta' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$, then

$$\theta'\theta = \begin{pmatrix} \alpha'\alpha + \beta'{\gamma}^{\gamma'(-\alpha)} & \alpha'\beta + \beta'{\delta}^{\gamma'(-\beta)} \\ \gamma'\alpha + \delta'\gamma & \gamma'\beta + \delta'\delta \end{pmatrix}, \tag{2.1}$$

where the maps $-\alpha$ and $-\beta$ are defined as $(-\alpha)(h) = \alpha(h)^{-1}$ and $(-\beta)(k) = \beta(k)^{-1}$ for all $h \in H$ and $k \in K$.

Corollary 2.1 *[2, Corollary 2.2, p. 490] Let $G = H \rtimes K$ be the semidirect product of H and K, where K is abelian. Then*

(i) $\gamma \in Hom(H/[H, K], K)$,

(ii) for all $h \in H$ and $k \in K$, $\alpha\left(h^{k^{-1}}\right)\beta(k)^{\gamma(h)} = \beta(k)\alpha(h)^{\delta(k)^{-1}}$.

Let K be abelian and

$$\begin{aligned} P =& \{(\alpha, \delta) \in Aut(H) \times Aut(K) \mid \alpha(h^k) = \alpha(h)^{\delta(k)}, \forall h \in H, k \in K\}, \\ S =& \{\beta \in Map(K, H) \mid \beta(kk') = \beta(k)\beta(k')^{k^{-1}}, h^{k^{-1}}\beta(k) = \beta(k)h^{k^{-1}}, \forall k, k' \in K\} \end{aligned}$$

and let the corresponding subsets of $\mathcal{A}$ be

$$R = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid (\alpha, \delta) \in P \right\} \text{ and } Q = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in S \right\}.$$

Theorem 2.3 *Let $G = H \rtimes K$ be the semidirect product of H and K, where H and K are abelian and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$. Then, if $Hom(H, K)$ is the trivial group, then $\mathcal{A}$ is a group and $Aut(G) \simeq \mathcal{A} \simeq Q \rtimes R$.*

Proof 1 *Let $Hom(H, K)$ be the trivial group. Then for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$, we have $\gamma = 0$. Now, using the Equation (2.1), one can easily observe that $\mathcal{A}$ is a group and the binary operation reduces to*

$$\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha & \alpha'\beta \\ 0 & \delta'\delta \end{pmatrix}.$$

Now, let $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{A}$. Then we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & \hat{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in QR,$$

where $\hat{\beta} = \beta\delta^{-1}$. Also, for any $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in Q$ and $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in R$, we have

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}^{-1} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & \alpha^{-1}\beta\delta \\ 0 & 1 \end{pmatrix} \in Q.$$

Thus, $Q \trianglelefteq \mathcal{A}$. Also, $Q \cap R = \{1\}$. Therefore, $\mathcal{A} \simeq Q \rtimes R$.

We will identify the automorphisms of the group G with the corresponding matrices in $\mathcal{A}$.

3. Groups of order p^2q^2, p and q are distinct primes

In [5], the authors have classified the groups of order p^2q^2, where p and q are distinct primes, as follows.

Theorem 3.1 *[5, Theorem 3.1, p. 91] Let G be a group of order p^2q^2. Then G is isomorphic to one of the following groups.*

(i) If $pq = 6$, then G is isomorphic to

1. $\mathbb{Z}_{36}$,
2. $\mathbb{Z}_{18} \times \mathbb{Z}_2$,
3. $\mathbb{Z}_6 \times \mathbb{Z}_6$,
4. $\mathbb{Z}_{12} \times \mathbb{Z}_3$,
5. $\mathbb{Z}_6 \times D_3 \simeq S_3 \times \mathbb{Z}_6 \simeq D_6 \times \mathbb{Z}_3$,
6. $D_3 \times D_3 \simeq S_3 \times S_3$,
7. $D_9 \times \mathbb{Z}_2$,
8. $A_4 \times \mathbb{Z}_3$,

9. D_{18},
10. $H \times \mathbb{Z}_3$ *where* $H = \langle a, b \mid a^4 = b^3 = 1, a^{-1}ba = b^{-1}\rangle$,
11. $K \times \mathbb{Z}_2$, *where* $K = \langle a, b, c, \mid a^2 = b^2 = c^2 = (abc)^2 = (ab)^3 = (ac)^3 = 1\rangle$,
12. $\langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = a^{-1}\rangle$,
13. $\langle a, b, c \mid a^2 = b^2 = c^9 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = ab\rangle$,
14. $\langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$,

(ii) If $pq \neq 6$, *then* G *is isomorphic to*

15. $\mathbb{Z}_{q^2} \times \mathbb{Z}_{p^2}$,
16. $\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_{p^2}$,
17. $\mathbb{Z}_{q^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$,
18. $\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_p \times \mathbb{Z}_p$,
19. $\langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r^q \equiv 1 \ (mod\ p^2)\rangle$, *where* q *divides* $p-1$,
20. $\langle a, b \mid a^{q^2} = b^{p^2} = 1, a^{-1}ba = b^r, r^{q^2} \equiv 1 \ (mod\ p^2)\rangle$, *where* q^2 *divides* $p-1$,
21. $\langle a, b, c \mid a^q = b^q = c^{p^2} = 1, [a, b] = [b, c] = 1, a^{-1}ca = c^r, r^q \equiv 1 \ (mod\ p^2)\rangle$, *where* q *divides* $p-1$,
22. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^r, a^{-1}ca = c^r, r^q \equiv 1 \ (mod\ p)\rangle$, *where* q *divides* $p-1$,
23. $\langle a, b, c \mid a^4 = b^p = c^p = [a, b] = [b, c] = 1, a^{-1}ca = c^{-1}\rangle$,
24. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [a, b] = [b, c] = 1, a^{-1}ca = c^r, r^q \equiv 1 \ (mod\ p)\rangle$, *where* $q \neq 2$ *divides* $p-1$,
25. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^r, a^{-1}ca = c^{r^{-1}}, r^q \equiv 1 \ (mod\ p)\rangle$, *where* $q \neq 2$ *divides* $p-1$,
26. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^r, a^{-1}ca = c^{r^{-1}}, r^{q^2} \equiv 1 \ (mod\ p)\rangle$, *where* q^2 *divides* $p-1$,
27. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^{r^{-1}}, a^{-1}ca = c^{r^n}, r^q \equiv 1 \ (mod\ p)\rangle$, *where* $q \neq 2$ *divides* $p-1$ *and* $n \in \{2, 3, \cdots, \frac{q-1}{2}\}$,
28. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^r, a^{-1}ca = c^{r^n}, r^{q^2} \equiv 1 \ (mod\ p)\rangle$, *where* q^2 *divides* $p-1$ *and* $2 \leq n \leq \frac{q^2-1}{2}$ *or* $i = mq(m \geq \frac{q+1}{2})$,
29. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^r, a^{-1}ca = c^r, r^{q^2} \equiv 1 \ (mod\ p)\rangle$, *where* q^2 *divides* $p-1$,
30. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^m c^{nD}, a^{-1}ca = b^n c^m\rangle$, *where* $m + n\sqrt{D} = \sigma^{\frac{p^2-1}{q}}$, σ *is a primitive root of Galois Field* $GF(p^2)$, $m, n, D \in GF(p)$, $n \neq 0$, D *is not a perfect square and* q *divides* $p+1$,
31. $\langle a, b, c \mid a^4 = b^p = c^p = [b, c] = 1, a^{-1}ba = b^m c^{nD}, a^{-1}ca = b^n c^m\rangle$, *where* $m+n\sqrt{D} = \sigma^{\frac{p^2-1}{4}}$, σ *is a primitive root of Galois Field* $GF(p^2)$, $\alpha, \beta, D \in GF(p)$, $\beta \neq 0$, D *is not a perfect square and* $p \equiv 3 \ (mod\ 4)$,
32. $\langle a, b, c \mid a^{q^2} = b^p = c^p = [b, c] = 1, a^{-1}ba = b^m c^{nD}, a^{-1}ca = b^n c^m\rangle$, *where* $m + n\sqrt{D} = \sigma^{\frac{p^2-1}{q^2}}$, σ *is a primitive root of Galois Field* $GF(p^2)$, $m, n, D \in GF(p)$, $\beta \neq 0$, D *is not a perfect square and* q^2 *divides* $p+1$,
33. $\langle a, b, c, d \mid a^q = b^q = c^p = d^p = [a, c] = [a, d] = [a, b] = [c, d] = 1, b^{-1}cb = c^r, b^{-1}db = d^r, r^q \equiv 1 \ (mod\ p)\rangle$, *where* q *divides* $p-1$,
34. $\langle a, b, c, d \mid a^2 = b^2 = c^p = d^p = [a, b] = [a, c] = [a, d] = [b, d] = [c, d] = 1, b^{-1}cb = c^{-1}\rangle$,

35. $\langle a, b, c, d \mid a^q = b^q = c^p = d^p = [a, b] = [a, d] = [b, c] = [c, d] = 1, a^{-1}ca = c^r, b^{-1}db = d^r, r^q \equiv 1 \ (mod\ p)\rangle$, *where* q *divides* $p - 1$,

36. $\langle a, b, c, d \mid a^q = b^q = c^p = d^p = [a, b] = [a, d] = [a, c] = [c, d] = 1, b^{-1}cb = c^u d^{vD}, b^{-1}db = c^v d^u\rangle$, *where* $u+v\sqrt{D} = \sigma^{\frac{p^2-1}{q}}$, σ *is a primitive root of Galois Field* $GF(p^2)$, $u, v, D \in GF(p)$, $v \neq 0$, D *is not a perfect square,* q *divides* $p + 1$ *and* $p \not\equiv 1 \ (mod\ q)$.

4. Automorohism Groups of Groups of order p^2q^2

In this section, we will compute the automorphism groups of the groups mentioned in the Theorem 3.1. Using GAP [4], we have found the automorphism groups for the case $pq = 6$, which are listed in the Table 1 in the last section.

Now, we will compute the automorphism group $Aut(G)$ of all groups of order p^2q^2 upto isomorphism in the Theorem 3.1 such that $pq \neq 6$. Note that, in the following cases, for $G = H \times K$ or $G = H \rtimes K$, we will take H to be isomorphic to either $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ and K to be isomorphic to either $\mathbb{Z}_q \times \mathbb{Z}_q$ or $\mathbb{Z}_{q^2}$. Since p and q are distinct primes, $Hom(H, K), Hom(K, H)$ and $Hom(H/[H, K], K)$ are all trivial. Therefore, in this case, the map $\gamma = 0$ always.

Type 15.

Let $G \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Then $G \simeq \mathbb{Z}_{p^2q^2}$, as p and q are distinct primes. Thus, $Aut(G) \simeq \mathbb{Z}_{p(p-1)} \times \mathbb{Z}_{q(q-1)}$.

Type 16.

Let $G \simeq \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_{p^2}$. Let $H \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and $K \simeq \mathbb{Z}_{p^2}$. Then $A \simeq Aut(H) \simeq GL(2, q)$ and $D \simeq Aut(K) \simeq \mathbb{Z}_{p(p-1)}$. Hence, using the Theorem 2.1, $Aut(G) \simeq A \times D \simeq GL(2, q) \times \mathbb{Z}_{p(p-1)}$.

Type 17.

Let $G \simeq \mathbb{Z}_{q^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then, using the Theorem 2.1, $Aut(G) \simeq \mathbb{Z}_{q(q-1)} \times GL(2, p)$.

Type 18.

Let $G \simeq \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_p \times \mathbb{Z}_p$. Let $H \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and $K \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Then using the similar argument as in 4, we get $A \simeq Aut(H) \simeq GL(2, q)$ and $D \simeq Aut(K) \simeq GL(2, p)$. Hence, $Aut(G) \simeq A \times D \simeq GL(2, q) \times GL(2, p)$.

Type 19.

Let $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$, $H = \langle b \mid b^{p^2} = 1\rangle \simeq \mathbb{Z}_{p^2}$ and $\phi : K \longrightarrow Aut(H)$ be the homomorphism defined by $\phi(a)(b) = a^{-1}ba = b^r$, where $r^q \equiv 1 \ (mod\ p^2)$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\alpha(b) = b^i$ and $\delta(a) = a^j$, where $1 \leq i \leq p^2 - 1, \gcd(i, p) = 1$ and $1 \leq j \leq q^2 - 1, \gcd(j, q) = 1$. Since $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$. Now, $b^{ri} = \alpha(b^r) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i)^{a^j} = b^{ir^j}$. Thus, $ri \equiv ir^j \ (mod\ p^2)$, which implies that $r^{j-1} \equiv 1 \ (mod\ p^2)$. Therefore, $j \equiv 1 \ (mod\ q)$ and so, $Aut(K) \simeq \mathbb{Z}_q$. Therefore, $R \simeq \mathbb{Z}_q \times \mathbb{Z}_{p(p-1)}$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda$, where $0 \leq \lambda \leq p^2 - 1$. Then $\beta(a^{-2}) = \beta(a^{-1})\beta(a^{-1})^a = b^\lambda(b^\lambda)^a = b^{\lambda(1+r)}$. Inductively, we get, $\beta(a^{-l}) = b^{\lambda(1+r+r^2+\cdots+r^{l-1})} = b^{\lambda\frac{r^l-1}{r-1}}$ for any $0 \leq l \leq q^2 - 1$. In particular, $\beta(a^q) = \beta(a^{q^2}) = 1$. Thus, $S \simeq Q \simeq \mathbb{Z}_{p^2}$. Hence, using the Theorem 2.3, $Aut(G) \simeq Q \rtimes R \simeq \mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_{p(p-1)} \times \mathbb{Z}_q)$.

Type 20.

Let $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$, $H = \langle b \mid b^{p^2} = 1\rangle \simeq \mathbb{Z}_{p^2}$ and $\phi : K \longrightarrow Aut(H)$ be the homomorphism defined by $\phi(a)(b) = a^{-1}ba = b^r$, where $r^{q^2} \equiv 1 \ (mod\ p^2)$. Then $G \simeq H \rtimes_\phi K$. Using the similar argument as in the Type 19, we get $R \simeq \mathbb{Z}_{p(p-1)}$ and $Q \simeq \mathbb{Z}_{p^2}$. Hence, using the Theorem 2.3, $Aut(G) \simeq Q \rtimes R \simeq \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p(p-1)}$.

Type 21.

Let $H = \langle c \mid c^{p^2} = 1\rangle \simeq \mathbb{Z}_{p^2}$, $K = \langle a, b \mid a^q = b^q = [a, b] = 1\rangle \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(c) = a^{-1}ca = c^r, \phi(b)(c) = b^{-1}cb = c$, where $r^q \equiv 1\ (mod\ p^2)$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\alpha \in Aut(H)$ is defined by $\alpha(c) = c^s$ and $\delta \in Aut(K) \simeq GL(2, q)$ is given by $\delta = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\delta(a) = a^i b^j, \delta(b) = a^l b^k$, where $1 \leq s \leq p^2 - 1, \gcd(s, p) = 1$, $0 \leq i, j, l, k \leq q - 1$ and $ik \not\equiv jl\ (mod\ q)$. Since $(\alpha, \delta) \in P$, $\alpha(c^a) = \alpha(c)^{\delta(a)}$ and $\alpha(c^b) = \alpha(c)^{\delta(b)}$.

Now, $c^{rs} = \alpha(c^a) = \alpha(c)^{\delta(a)} = (c^s)^{a^i b^j} = c^{sr^i}$. Then $rs \equiv sr^i\ (mod\ p^2)$ which implies that $r^{i-1} \equiv 1\ (mod\ p^2)$. So, $i \equiv 1\ (mod\ q)$. Also, $c^s = \alpha(c^b) = \alpha(c)^{\delta(b)} = (c^s)^{a^l b^k} = c^{sr^l}$. Then $sr^l \equiv s\ (mod\ p^2)$ which implies that $r^l \equiv 1\ (mod\ p^2)$. Therefore, $l \equiv 0\ (mod\ q)$. Thus $P \sim R \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1})$. Let $\beta \in S$ be defined by $\beta(a^{-1}) = c^\lambda$ and $\beta(b^{-1}) = c^\rho$, where $0 \leq \lambda, \rho \leq p^2 - 1$. Now, $\beta(a^{-2}) = \beta(a^{-1})\beta(a^{-1})^a = c^\lambda (c^\lambda)^a = c^{\lambda(1+r)}$. Inductively, we get for $0 \leq l \leq q - 1$, $\beta(a^{-l}) = c^{\lambda(1+r+r^2+\cdots+r^{l-1})} = c^{\lambda\frac{r^l-1}{r-1}}$. In particular, $\beta(a^q) = c^{\lambda\frac{r^{-q}-1}{r-1}} = 1$. Also, $\beta(b^{-2}) = c^{-2\rho}$ and $\beta(b^{-q}) = c^{q\rho}$. Since $\beta(b^{-q}) = 1$, $q\rho \equiv 0\ (mod\ p^2)$ which implies that $\rho = 0$. Thus, we have $\beta(a^{-1}) = c^\lambda$ and $\beta(b^{-1}) = 1$, where $0 \leq \lambda \leq p^2 - 1$ and so, $S \simeq Q \simeq \mathbb{Z}_{p^2}$. Hence, using the Theorem 2.3, $Aut(G) \simeq Q \rtimes R \simeq \mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}))$.

Type 22.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi_r : K \longrightarrow Aut(H)$ be defined as $\phi_r(a)(b) = a^{-1}ba = b^r, \phi_r(a)(c) = a^{-1}ca = c^r$, where $r^q \equiv 1\ (mod\ p^2)$. Then $G \simeq H \rtimes_{\phi_r} K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \leq s \leq q^2 - 1, \gcd(s, q) = 1$, $0 \leq i, j, l, k \leq p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^{ir}c^{jr} = \alpha(b^r) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^{ir^s}c^{jr^s}$. Then $ir \equiv ir^s\ (mod\ p)$ and $jr \equiv jr^s\ (mod\ p)$ which implies that $r^{s-1} \equiv 1\ (mod\ p)$ and so, $s \equiv 1\ (mod\ q)$. Also, using $\alpha(c^a) = \alpha(c)^{\delta(a)}$, we get $s \equiv 1\ (mod\ q)$. So, we get $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta(a) = a^s$, where $s \equiv 1\ (mod\ q)$, $0 \leq i, j, l, k \leq p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Thus $P \simeq R \simeq GL(2, p) \times \mathbb{Z}_q$. Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$. Then $\beta\left(a^{-2}\right) = b^{\lambda(1+r)}c^{\rho(1+r)}$ and $\beta(a^{-q}) = b^{\lambda(1+r+r^2+\cdots+r^{q-1})}c^{\rho(1+r+r^2+\cdots+r^{q-1})} = b^{\lambda\frac{r^q-1}{r-1}}c^{\rho\frac{r^q-1}{r-1}} = 1 = \beta(a^{q^2})$. Thus $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$ and so, $S \simeq Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (GL(2, p) \times \mathbb{Z}_q)$.

Type 23.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^4 = 1\rangle \simeq \mathbb{Z}_4$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b, \phi(a)(c) = a^{-1}ca = c^{-1}$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $s \in \{1, 3\}$, $0 \leq i, j, l, k \leq p-1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^i c^j = \alpha(b) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^i c^{-j}$. Then $j \equiv -j\ (mod\ p)$ which implies that $j = 0$. Also, $b^{-l}c^{-k} = \alpha(c^{-1}) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^l c^{-k}$. Then $-l \equiv l\ (mod\ p)$ which implies that $l = 0$. Thus, $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta(a) = a^s$, where $1 \leq i, k \leq p - 1$ and $s \in \{1, 3\}$ and so, $R \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2$. Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$. Then $\beta(a^{-2}) = b^{2\lambda}$, $\beta(a^{-3}) = b^{3\lambda}c^\rho$ and $1 = \beta(a^{-4}) = b^{4\lambda}$. Thus $4\lambda \equiv 0\ (mod\ p)$ which implies that $\lambda = 0$ and so, $\beta(a^{-1}) = c^\rho$, where $0 \leq \rho \leq p - 1$ and so, $Q \simeq \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq \mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2)$.

Type 24.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b, \phi(a)(c) = a^{-1}ca = c^r$, where $r^q \equiv 1\ (mod\ p)$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \leq s \leq q^2 - 1, \gcd(s, q) = 1$, $0 \leq i, j, l, k \leq p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^i c^j = \alpha(b) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^i c^{jr^s}$. Then $j \equiv jr^s\ (mod\ p)$. Since, $\gcd(s, q) = 1$, $r^s \not\equiv 1\ (mod\ p)$. Therefore, $j = 0$. Also, $b^{rl} c^{rk} = \alpha(c^r) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^l c^{kr^s}$. Then, $rl \equiv l\ (mod\ p)$ and $rk \equiv kr^s\ (mod\ p)$ which implies that $l = 0$ and $(r^{s-1} - 1)k \equiv 0\ (mod\ p)$. So, $k \neq 0$, otherwise $\alpha \notin GL(2, p)$. Therefore, $r^{s-1} \equiv 1\ (mod\ p)$ and so, $s \equiv 1\ (mod\ q)$. Thus, we have $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta(a) = a^s$, where $s \equiv 1\ (mod\ q)$ and $1 \leq i, k \leq p - 1$ and so, $R \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q$. Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$. Then $\beta(a^{-2}) = b^{2\lambda} c^{\rho(1+r)}$ and for any $0 \leq l \leq q^2 - 1$, $\beta(a^{-l}) = b^{l\lambda} c^{\rho\frac{r^l - 1}{r-1}}$. In particular, $1 = \beta(a^{-q^2}) = b^{q^2\lambda}$. Thus $q^2\lambda \equiv 0\ (mod\ p)$ which implies that $\lambda = 0$. Therefore, $\beta(a^{-1}) = c^\rho$, where $0 \leq \rho \leq p - 1$ and so, $Q \simeq \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq \mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q)$.

Type 25.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi_{r^{-1}} : K \longrightarrow Aut(H)$ be defined as $\phi_{r^{-1}}(a)(b) = a^{-1}ba = b^r, \phi_{r^{-1}}(a)(c) = a^{-1}ca = c^{r^{-1}}$, where $r^q \equiv 1\ (mod\ p)$. Then $G \simeq H \rtimes_{\phi_{r^{-1}}} K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \leq s \leq q^2 - 1, \gcd(s, q) = 1$, $0 \leq i, j, l, k \leq p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^{ri} c^{rj} = \alpha(b^r) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^{ir^s} c^{jr^{-s}}$ and $b^{lr^{-1}} c^{kr^{-1}} = \alpha(c^{r^{-1}}) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^{lr^s} c^{kr^{-s}}$. Then $ri \equiv ir^s\ (mod\ p)$, $rj \equiv jr^{-s}\ (mod\ p)$, $lr^{-1} \equiv lr^s\ (mod\ p)$ and $kr^{-1} \equiv kr^{-s}\ (mod\ p)$. Since both i, j can not vanish together. We have two cases, namely either $i \not\equiv 0\ (mod\ p)$ or $j \not\equiv 0\ (mod\ p)$. Also, we will observe that both i and j can not be non-zero at the same time.

First, let $i \not\equiv 0\ (mod\ p)$. Then $r^{s-1} \equiv 1\ (mod\ p)$ which implies that $s \equiv 1\ (mod\ q)$. Also, using $rj \equiv jr^{-s}\ (mod\ p)$, we get $(r^2 - 1)j \equiv 0\ (mod\ p)$ which implies that $j = 0$ as $r^2 \not\equiv 1\ (mod\ p)$. Similarly, $l = 0$. So, in this case $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta(a) = a^s$, where $s \equiv 1\ (mod\ q)$ and $1 \leq i, k \leq p - 1$. Now, let $j \not\equiv 0\ (mod\ p)$. Then $r^{s+1} \equiv 1\ (mod\ p)$ which implies that $s \equiv -1\ (mod\ q)$. Also, using $ri \equiv ir^s\ (mod\ p)$, we get $(r^2 - 1)i \equiv 0\ (mod\ p)$ which implies that $i = 0$ as $r^2 \not\equiv 1\ (mod\ p)$. Similarly, $k = 0$. So, in this case $\alpha = \begin{pmatrix} 0 & j \\ l & 0 \end{pmatrix}$ and $\delta(a) = a^s$, where $s \equiv -1\ (mod\ q)$ and $1 \leq i, k \leq p - 1$. From both the cases, we have $R \simeq \left\langle \left(\begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}, \delta_s\right), \left(\begin{pmatrix} 0 & j \\ l & 0 \end{pmatrix}, \delta_{-s}\right) \right\rangle \simeq ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2$, where $\delta_s(a) = a^s$ and $s \equiv \pm 1\ (mod\ p)$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$. Then $\beta(a^{-2}) = b^{\lambda(1+r)} c^{\rho(1+r^{-1})}$ and for any $0 \leq l \leq q^2 - 1$, $\beta(a^{-l}) = b^{\lambda\frac{r^l - 1}{r-1}} c^{\rho\frac{r^{-l} - 1}{r-1}}$. In particular, $\beta(a^{-q^2}) = b^{\lambda\frac{r^{q^2} - 1}{r-1}} c^{\rho\frac{r^{-q^2} - 1}{r-1}} = 1$. Thus $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \leq \lambda, \rho \leq p - 1$ and so, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2)$.

Type 26.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi_{r^{-1}} : K \longrightarrow Aut(H)$ be defined as $\phi_{r^{-1}}(a)(b) = a^{-1}ba = b^r, \phi_{r^{-1}}(a)(c) = a^{-1}ca = c^{r^{-1}}$, where $r^{q^2} \equiv 1\ (mod\ p)$.

Then $G \simeq H \rtimes_{\phi_{r^{-1}}} K$. Using the similar argument as in the Type (25), we get $R \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$ and $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2)$.

Type 27.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi_{r^n} : K \longrightarrow Aut(H)$ be defined as $\phi_{r^n}(a)(b) = a^{-1}ba = b^r, \phi_{r^n}(a)(c) = a^{-1}ca = c^{r^n}$, where $r^q \equiv 1\ (mod\ p)$ and $n \in \{2, 3, \cdots, \frac{q-1}{2}\}$. Then $G \simeq H \rtimes_{\phi_{r^n}} K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \le s \le q^2 - 1, \gcd(s, q) = 1$, $0 \le i, j, l, k \le p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^{ri}c^{rj} = \alpha(b^r) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^{ir^s} c^{jr^{ns}}$ and $b^{lr^n} c^{kr^n} = \alpha(c^{r^n}) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^{lr^s} c^{kr^{ns}}$. Then $ri \equiv ir^s\ (mod\ p)$, $rj \equiv jr^{ns}\ (mod\ p)$, $lr^n \equiv lr^s\ (mod\ p)$ and $kr^n \equiv kr^{ns}\ (mod\ p)$. Since both i, j can not vanish together. We have two cases, namely either $i \not\equiv 0\ (mod\ p)$ or $j \not\equiv 0\ (mod\ p)$. Also, we will observe that both i and j can not be non-zero at the same time.

First, let $i \not\equiv 0\ (mod\ p)$. Then $r^{s-1} \equiv 1\ (mod\ p)$ which implies that $s \equiv 1\ (mod\ q)$. Also, using $rj \equiv jr^{ns}\ (mod\ p)$, we get $(r^{n-1} - 1)j \equiv 0\ (mod\ p)$. If $j \not\equiv 0\ (mod\ p)$, then $r^{n-1} \equiv 1\ (mod\ p)$ and so, $n \equiv 1\ (mod\ q)$ which is a contradiction. Therefore, $j = 0$. Similarly, $l = 0$. So, in this case $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta(a) = a^s$, where $s \equiv 1\ (mod\ q)$ and $1 \le i, k \le p - 1$.

Now, assume that $j \not\equiv 0\ (mod\ p)$. Then $r^{ns-1} \equiv 1\ (mod\ p)$ which implies that $ns \equiv 1\ (mod\ q)$. Using $kr^n \equiv kr^{ns}\ (mod\ p)$, we get $k \equiv 0\ (mod\ p)$. Now, using $lr^n \equiv lr^s\ (mod\ p)$, $lr^{n^2} \equiv lr^{ns}\ (mod\ p)$ which implies that $l \equiv 0\ (mod\ p)$, otherwise $n^2 \equiv 1\ (mod\ q)$ which is a contradiction. But, then $\alpha \notin GL(2, p)$. Therefore, the case $j \not\equiv 0\ (mod\ p)$ is not possible. Thus, we have $R \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q$. Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \le \lambda, \rho \le p - 1$. Then for any $l(0 \le l \le q^2 - 1)$, we have $\beta(a^{-l}) = b^{\lambda \frac{r^l - 1}{r-1}} c^{\rho \frac{r^{nl}-1}{r-1}}$. Clearly, $\beta(a^{q^2}) = 1$ and so, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_q)$.

Type 28.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi_{r^n} : K \longrightarrow Aut(H)$ be defined as $\phi_{r^n}(a)(b) = a^{-1}ba = b^r, \phi_{r^n}(a)(c) = a^{-1}ca = c^{r^n}$, where $r^{q^2} \equiv 1\ (mod\ p)$ and $2 \le n \le \frac{q^2-1}{2}$ or $n = mq (m \ge \frac{q+1}{2})$. Then $G \simeq H \rtimes_{\phi_{r^n}} K$. Using the similar argument as in the Type (27), we get $R \simeq \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ and $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$.

Type 29.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b^r, \phi(a)(c) = a^{-1}ca = c^r$, where $r^{q^2} \equiv 1\ (mod\ p)$ and $n \in \{2, 3, \cdots, \frac{q-1}{2}\}$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \le s \le q^2 - 1, \gcd(s, q) = 1$, $0 \le i, j, l, k \le p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^{ri}c^{rj} = \alpha(b^r) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^{ir^s} c^{jr^s}$ and $b^{lr} c^{kr} = \alpha(c^r) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^{lr^s} c^{kr^s}$. Then $ri \equiv ir^s\ (mod\ p)$, $rj \equiv jr^s\ (mod\ p)$, $lr \equiv lr^s\ (mod\ p)$ and $kr \equiv kr^s\ (mod\ p)$. Since all i, j, l, k can not vanish together, take $i \not\equiv 0\ (mod\ p)$. Then we get, $r^{s-1} \equiv 1\ (mod\ p)$ which implies that $s \equiv 1\ (mod\ q^2)$. Therefore, $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta(a) = a$, where $0 \le i, j, l, k \le p - 1$ and $ik \not\equiv jl\ (mod\ p)$. Thus $R \simeq GL(2, p)$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = b^\lambda c^\rho$, where $0 \le \lambda, \rho \le p - 1$. Then for any $l(0 \le l \le q^2 - 1)$, we have $\beta(a^{-l}) = b^{\lambda \frac{r^l-1}{r-1}} c^{\rho \frac{r^l-1}{r-1}}$. Clearly, $\beta(a^{q^2}) = 1$ and so, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3,

$Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes GL(2,p)$.

Type 30.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1\rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b^m c^{nD}, \phi(a)(c) = a^{-1}ca = b^n c^m$, where $m + n\sqrt{D} = \sigma^{\frac{p^2-1}{q}}$, σ is a primitive root of Galois Field $GF(p^2)$, $m, n, D \in GF(p)$, $n \neq 0$, D is not a perfect square and q divides $p + 1$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2,p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $1 \leq s \leq q^2 - 1, \gcd(s,q) = 1$, $0 \leq i, j, l, k \leq p-1$ and $ik \not\equiv jl\ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $b^{mi+lnD}c^{jm+knD} = \alpha(b^m c^{nD}) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = b^{iM+jN}c^{iDN+jM}$, and $b^{in+lm}c^{jn+km} = \alpha(b^n c^m) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = b^{lM+kN}c^{lDN+kM}$, where $M = \sum_{t=0}^{[\frac{n}{2}]} {}^sC_{2t} m^{s-2t} n^{2t} D^t$ and $N = \sum_{t=0}^{[\frac{n}{2}]} {}^sC_{2t+1} m^{s-2t-1} n^{2t+1} D^t$. Then

$$mi + lnD \equiv iM + jN\ (mod\ p) \tag{4.1}$$
$$jm + knD \equiv iDN + jM\ (mod\ p) \tag{4.2}$$
$$in + lm \equiv lM + kN\ (mod\ p) \tag{4.3}$$
$$jn + km \equiv lDN + kM\ (mod\ p). \tag{4.4}$$

Note that $(m + n\sqrt{D})^s = M + \sqrt{D}N$ and so, $M = (m + n\sqrt{D})^s - \sqrt{D}N$. Substituting the value of M in the Congruence relations (4.1)-(4.4), we get

$$mi + lnD \equiv i(m + n\sqrt{D})^s + N(j - i\sqrt{D})\ (mod\ p) \tag{4.5}$$
$$jm + knD \equiv j(m + n\sqrt{D})^s - N\sqrt{D}(j - i\sqrt{D})\ (mod\ p) \tag{4.6}$$
$$in + lm \equiv l(m + n\sqrt{D})^s + N(k - l\sqrt{D})\ (mod\ p) \tag{4.7}$$
$$jn + km \equiv k(m + n\sqrt{D})^s - N\sqrt{D}(k - l\sqrt{D})\ (mod\ p). \tag{4.8}$$

Now, applying (4.6) + $\sqrt{D}$(4.5), and (4.8) + $\sqrt{D}$(4.7) we get,

$$m(j + i\sqrt{D}) + nD(k + l\sqrt{D}) \equiv (j + i\sqrt{D})(m + n\sqrt{D})^s\ (mod\ p) \tag{4.9}$$
$$n(j + i\sqrt{D}) + m(k + l\sqrt{D}) \equiv (k + l\sqrt{D})(m + n\sqrt{D})^s\ (mod\ p). \tag{4.10}$$

Applying $(j + i\sqrt{D}) \times$ (4.10) $- (k + l\sqrt{D}) \times$ (4.9), we get

$$0 \equiv n((j + i\sqrt{D})^2 - D(k + l\sqrt{D})^2)(m + n\sqrt{D})^s\ (mod\ p) \tag{4.11}$$
$$\equiv (j^2 + Di^2 - Dk^2 - D^2l^2) + \sqrt{D}(2ij - 2klD)\ (mod\ p). \tag{4.12}$$

This implies that $j^2 + Di^2 - Dk^2 - D^2l^2 \equiv 0\ (mod\ p)$ and $2ij - 2klD \equiv 0\ (mod\ p)$. So, letting $k = ri$, $j = rlD$ for some $r \in GF(p)$, we get $r^2l^2D^2 + Di^2 - Dr^2i^2 - D^2l^2 \equiv 0\ (mod\ p)$. Then, $(r^2-1)(l^2D^2 - Di^2) \equiv 0\ (mod\ p)$. Note that, $l^2D^2 - Di^2 = -D(i + l\sqrt{D})(i - l\sqrt{D})$. So, if $l^2D^2 - Di^2 \equiv 0\ (mod\ p)$, then either $i + l\sqrt{D} \equiv 0\ (mod\ p)$ or $i - l\sqrt{D} \equiv 0\ (mod\ p)$, but this is not possible. So, $r^2 \equiv 1\ (mod\ p)$ which implies $r \equiv \pm 1\ (mod\ p)$. Therefore, $k = \pm i$, $j = \pm lD$.

Now, applying $m \times$ (4.10) $- n \times$ (4.9), we get

$$(k + l\sqrt{D})(m^2 - Dn^2) \equiv (m + n\sqrt{D})^s(m(k + l\sqrt{D}) - n(j + i\sqrt{D})\ (mod\ p).$$

Then, $(\pm i + l\sqrt{D})(m^2 - Dn^2) \equiv (m + n\sqrt{D})^s(m(\pm i + l\sqrt{D}) - n(\pm lD + i\sqrt{D}))\ (mod\ p)$ which implies $m \pm n\sqrt{D} \equiv (m + n\sqrt{D})^s\ (mod\ p)$. Since, $m \pm n\sqrt{D} \in \langle m + n\sqrt{D}\rangle$, $m^2 - Dn^2 \in \langle m + n\sqrt{D}\rangle$. But, one can easily observe that $1 \in \langle m + n\sqrt{D}\rangle$ is the only element without involving $\sqrt{D}$, as $n \neq 0$. So, $m^2 - Dn^2 = 1$ and $(m + n\sqrt{D})^{-1} = m - n\sqrt{D}$. Thus, $(m + n\sqrt{D})^s \equiv (m + n\sqrt{D})^{\pm 1}\ (mod\ p)$

and so, $s \equiv \pm 1 \ (mod\ q)$. Hence, $\alpha = \begin{pmatrix} i & l \\ \pm lD & \pm i \end{pmatrix}$ and $\delta(a) = a^s$, where $1 \le l, j \le p-1$ and $s \equiv \pm 1 \ (mod\ q)$ and so, $R \simeq \langle \left(\begin{pmatrix} i & l \\ ld & i \end{pmatrix}, \delta_s \right), \left(\begin{pmatrix} i & l \\ -ld & -i \end{pmatrix}, \delta_{-s} \right) \rangle \simeq (\mathbb{Z}_{p^2-1} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2$, where $\delta_s(a) = a^s$ and $s \equiv \pm 1 \ (mod\ p)$.

Now. let $\beta \in S$ be defined by $\beta(a^{-1}) = b^{\lambda} c^{\rho}$, where $0 \le \lambda, \rho \le p-1$. Then for any $0 \le l \le q^2 - 1$, we get

$$\beta(a^{-l}) = b^{\lambda X_l + \rho Y_l} c^{\rho X_l + \lambda D Y_l},$$

where

$$X_l = \sum_{u=0}^{[\frac{l-1}{2}]} n^{2u} D^u \sum_{v=2u}^{l-1} {}^vC_{2u} m^{v-2u} \text{ and } Y_l = \sum_{u=1}^{[\frac{l-1}{2}]} n^{2u-1} D^{u-1} \sum_{v=2u-1}^{l-1} {}^vC_{2u-1} m^{v-2u+1}.$$

Now, $1 = \beta(a^{-q^2}) = b^{\lambda X_{q^2} + \rho Y_{q^2}} c^{\rho X_{q^2} + \lambda D Y_{q^2}}$. Then $\lambda X_{q^2} + \rho Y_{q^2} \equiv 0 \ (mod\ p)$ and $\rho X_{q^2} + \lambda D Y_{q^2} \equiv 0 \ (mod\ p)$. Therefore, $(\rho^2 - D\lambda^2)(Y_{q^2}) \equiv 0 \ (mod\ p)$ which implies that $Y_{q^2} \equiv 0 \ (mod\ p)$. Similarly, $X_{q^2} \equiv 0 \ (mod\ p)$ and so, $\beta(a^{-q^2}) = 1$. Thus $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p^2-1} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2)$.

Type 31.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^4 = 1 \rangle \simeq \mathbb{Z}_4$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b^m c^{nD}, \phi(a)(c) = a^{-1}ca = b^n c^m$, where $m + n\sqrt{D} = \sigma^{\frac{p^2-1}{4}}$, σ is a primitive root of Galois Field $GF(p^2)$, $m, n, D \in GF(p)$, $n \ne 0$, D is not a perfect square and $p \equiv 3 \ (mod\ 4)$. Then $G \simeq H \rtimes_\phi K$. Note that, $(m + n\sqrt{D})^2 \equiv -1 \ (mod\ p)$ which implies that $m^2 + n^2 D + 2mn\sqrt{D} \equiv -1 \ (mod\ p)$. Therefore, $m^2 + n^2 D = -1$ and $2mn = 0$. Since $n \ne 0$, $m = 0$ and $n^2 D = -1$. Thus, $b^a = c^{nD}$ and $c^a = b^n$. Now, let $(\alpha, \delta) \in P$. Then $\delta \in Aut(K)$ is defined by $\delta(a) = a^s$ and $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ such that $\alpha(b) = b^i c^j, \alpha(c) = b^l c^k$, where $s \in \{1, 3\}$, $0 \le i, j, l, k \le p-1$ and $ik \not\equiv jl \ (mod\ p)$. Since, $(\alpha, \delta) \in P$, $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and $\alpha(c^a) = \alpha(c)^{\delta(a)}$.

Now, $\alpha(b^a) = \alpha(c^{nD}) = b^{lnD} c^{knD}$ and $\alpha(c^a) = \alpha(b^n) = b^{in} c^{jn}$, $\alpha(b)^{\delta(a)} = (b^i c^j)^{a^s} = \begin{cases} b^{jn} c^{inD} & \text{if } s = 1 \\ b^{-jn} c^{-inD} & \text{if } s = 3 \end{cases}$ and $\alpha(c)^{\delta(a)} = (b^l c^k)^{a^s} = \begin{cases} b^{kn} c^{lnD} & \text{if } s = 1 \\ b^{-kn} c^{-lnD} & \text{if } s = 3 \end{cases}$.

If $s = 1$, then $j = lD$ and $k = i$, and if $s = 3$, then $j = -lD$ and $k = -i$. Thus, by the similar argument in the Type (30), we get $R \simeq \mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2$ and $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2)$.

Type 32.

Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a \mid a^{q^2} = 1 \rangle \simeq \mathbb{Z}_{q^2}$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(b) = a^{-1}ba = b^m c^{nD}, \phi(a)(c) = a^{-1}ca = b^n c^m$, where $m + n\sqrt{D} = \sigma^{\frac{p^2-1}{q^2}}$, σ is a primitive root of Galois field $GF(p^2)$, $m, n, D \in GF(p)$, $n \ne 0$, D is not a perfect square and $p \equiv 3 \ (mod\ 4)$. Then $G \simeq H \rtimes_\phi K$. Using the similar argument as in the Type (31), we get $R \simeq \mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2$ and $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2)$.

Type 33.

Let $H = \langle c, d \mid c^p = d^p = 1, cd = dc \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a, b \mid a^q = b^q = 1, ab = ba \rangle \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(c) = a^{-1}ca = c, \phi(a)(d) = a^{-1}da = d, \phi(b)(c) = b^{-1}cb = c^r, \phi(b)(d) = b^{-1}db = d^r$, where $r^q \equiv 1 \ (mod\ p)$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha, \delta) \in P$. Then $\alpha \in Aut(H) \simeq GL(2, p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta \in Aut(K) \simeq GL(2, q)$ is given by $\delta = \begin{pmatrix} m & s \\ n & t \end{pmatrix}$ such that $\alpha(c) = c^i d^j, \alpha(d) = c^l d^k, \delta(a) = a^m b^n, \delta(b) = a^s b^t$, where $0 \le i, j, l, k \le p-1$

and $ik \not\equiv jl\ (mod\ p)$, and $0 \leq m,n,s,t \leq q-1$ and $mt \not\equiv ns\ (mod\ q)$. Since, $(\alpha,\delta) \in P$, $\alpha(c^a) = \alpha(c)^{\delta(a)}$, $\alpha(d^a) = \alpha(d)^{\delta(a)}$, $\alpha(c^b) = \alpha(c)^{\delta(b)}$ and $\alpha(d^b) = \alpha(d)^{\delta(b)}$.

Now, $c^id^j = \alpha(c) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (c^id^j)^{a^mb^n} = c^{ir^n}d^{jr^n}$, $c^ld^k = \alpha(d) = \alpha(d^a) = \alpha(d)^{\delta(a)} = (c^ld^k)^{a^mb^n} = c^{lr^n}d^{kr^n}$, $c^{ir}d^{jr} = \alpha(c^r) = \alpha(c^b) = \alpha(c)^{\delta(b)} = (c^id^j)^{a^sb^t} = c^{ir^t}d^{jr^t}$ and $c^{lr}d^{kr} = \alpha(d^r) = \alpha(d^b) = \alpha(d)^{\delta(b)} = (c^ld^k)^{a^sb^t} = c^{lr^t}d^{kr^t}$. Then $i \equiv ir^n\ (mod\ p)$, $j \equiv jr^n\ (mod\ p)$, $l \equiv lr^n\ (mod\ p)$, $k \equiv kr^n\ (mod\ p)$, $ir \equiv ir^t\ (mod\ p)$, $jr \equiv jr^t\ (mod\ p)$, $lr \equiv lr^t\ (mod\ p)$ and $kr \equiv kr^t\ (mod\ p)$. Since all i,j,k,l can not vanish together, let $i \neq 0$. Then $r^n \equiv 1\ (mod\ p)$ and $r^t \equiv r\ (mod\ p)$ which implies that $n \equiv 0\ (mod\ q)$ and $t \equiv 1\ (mod\ q)$. Thus, $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta = \begin{pmatrix} m & s \\ 0 & 1 \end{pmatrix}$, where $0 \leq i,j,k,l \leq p-1$, $ik \not\equiv jl\ (mod\ p)$, $1 \leq m \leq q-1$ and $0 \leq s \leq q-1$. Therefore, $R \simeq GL(2,p) \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1})$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = c^\lambda d^\mu$ and $\beta(b^{-1}) = c^\rho d^\nu$, where $0 \leq \lambda,\mu,\rho,\nu \leq p-1$. Now for any $0 \leq u \leq q-1$, we have $\beta(a^{-u}) = c^{u\lambda}d^{u\mu}$ and $\beta(b^{-u}) = c^{\lambda\frac{r^u-1}{r-1}}d^{\mu\frac{r^u-1}{r-1}}$. Then clearly, $\beta(a^{-q}) = c^{q\lambda}d^{q\mu}$ and $\beta(b^{-q}) = c^{\lambda\frac{r^q-1}{r-1}}d^{\mu\frac{r^q-1}{r-1}} = 1$. Since $\beta(a^{-q}) = 1$, both $q\lambda, q\mu \equiv 0\ (mod\ p)$ which implies that $\lambda,\mu \equiv 0\ (mod\ p)$. Thus, $\beta(a^{-1}) = 1$ and $\beta(b^{-1}) = c^\rho d^\nu$, where $0 \leq \rho,\nu \leq p-1$ and so, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (GL(2,p) \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}))$.

Type 34.

Let $H = \langle c,d \mid c^p = d^p = 1, cd = dc\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a,b \mid a^2 = b^2 = 1, ab = ba\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(c) = a^{-1}ca = c, \phi(a)(d) = a^{-1}da = d, \phi(b)(c) = b^{-1}cb = c^{-1}, \phi(b)(d) = b^{-1}db = d$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha,\delta) \in P$. Then $\alpha \in Aut(H) \simeq GL(2,p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta \in Aut(K) \simeq GL(2,q)$ is given by $\delta = \begin{pmatrix} m & s \\ n & t \end{pmatrix}$ such that $\alpha(c) = c^id^j, \alpha(d) = c^ld^k, \delta(a) = a^mb^n, \delta(b) = a^sb^t$, where $0 \leq i,j,k,l \leq p-1$, $ik \not\equiv jl\ (mod\ p)$, $0 \leq m,n,s,t \leq 1$ and $mt \not\equiv ns\ (mod\ 2)$. Since, $(\alpha,\delta) \in P$, $\alpha(c^a) = \alpha(c)^{\delta(a)}$, $\alpha(d^a) = \alpha(d)^{\delta(a)}$, $\alpha(c^b) = \alpha(c)^{\delta(b)}$ and $\alpha(d^b) = \alpha(d)^{\delta(b)}$.

Now, $c^id^j = \alpha(c) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (c^id^j)^{a^mb^n} = c^{i(-1)^n}d^j$, $c^ld^k = \alpha(d) = \alpha(d^a) = \alpha(d)^{\delta(a)} = (c^ld^k)^{a^mb^n} = c^{l(-1)^n}d^k$, $c^{-i}d^{-j} = \alpha(c^{-1}) = \alpha(c^b) = \alpha(c)^{\delta(b)} = (c^id^j)^{a^sb^t} = c^{i(-1)^t}d^j$ and $c^ld^k = \alpha(d) = \alpha(d^b) = \alpha(d)^{\delta(b)} = (c^ld^k)^{a^sb^t} = c^{l(-1)^t}d^k$. Then $i \equiv i(-1)^n\ (mod\ p)$, $l \equiv l(-1)^n\ (mod\ p)$, $-i \equiv i(-1)^t\ (mod\ p)$, $-j \equiv j\ (mod\ p)$, $l \equiv l(-1)^t\ (mod\ p)$. Clearly, $j = 0$. So, $i \neq 0$. Therefore, $(-1)^n \equiv 1\ (mod\ p)$ and $(-1)^t \equiv -1\ (mod\ p)$ which implies that $n = 0$ and $t = 1$ and so, $l = 0$ and $m = 1$. Thus, $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, where $1 \leq i,k \leq p-1$ and $s \in \{0,1\}$. Therefore, $R \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = c^\lambda d^\mu$ and $\beta(b^{-1}) = c^\rho d^\nu$, where $0 \leq \lambda,\mu,\rho,\nu \leq p-1$. Then $\beta(a^{-2}) = c^{2\lambda}d^{2\mu}$. Since, $\beta(a^{-2}) = 1$, $\lambda,\mu \equiv 0\ (mod\ p)$. Also, $\beta(b^{-2}) = d^{2\nu}$, and $\beta(b^{-2}) = 1$. So, $\nu \equiv 0\ (mod\ p)$. Thus, $\beta(a^{-1}) = 1$ and $\beta(b^{-1}) = c^\rho$, where $0 \leq \rho \leq p-1$ and so, $Q \simeq \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq \mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2)$.

Type 35.

Let $H = \langle c,d \mid c^p = d^p = 1, cd = dc\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a,b \mid a^q = b^q = 1, ab = ba\rangle \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(c) = a^{-1}ca = c^r, \phi(a)(d) = a^{-1}da = d, \phi(b)(c) = b^{-1}cb = c, \phi(b)(d) = b^{-1}db = d^r$, where $r^q \equiv 1\ (mod\ p)$. Then $G \simeq H \rtimes_\phi K$. Now, let $(\alpha,\delta) \in P$. Then $\alpha \in Aut(H) \simeq GL(2,p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta \in Aut(K) \simeq GL(2,q)$ is given by $\delta = \begin{pmatrix} m & s \\ n & t \end{pmatrix}$ such that $\alpha(c) = c^id^j, \alpha(d) = c^ld^k, \delta(a) = a^mb^n, \delta(b) = a^sb^t$, where $0 \leq i,j,k,l \leq p-1$, $ik \not\equiv jl\ (mod\ p)$, $0 \leq m,n,s,t \leq 1$ and $mt \not\equiv ns\ (mod\ 2)$. Since, $(\alpha,\delta) \in P$, $\alpha(c^a) = \alpha(c)^{\delta(a)}$, $\alpha(d^a) = \alpha(d)^{\delta(a)}$, $\alpha(c^b) = \alpha(c)^{\delta(b)}$ and $\alpha(d^b) = \alpha(d)^{\delta(b)}$.

Now, $c^{ri}d^{rj} = \alpha(c^r) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (c^id^j)^{a^mb^n} = c^{ir^m}d^{jr^n}$, $c^ld^k = \alpha(d) = \alpha(d^a) = \alpha(d)^{\delta(a)} = (c^ld^k)^{a^mb^n} = c^{lr^m}d^{kr^n}$, $c^id^j = \alpha(c) = \alpha(c^b) = \alpha(c)^{\delta(b)} = (c^id^j)^{a^sb^t} = c^{ir^s}d^{jr^t}$ and $c^{lr}d^{kr} = \alpha(d^r) = \alpha(d^b) = \alpha(d)^{\delta(b)} = (c^ld^k)^{a^sb^t} = c^{lr^s}d^{kr^t}$. Then $ri \equiv ir^m\ (mod\ p)$, $rj \equiv jr^n\ (mod\ p)$, $l \equiv lr^m\ (mod\ p)$, $k \equiv kr^n\ (mod\ p)$, $i \equiv ir^s\ (mod\ p)$, $j \equiv jr^t\ (mod\ p)$, $lr \equiv lr^s\ (mod\ p)$ and $kr \equiv kr^t\ (mod\ p)$. Since i and j both can not vanish together, we have two cases namely, $i \neq 0$ and $j \neq 0$. Also, we will observe that one of i or j must be zero.

If $i \neq 0$, then $m = 1$, $s = 0$, $l = 0$. Since, $l = 0$, $k \neq 0$ and so, $n = 0$ and $t = 1$ which implies that $j = 0$. Thus in this case, $\alpha = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now, if $j \neq 0$, then $n = 1$, $t = 0$ and so, $k = 0$. Since, $k = 0$, $l \neq 0$ and so, $s = 1$, $m = 0$ which implies that $i = 0$. Thus in this case, $\alpha = \begin{pmatrix} 0 & l \\ j & 0 \end{pmatrix}$ and $\delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, $R \simeq \langle \left(\begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & l \\ j & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \rangle \simeq (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$.

Now, let $\beta \in S$ be defined by $\beta(a^{-1}) = c^{\lambda} d^{\mu}$ and $\beta(b^{-1}) = c^{\rho} d^{\nu}$, where $0 \leq \lambda, \mu, \rho, \nu \leq p-1$. Now for any $0 \leq u \leq q-1$, we have $\beta(a^{-u}) = c^{\lambda \frac{r^u-1}{r-1}} d^{u\mu}$ and $\beta(b^{-u}) = c^{u\rho} d^{\nu \frac{r^u-1}{r-1}}$. Then clearly, $\beta(a^{-q}) = d^{q\mu}$ and $\beta(b^{-q}) = c^{q\rho}$. Since $\beta(a^{-q}) = 1 = \beta(b^{-q})$, $q\rho, q\mu \equiv 0 \ (mod\ p)$ which implies that $\rho, \mu \equiv 0 \ (mod\ p)$. Thus, $\beta(a^{-1}) = c^{\lambda}$ and $\beta(b^{-1}) = d^{\nu}$, where $0 \leq \lambda, \nu \leq p-1$ and so, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2)$.

Type 36.

Let $H = \langle c, d \mid c^p = d^p = 1, cd = dc \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, $K = \langle a, b \mid a^q = b^q = 1, ab = ba \rangle \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and the homomorphism $\phi : K \longrightarrow Aut(H)$ be defined as $\phi(a)(c) = a^{-1}ca = c, \phi(a)(d) = a^{-1}da = d, \phi(b)(c) = b^{-1}cb = c^u d^{vD}, \phi(b)(d) = b^{-1}db = c^v d^u$, where $u + v\sqrt{D} = \sigma^{\frac{p^2-1}{q}}$, σ is a primitive root of Galois Field $GF(p^2)$, $u, v, D \in GF(p)$, $v \neq 0$, D is not a perfect square, q divides $p+1$ and $p \not\equiv 1 \ (mod\ q)$. Then $G \simeq H \rtimes_{\phi} K$. Now, let $(\alpha, \delta) \in P$. Then $\alpha \in Aut(H) \simeq GL(2,p)$ is given by $\alpha = \begin{pmatrix} i & l \\ j & k \end{pmatrix}$ and $\delta \in Aut(K) \simeq GL(2,q)$ is given by $\delta = \begin{pmatrix} m & s \\ n & t \end{pmatrix}$ such that $\alpha(c) = c^i d^j, \alpha(d) = c^l d^k, \delta(a) = a^m b^n, \delta(b) = a^s b^t$, where $0 \leq i, j, k, l \leq p-1$, $ik \not\equiv jl \ (mod\ p)$, $0 \leq m, n, s, t \leq 1$ and $mt \not\equiv ns \ (mod\ q)$. Since, $(\alpha, \delta) \in P$, $\alpha(c^a) = \alpha(c)^{\delta(a)}$, $\alpha(d^a) = \alpha(d)^{\delta(a)}$, $\alpha(c^b) = \alpha(c)^{\delta(b)}$ and $\alpha(d^b) = \alpha(d)^{\delta(b)}$.

Now, $c^i d^j = \alpha(c) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (c^i d^j)^{a^m b^n} = c^{iM+jN} d^{iND+jM}$, $c^l d^k = \alpha(d) = \alpha(d^a) = \alpha(d)^{\delta(a)} = (c^l d^k)^{a^m b^n} = c^{lM+kN} d^{lND+kM}$, $c^{iu+lvD} d^{ju+kvD} = \alpha(c^u d^{vD}) = \alpha(c^b) = \alpha(c)^{\delta(b)} = (c^i d^j)^{a^s b^t} = c^{iX+jY} d^{iYD+jX}$, and $c^{iv+lu} d^{jv+ku} = \alpha(c^v d^u) = \alpha(d^b) = \alpha(d)^{\delta(b)} = (c^l d^k)^{a^s b^t} = c^{lX+kY} d^{lYD+kX}$, where

$$M = \sum_{x=0}^{[\frac{n}{2}]} {}^nC_{2x} u^{n-2x} v^{2x} D^x, N = \sum_{x=0}^{[\frac{n}{2}]} {}^nC_{2x+1} v^{n-2x-1} v^{2x+1} D^x,$$

$$X = \sum_{e=0}^{[\frac{t}{2}]} {}^tC_{2e} u^{t-2e} v^{2e} D^e, \text{ and } Y = \sum_{e=0}^{[\frac{t}{2}]} {}^tC_{2e+1} m^{t-2e-1} n^{2e+1} D^e.$$

Then

$$i \equiv iM + jN \ (mod\ p) \tag{4.13}$$
$$j \equiv iND + jM \ (mod\ p) \tag{4.14}$$
$$l \equiv lM + kN \ (mod\ p) \tag{4.15}$$
$$k \equiv lND + kM \ (mod\ p) \tag{4.16}$$
$$iu + lvD \equiv iX + jY \ (mod\ p) \tag{4.17}$$
$$ju + kvD \equiv iYD + jX \ (mod\ p) \tag{4.18}$$
$$iv + lu \equiv lX + kY \ (mod\ p) \tag{4.19}$$
$$jv + ku \equiv lYD + kX \ (mod\ p). \tag{4.20}$$

Note that $(u + v\sqrt{D})^n = M + \sqrt{D}N$ and $(u + v\sqrt{D})^t = X + \sqrt{D}Y$. Now, substituting the value of M

and X in the Congruence relations (4.13)-(4.20), we get

$$
\begin{aligned}
i &\equiv i(u+v\sqrt{D})^n + N(j-i\sqrt{D}) \ (mod\ p) && (4.21)\\
j &\equiv j(u+v\sqrt{D})^n - N\sqrt{D}(j-i\sqrt{D}) \ (mod\ p) && (4.22)\\
l &\equiv l(u+v\sqrt{D})^n + N(k-l\sqrt{D}) \ (mod\ p) && (4.23)\\
k &\equiv k(u+v\sqrt{D})^n - N\sqrt{D}(k-l\sqrt{D}) \ (mod\ p) && (4.24)\\
iu+lvD &\equiv i(u+v\sqrt{D})^t + Y(j-i\sqrt{D}) \ (mod\ p) && (4.25)\\
ju+kvD &\equiv j(u+v\sqrt{D})^t - Y\sqrt{D}(j-i\sqrt{D}) \ (mod\ p) && (4.26)\\
iv+lu &\equiv l(u+v\sqrt{D})^t + Y(k-l\sqrt{D}) \ (mod\ p) && (4.27)\\
jv+ku &\equiv k(u+v\sqrt{D})^t - Y\sqrt{D}(k-l\sqrt{D}) \ (mod\ p). && (4.28)
\end{aligned}
$$

Solving the congruence relations (4.21 - 4.28) we get,

$$
\begin{aligned}
j+i\sqrt{D} &\equiv (j+i\sqrt{D})(u+v\sqrt{D})^n \ (mod\ p) && (4.29)\\
k+l\sqrt{D} &\equiv (k+l\sqrt{D})(u+v\sqrt{D})^n \ (mod\ p) && (4.30)\\
u(j+i\sqrt{D}) + vD(k+l\sqrt{D}) &\equiv (j+i\sqrt{D})(u+v\sqrt{D})^t \ (mod\ p) && (4.31)\\
v(j+i\sqrt{D}) + u(k+l\sqrt{D}) &\equiv (k+l\sqrt{D})(u+v\sqrt{D})^t \ (mod\ p). && (4.32)
\end{aligned}
$$

From the congruence relations (4.29) and (4.30) we get, $(u+v\sqrt{D})^n \equiv 1 \ (mod\ p)$ which implies that $n \equiv 0 \ (mod\ q)$.

Applying $(j+i\sqrt{D}) \times$ (4.32) $- (k+l\sqrt{D}) \times$ (4.31), we get

$$
\begin{aligned}
0 \equiv & v((j+i\sqrt{D})^2 - D(k+l\sqrt{D})^2)(u+v\sqrt{D})^t \ (mod\ p) && (4.33)\\
\equiv & (j^2+Di^2-Dk^2-D^2l^2) + \sqrt{D}(2ij-2lkD) \ (mod\ p). && (4.34)
\end{aligned}
$$

This implies that $j^2+Di^2-Dk^2-D^2l^2 \equiv 0 \ (mod\ p)$ and $2ij-2lkD \equiv 0 \ (mod\ p)$. So, letting $k = ri$, $j = rlD$ for some $r \in GF(p)$, we get $r^2l^2D^2+Di^2-Dr^2i^2-D^2l^2 \equiv 0 \ (mod\ p)$. Then, $(r^2-1)(l^2D^2-Di^2) \equiv 0 \ (mod\ p)$. Note that, $l^2D^2-Di^2 = -D(i+l\sqrt{D})(i-l\sqrt{D})$. So, if $l^2D^2-Di^2 \equiv 0 \ (mod\ p)$, then either $i+l\sqrt{D} \equiv 0 \ (mod\ p)$ or $i-l\sqrt{D} \equiv 0 \ (mod\ p)$, but this is not possible. So, $r^2 \equiv 1 \ (mod\ p)$ which implies $r \equiv \pm 1 \ (mod\ p)$. Therefore, $k = \pm i$, $j = \pm lD$.

Now, applying $u \times$ (4.32) $- v \times$ (4.31), we get $(k+l\sqrt{D})(u^2-Dv^2) \equiv (u+v\sqrt{D})^t(u(k+l\sqrt{D}) - v(j+i\sqrt{D})) \ (mod\ p)$. Then, $(\pm i+l\sqrt{D})(u^2-Dv^2) \equiv (u+v\sqrt{D})^t(u(\pm i+l\sqrt{D}) - v(\pm lD+i\sqrt{D})) \ (mod\ p)$ which implies $u \pm v\sqrt{D} \equiv (u+v\sqrt{D})^t \ (mod\ p)$. Since, $u \pm v\sqrt{D} \in \langle u+v\sqrt{D}\rangle$, $u^2-Dv^2 \in \langle u+v\sqrt{D}\rangle$. But, one can easily observe that $1 \in \langle u+v\sqrt{D}\rangle$ is the only element without involving $\sqrt{D}$, as $v \neq 0$. So, $u^2-Dv^2 = 1$ and $(u+v\sqrt{D})^{-1} = u-v\sqrt{D}$. Thus, $(u+v\sqrt{D})^t \equiv (u+v\sqrt{D})^{\pm 1} \ (mod\ p)$ and so, $t \equiv \pm 1 \ (mod\ q)$. Hence, $\alpha = \begin{pmatrix} i & l \\ \pm lD & \pm i \end{pmatrix}$ and $\delta = \begin{pmatrix} m & s \\ 0 & \pm 1 \end{pmatrix}$, where $1 \le i, l \le p-1$, $1 \le m \le q-1$ and $0 \le s \le q-1$. Thus $R \simeq \mathbb{Z}_2 \times (\mathbb{Z}_{p^2-1} \times (\mathbb{Z}_{q-1} \rtimes \mathbb{Z}_q))$.

Now. let $\beta \in S$ be defined by $\beta(a^{-1}) = c^\lambda d^\mu$, and $\beta(b^{-1}) = c^\rho d^\nu$ where $0 \le \lambda, \mu, \rho, \nu \le p-1$. Then for any $0 \le w \le q-1$, we get

$$\beta(a^{-w}) = c^{w\lambda}d^{w\mu} \text{ and } \beta(b^{-w}) = c^{\rho X_w + \nu Y_w} d^{\nu X_w + \rho D Y_w},$$

where

$$X_w = \sum_{\theta=0}^{[\frac{w-1}{2}]} v^{2\theta} D^\theta \sum_{z=2\theta}^{w-1} {}^zC_{2\theta} u^{z-2\theta} \text{ and } Y_w = \sum_{\theta=1}^{[\frac{w-1}{2}]} v^{2\theta-1} D^{\theta-1} \sum_{z=2\theta-1}^{w-1} {}^zC_{2\theta-1} u^{z-2\theta+1}.$$

Now, $1 = \beta(a^{-q}) = c^{q\lambda}d^{q\mu}$. Then $q\lambda, q\mu \equiv 0 \ (mod\ p)$ which implies that $\lambda, \mu \equiv 0 \ (mod\ p)$. Also, $1 = \beta(b^{-q}) = c^{\rho X_q + \nu Y_q} d^{\nu X_q + \rho D Y_q}$. Then $\rho X_q + \nu Y_q \equiv 0 \ (mod\ p)$ and $\nu X_q + \rho D Y_q \equiv 0 \ (mod\ p)$ which implies that $(\nu^2 - D\rho^2)Y_q \equiv 0\ (mod\ p)$ and so, $Y_q \equiv 0\ (mod\ p)$. Similarly, $X_q \equiv 0\ (mod\ p)$. Thus $\beta(a^{-1}) = 1$ and $\beta(b^{-1}) = c^\rho d^\nu$ where $0 \leq \rho, \nu \leq p-1$. Therefore, $Q \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, using the Theorem 2.3, $Aut(G) \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_2 \times \mathbb{Z}_{p^2-1}) \times (\mathbb{Z}_{q-1} \rtimes \mathbb{Z}_q))$.

Thus the automorphism groups for all groups of order p^2q^2 upto isomorphism are listed in the following Tables 1 and 2.

Table 1: Structure of $Aut(G)$ of all groups G of order p^2q^2 such that $pq = 6$.

Type	Group Description (G)	Structure of $Aut(G)$
1	$\mathbb{Z}_{36}$	$U_{36} \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$
2	$\mathbb{Z}_{18} \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times S_3$
3	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$S_3 \times Gl(2,3)$
4	$\mathbb{Z}_{12} \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times GL(2,3)$
5	$\mathbb{Z}_6 \times D_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$
6	$D_3 \times D_3$	$(S_3 \times S_3) \rtimes \mathbb{Z}_2$
7	$D_9 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times (\mathbb{Z}_9 \rtimes \mathbb{Z}_6)$
8	$A_4 \times \mathbb{Z}_3$	$S_4 \times S_3$
9	D_{18}	$\mathbb{Z}_2 \times (\mathbb{Z}_9 \rtimes \mathbb{Z}_6)$
10	$(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times S_3$
11	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times ((((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$
12	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (\mathbb{Z}_8 \rtimes \mathbb{Z}_2)$
13	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$	$\mathbb{Z}_3 \times S_4$
14	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (GL(2,3) \times \mathbb{Z}_2)$

Table 2: Structure of $Aut(G)$ of all groups G of order p^2q^2 such that $pq \neq 6$.

Type	Conditions	Group Description (G)	Structure of $Aut(G)$
15		$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$	$\mathbb{Z}_{p(p-1)} \times \mathbb{Z}_{q(q-1)}$
16		$\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_{p^2}$	$GL(2,q) \times \mathbb{Z}_{p(p-1)}$
17		$\mathbb{Z}_{q^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_{q(q-1)} \times GL(2,p)$
18		$\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_p \times \mathbb{Z}_p$	$GL(2,q) \times GL(2,p)$
19	$q \mid p-1$	$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{q^2}$	$\mathbb{Z}_{p^2} \rtimes (\mathbb{Z}p(p-1) \times \mathbb{Z}_q)$
20	$q^2 \mid p-1$	$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{q^2}$	$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}p(p-1)$
21	$q \mid p-1$	$\mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$	$\mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}))$
22	$q \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi_r} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (GL(2,p) \times \mathbb{Z}_q)$
23		$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_4$	$\mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2)$
24	$q \mid p-1, q \neq 2$	$\mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_{q^2})$	$\mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q)$
25	$q \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi_{r^{-1}}} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2)$
26	$q^2 \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi_{r^{-1}}} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2)$
27	$q \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi_{r^n}} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_q)$
28	$q^2 \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi_{r^n}} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$
29	$q^2 \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\phi} \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes GL(2,p)$
30	$q \mid p+1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p^2-1} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_2)$
31	$p \equiv 3 \ (mod\ 4)$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_4$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2)$
32	$q^2 \mid p+1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p^2-1} \rtimes \mathbb{Z}_2)$
33	$q \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_q$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (GL(2,p) \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}))$
34		$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	$\mathbb{Z}_p \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2)$
35	$q \mid p-1$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_2)$
36	$q \mid p+1$, $p \not\equiv 1 \ (mod\ q)$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes ((\mathbb{Z}_2 \times \mathbb{Z}_{p^2-1}) \times (\mathbb{Z}_{q-1} \rtimes \mathbb{Z}_q))$

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